Oscillatory motions, parabolic orbits and collision orbits in the planar circular restricted three-body problem

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Abstract

In this paper we consider the planar circular restricted three body problem (PCRTBP), which models the motion of a massless body under the attraction of other two bodies, the primaries, which describe circular orbits around their common center of mass. In a suitable system of coordinates, this is a two degrees of freedom Hamiltonian system. The orbits of this system are either defined for all (future or past) time or eventually go to collision with one of the primaries. For orbits defined for all time, Chazy provided a classification of all possible asymptotic behaviors, usually called final motions.

By considering a sufficiently small mass ratio between the primaries, we analyze the interplay between collision orbits and various final motions and construct several types of dynamics.

In particular, we show that orbits corresponding to any combination of past and future final motions can be created to pass arbitrarily close to the massive primary. Additionally, we construct arbitrarily large ejection-collision orbits (orbits which experience collision in both past and future times) and periodic orbits that are arbitrarily large and get arbitrarily close to the massive primary. Furthermore, we also establish oscillatory motions in both position and velocity, meaning that as time tends to infinity, the superior limit of the position or velocity is infinity while the inferior limit remains a real number.

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Contents

1 Introduction

Understanding the long term dynamics of the 3 body problem is one of the longstanding questions in dynamical systems. Following the seminal works by Painlevé [46], it is well known that orbits are either defined for all positive time or either two or three bodies collide (same happens regarding the past behavior).

Chazy in 1922 classified the (future and past) final motions of the 3 Body Problem. They are defined as the possible asymptotic behaviors that the orbits which are defined for all positive (negative) time can have as time tends to infinity (minus infinity).

Let us denote by q_0, q_1, q_2 the positions of the three bodies and by r_k the vector from q_i to q_j for $i \neq k$, $j \neq k$, $i < j$. The following theorem classifies the possible final motions of the 3 body problem in terms of the mutual distances r_i.

Theorem 1.1 (Chazy, 1922, see also [5])**.** *Every solution of the 3 Body Problem defined for all (future) time belongs to one of the following seven classes.*

- *Hyperbolic (H):* $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i = 0, 1, 2,$ *as* $t \to \infty$.
- *Hyperbolic–Parabolic (HP_k):* $|r_i| \to \infty$, $i = 0, 1, 2$, $|\dot{r}_k| \to 0$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$, *as* $t \to \infty$.
- *Hyperbolic–Elliptic, (HE_k):* $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i = 0, 1, 2$, $i \neq k$, *as* $t \to \infty$, $\sup_{t \geq t_0} |r_k| < \infty$.
- *Parabolic-Elliptic (PE_k):* $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i = 0, 1, 2$, $i \neq k$, *as* $t \to \infty$, $\sup_{t \geq t_0} |r_k| < \infty$.
- *Parabolic (P):* $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i = 0, 1, 2$, *as* $t \to \infty$.
- *Bounded (B):* $\sup_{t \geq t_0} |r_i| < \infty$, $i = 0, 1, 2$.
- *Oscillatory (OS):* $\limsup_{t \to \infty} \sup_{i=0,1,2} |r_i| = \infty$ *and* $\liminf_{t \to \infty} \sup_{i=0,1,2} |r_i| < \infty$.

Note that this classification applies both when $t \to +\infty$ or $t \to -\infty$. To distinguish both cases we add a superindex $+$ or $-$ to each of the cases, e.g H^+ and H^-. Among the admissible final motions, it is known that any future and past combination of them can be obtained for almost all choices of masses (see [1, 28, 42]).

Besides the question of existence of such motions, there is the question about their "abundance". It turns out that for any combination of past and future final motion it is known whether the set has zero or positive measure except for $\mathcal{OS}^+ \cap \mathcal{OS}^-$ (see [5]). As is pointed out in [25], V. Arnol'd, in the conference in honor of the 70th anniversary of Alekseev, posed the following question.

Conjecture 1.2. *The Lebesgue measure of the set* $\mathcal{OS}^+ \cap \mathcal{OS}^-$ *is equal to zero.*

This conjecture is wide open. Another fundamental question is what is the topology of this set. For instance, in some restricted 3 body problems, the set of oscillatory motions has maximal Hausdorff dimension (see [25]). On the other hand, a famous conjecture by Herman (see [29]), if true, would imply that the set of bounded orbits is nowhere dense.

The mentioned classification and conjectures refer to orbits defined for all time. That is, they exclude orbits hitting collisions. How "large" is the set of such orbits? Saari [50, 49] (see also [20, 21]) proved that the set of orbits hitting a collision has zero measure. However, it may form a topologically "rich" set as stated in the following conjecture by Alekseev [1] (it actually might be traced back to Siegel, Sec. 8, P. 49 in [51]).

Conjecture 1.3. *Is there an open subset $\mathcal{U}$ of the phase space such that for a dense subset of initial conditions the associated trajectories go to a collision?*

The purpose of this paper is to analyze the interplay between the different types of final motions and collision orbits in the planar circular restricted 3-body problem (PCRTBP).

1.1 Main results

The PCRTBP models the motion of an infinitesimal body P_3 under the attraction of two massive bodies called primaries, labeled as P_1 and P_2. Since P_3 is considered to have zero mass, it does not exert any force onto the other two bodies, which are assumed to perform circular motions around their common center of mass and are coplanar with the motion of P_3. In this configuration one can normalize the masses of P_1 and P_2 as $m_1 = 1 - \mu$ and $m_2 = \mu$ with $\mu \in (0, 1/2]$. Since in this paper we will consider μ small, we will refer to P_1 as the Sun and to P_2 as Jupiter.

Taking the appropriate units and considering rotating coordinates, the PCRTBP is Hamiltonian with respect to

$$\mathcal{H}(q,p;\mu) = \frac{||p||^2}{2} - q_1p_2 + q_2p_1 - \frac{1-\mu}{||q+(\mu,0)||} - \frac{\mu}{||q-(1-\mu,0)||}, \tag{1.1}$$

where $q, p \in \mathbb{R}^2$ and $P_1 = (-\mu, 0)$ and $P_2 = (1-\mu, 0)$ are the position of the primaries, which are fixed.

Remark 1.4. *From now on, the Hamiltonian $\mathcal{H}$ will be referred to as the energy. It should be noted that this is a shorthand for Jacobi's integral.*

The PCRTBP is reversible with respect to the symmetry

$$(q_1, q_2, p_1, p_2; t) \to (q_1, -q_2, -p_1, p_2; -t). \tag{1.2}$$

We define the collision with the Sun as the set

$$\mathcal{S} = \{(q,p) \in \mathbb{R}^4 \colon q = (-\mu, 0)\}. \tag{1.3}$$

For the PCRTBP the Chazy classification is reduced to

- Hyperbolic ($\mathcal{H}^\pm$) : $\lim_{t\to\pm\infty} ||q(t)|| = \infty$ and $\lim_{t\to\pm\infty} ||\dot{q}(t)|| = c > 0$.
- Parabolic ($\mathcal{P}^\pm$) : $\lim_{t\to\pm\infty} ||q(t)|| = \infty$ and $\lim_{t\to\pm\infty} ||\dot{q}(t)|| = 0$.
- Bounded ($\mathcal{B}^\pm$) : $\limsup_{t\to\pm\infty} ||q(t)|| < \infty$.
- Oscillatory ($\mathcal{OS}^\pm$) : $\limsup_{t\to\pm\infty} ||q(t)|| = \infty$ and $\liminf_{t\to\pm\infty} ||q(t)|| < \infty$.

For any $\mu \in (0, 1/2]$, it is known (see [27, 34]) that

$$X^+ \cap Y^- \neq \emptyset \quad \text{where} \quad X, Y = \mathcal{H}, \mathcal{P}, \mathcal{B}, \mathcal{OS}.$$

Note that, when $\mu = 0$, the PCRTBP is reduced to a Kepler problem and therefore in this case $\mathcal{OS}^\pm = \emptyset$ and $\mathcal{H}^+ = \mathcal{H}^-$, $\mathcal{P}^+ = \mathcal{P}^-$, $\mathcal{B}^+ = \mathcal{B}^-$. On the contrary, for any $\mu > 0$, any combination of past and future final motion is possible.

Let us define also the ejection and collision orbits.

Definition 1.5. *Consider an orbit* $(q(t), p(t))$ *of the Hamiltonian* (1.1)*. Then, we call this orbit*

- *Ejection orbit (from the Sun) if there exists* $t_0 \in \mathbb{R}$ *such that* $\lim_{t\to t_0^-} q(t) = (-\mu, 0)$.
- *Collision orbit (to the Sun) if there exists* $t_1 \in \mathbb{R}$ *such that* $\lim_{t\to t_1^+} q(t) = (-\mu, 0)$.
- *Ejection-collision orbit if it is both collision and ejection orbit.*

The main results of the present paper are the following. The first one analyzes the interplay between the different final motions and ejection/collision orbits.

Theorem 1.6. *There exists* $\mu_0 > 0$ *such that, for any* $\mu \in (0, \mu_0)$,

$$\overline{X^+ \cap Y^-} \cap \mathcal{S} \neq \emptyset \qquad \text{where} \qquad X, Y = \mathcal{H}, \mathcal{P}, \mathcal{B}, \mathcal{OS}.$$

Moreover,

- *There exist orbits* $(q(t), p(t))$ *of* (1.1) *which are oscillatory and get arbitrarily close to collision with the Sun. Namely, they satisfy*

$$\limsup_{t\to\pm\infty} ||q(t)|| = \infty \quad \text{and} \quad \liminf_{t\to\pm\infty} ||q(t) + (\mu, 0)|| = 0.$$

In particular, this also implies that

$$\limsup_{t\to\pm\infty} ||\dot{q}(t)|| = \infty \quad \text{and} \quad \liminf_{t\to\pm\infty} ||\dot{q}(t)|| < \infty. \tag{1.4}$$

- *For any* $\varepsilon > 0$*, there exists a periodic orbit* $(q(t), p(t))$ *of* (1.1) *satisfying*

$$\sup_{t\in\mathbb{R}} ||q(t)|| \geq \varepsilon^{-1} \quad \text{and} \quad \inf_{t\in\mathbb{R}} ||q(t) + (\mu, 0)|| \leq \varepsilon.$$

The second main theorem shows the existence of hyperbolic sets (whose dynamics is conjugated to the infinite symbols shift) which are unbounded and contain the Sun in its closure.

Theorem 1.7. *There exists* $\eta > 0$ *and* $\mu_0 > 0$ *such that, for any* $\mu \in (0, \mu_0)$ *and any energy* $h \in (-\eta\mu, \eta\mu)$ *there exists, in the energy level* $\mathcal{H}(q, p; \mu) = h$*, a section* Π *transverse to the flow of* (1.1) *such that the induced Poincaré map*

$$\mathbb{P} : U \subset \Pi \to \Pi$$

has an invariant set $\mathcal{X}$ *which is homeomorphic to* $\mathbb{N}^{\mathbb{Z}}$ *and whose dynamics* $\mathbb{P} : \mathcal{X} \to \mathcal{X}$ *is topologically conjugated to the shift* $\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$, $(\sigma\omega)_k = \omega_{k+1}$*. Moreover, this invariant set satisfies that its closure intersects* $\mathcal{S}$ *and contains ejection and collision orbits to* $\mathcal{S}$*, orbits in* $\mathcal{P}^+$ *and orbits in* $\mathcal{P}^-$.

This theorem implies that the "set of chaotic motions" accumulate both at collision with the Sun and at infinity. The construction of these hyperbolic sets accumulating to $\mathcal{P}^\pm$ was already achieved in [27, 34]. The main novelty of this theorem is that such sets can be also constructed accumulating to ejection and collision orbits.

Theorems 1.6 and 1.7 are proved together in Section 6. In fact, the existence of the periodic and oscillatory motions described in Theorem 1.6 are a direct consequence of Theorem 1.7.

We are also able to construct "large" ejection-collision orbits.

Theorem 1.8. *There exists $\eta > 0$ and $\mu_0 > 0$ such that, for any $\mu \in (0, \mu_0)$ and any energy $h \in (-\eta\mu, \eta\mu)$, one can find a sequence of ejection-collision orbits $\{z_k(t)\}_{k\in\mathbb{N}}$, $z_k(t) = (q_k(t), p_k(t))$ in the energy level $\mathcal{H}(q, p; \mu) = h$ such that*

$$\limsup_{k\to\infty} \left(\sup_{t\in\mathbb{R}} ||q_k(t)|| \right) = +\infty.$$

Theorem 1.6 gives the existence of periodic orbits and oscillatory orbits which can pass as close to the Sun as determined and also as large as determined. It also implies, for instance, that, for any $\varepsilon > 0$, there exists a (forward and backward) parabolic orbit $(q(t), p(t))$ such that

$$\inf_{t\in\mathbb{R}} ||q(t) + (\mu, 0)|| \leq \varepsilon.$$

Concerning Theorem 1.8, it provides the first proof of existence of "large" ejection-collision orbits. Indeed, all those provided by the previous results (see Section 1.2 below) were arbitrarily small.

1.2 Literature and previous results

The literature on the analysis of final motions and collision orbits is abundant.

Concerning the combination of different past and future final motions, it can be traced back to the work by Sitnikov for the nowadays called Sitnikov problem [53], who showed in this model that all combinations of past and future motion was possible, and in particular constructed oscillatory motions. A decade later Moser [43] gave a new proof of the same results relying on dynamical systems tools and relating them to chaotic motions (Smale horseshoes). After Moser, his approach has been implemented in other restricted three body problems [27, 28, 34] (see [22, 23, 24, 47] for results using other methods). The first results for the (non-restricted) 3 body problem were obtained by Alekseev [2, 3, 4], which has recently been generalized in [28].

Concerning collision dynamics, as already mentioned, Saari proved that the set of colliding orbits has zero Lebesgue measure [50, 49]. Even if Conjecture 1.3 is wide open, a partial answer was given in [26], where the authors prove that the set of orbits leading to collision in the PCRTBP is μ^α-dense (for some $\alpha > 0$) in some open set of phase space.

The first proof of existence of ejection-collision orbits for the PCRTBP was achieved by Lacomba and Llibre [31, 33] (see also [48]), where they prove their existence for small mass ratio and large Jacobi constant, which implies that the orbits are very close to collision (in contraposition to those obtained in Theorem 1.8 which can make arbitrarily large excursions).

These results have been later generalized in [44, 45, 35]. More recently, relying on computer assisted proofs, Capiński, Kepley and Mireles [9] have constructed ejection-collision orbits involving the two primaries (and other orbits which close passages to both collisions) and Capiński and Pasiut [10] have constructed orbits which oscillate between collision and a compact set of phase space away from collision (these orbits are oscillatory in velocity in the sense of (1.4) but not oscillatory in the sense of Chazy).

Another dynamics associated to collisions are the so-called punctured tori, that is invariant quasiperiodic tori (for the Levi-Civita regularized three body problems) which contain binary collisions [11, 15, 16, 55].

Passages close to collision also allow constructing the so-called second species periodic solutions, which are periodic orbits such that the massless body has a certain number of close encounters with the small primary (see [36, 8, 7, 6]).

Triple collision is not possible in the restricted 3 body problem, but it is indeed possible in the full 3 body problem. It has been widely studied since the pioneering results by McGehee [39] (see also [13, 14, 32, 41, 52]). The analysis of triple collisions has lead to the construction of a large variety of motions in the 3 body problem. In particular, to oscillatory motions both in position and velocity (see Moeckel [40, 42]). Note that this analysis requires close passages to triple collision and therefore one must be in the regime of total angular momentum very close to zero. On the contrary, in the present paper there are two bodies performing circular motion (and therefore have large angular momenta) whereas the other has small angular momentum.

1.3 Main ideas for the proofs of Theorems 1.6, 1.7 and 1.8

The orbits constructed in Theorems 1.8 and 1.6 rely on developing an invariant manifold theory for "singular" invariant objects that the PCRTBP possesses. Those are the collision (with the Sun) and infinity, which after some compactification can be seen as invariant objects for the regularized flow. This will require certain changes of coordinates (and time reparameterizations): one to deal with binary collision (similar to that considered by McGehee to analyze triple collision, see [39]) and a different one to compactify infinity, also first used by McGehee (see [38]). These changes of variables are explained in Section 2.

After these transformations, on the one hand the collision set $\mathcal{S}$ (see (1.3)) becomes an invariant torus which contains two circles that are normally hyperbolic invariant manifolds and are foliated by critical points. On the other hand, the "parabolic infinity", that is the limit of parabolic orbits, at a fixed energy level becomes a periodic orbit. This periodic orbit is degenerate (the linearization of the vector field at it vanishes) but it is well known that it possesses stable and unstable invariant manifolds (see [37]).

We analyze these invariant manifolds and, relying on perturbative methods (suitable versions of Poincaré-Melnikov Theory) we prove that they intersect transversally. These intersections plus the local analysis close to collision and infinity lead to the construction of the different types of motions provided by Theorems 1.6 and 1.8.

Let us be more precise.

1. We prove that the stable manifold of infinity intersects transversally the unstable manifold of the collision and the unstable manifold of infinity intersects transversally the stable manifold of the collision (see Section 3).

2. Relying on the local analysis close to infinity (at the $\mathcal{C}^1$ level) done in [43], one can prove that the stable and unstable invariant manifolds of the collision set intersect transversally and that these intersections can be arbitrarily far away from the Sun (see Section 4). This proves Theorem 1.8.

3. We analyze the dynamics close to the collision set, and we prove a $\mathcal{C}^1$ Lambda lemma type statement for the passage close to collision (see Section 5). This local analysis leads to transverse intersections between the stable and unstable manifolds of infinity close (but at a fixed distance) to

collision. Proceeding as in [43], one can construct hyperbolic sets with symbolic dynamics which contain the homoclinic points to infinity in its closure (but not containing the Sun in its closure). This leads to oscillatory motions passing close to collision (and combination of past and future different final motions), but not to oscillatory motions which have the Sun at its closure, and it does not imply Theorems 1.7 and 1.6.

4. To prove these theorems we have to further analyze the invariant manifolds of infinity and the collision. We rely on what we call, with a strong abuse of language, triple intersection of invariant manifolds. It is well known that stable invariant manifolds of different objects cannot intersect. So let us explain what do we mean by that. For an open interval of energy levels, we have transverse intersections of the stable manifold of collision with the unstable of infinity (and also the other way around). We say that we have triple intersection if, moreover, these two intersections belong to the stable/unstable leaf *of the same equilibrium* point in the collision set. We prove that there exists an energy level where this happens (see Section 6). Then, relying on the local analysis close to collision and the tools developed in Moser [43] one can construct the behaviors provided by Theorems 1.7 and 1.6.

2 Analysis of the invariant manifolds

The first step towards a proof of Theorems 1.6, 1.7 and 1.8 consists on the analysis of the collision and infinity "invariant sets". To this end, in Section 2.1, we introduce the so-called *McGehee coordinates at infinity* (see for instance [34, 38, 43]) to give a proper definition of the *parabolic infinity* set and analyze the dynamics "close" to it. On the other hand, the Hamiltonian (1.1) is singular at the collision set $\mathcal{S}$. To regularize it, in Section 2.2 we use the *McGehee coordinates at collision* (see [39, 45]).

2.1 The McGehee coordinates at infinity

To define the McGehee infinity coordinates, we first express the Hamiltonian (1.1) in (synodical) polar coordinates centered at the center of mass, defined by

$$\begin{aligned} q_1 &= \hat{r}\cos\hat{\theta} \quad p_1 = \hat{R}\cos\hat{\theta} - \frac{\hat{\Theta}}{\hat{r}}\sin\hat{\theta}, \\ q_2 &= \hat{r}\sin\theta \quad p_2 = \hat{R}\sin\hat{\theta} + \frac{\hat{\Theta}}{\hat{r}}\cos\hat{\theta}, \end{aligned} \tag{2.1}$$

which leads to the Hamiltonian

$$\hat{\mathcal{H}}(\hat{r},\hat{\theta},\hat{R},\hat{\Theta}) = \frac{1}{2}\left(\hat{R}^2 + \frac{\hat{\Theta}^2}{\hat{r}^2}\right) - \frac{1}{\hat{r}} - \hat{\Theta} - \hat{V}(\hat{r},\hat{\theta};\mu), \tag{2.2}$$

where

$$\hat{V}(\hat{r},\hat{\theta};\mu) = \frac{1-\mu}{\left(\hat{r}^2 + 2\hat{r}\mu\cos\hat{\theta} + \mu^2\right)^{\frac{1}{2}}} + \frac{\mu}{\left(\hat{r}^2 - 2\hat{r}(1-\mu)\cos\hat{\theta} + (1-\mu)^2\right)^{\frac{1}{2}}} - \frac{1}{\hat{r}}, \tag{2.3}$$

and the symmetry (1.2) becomes

$$\left(\hat{r},\hat{\theta},\hat{R},\hat{\Theta};t\right) \to \left(\hat{r},-\hat{\theta},-\hat{R},\hat{\Theta};-t\right). \tag{2.4}$$

Then, we define the McGehee change of coordinates (see [38]) as

$$\hat{r} = 2\xi^{-2}, \tag{2.5}$$

in which the parabolic infinity

$$\mathrm{A} = \left\{(\hat{r}, \hat{\theta}, \hat{R}, \hat{\Theta}) \colon \hat{r} = +\infty, \hat{R} = 0, \hat{\theta} \in \mathbb{T}, \hat{\Theta} \in \mathbb{R}\right\}$$

becomes

$$\mathrm{A} = \left\{(\xi, \hat{\theta}, \hat{R}, \hat{\Theta}) \colon \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}^2 \colon \xi = 0, \hat{R} = 0\right\}. \tag{2.6}$$

Applying the change of coordinates (2.5) to the equations of motion associated to Hamiltonian $\hat{\mathcal{H}}$ in (2.2) leads to

$$\begin{aligned} \frac{d\xi}{dt} &= -\frac{\hat{R}\xi^3}{4} \\ \frac{d\hat{\theta}}{dt} &= \frac{\hat{\Theta}}{4}\xi^4 - 1 \\ \frac{d\hat{R}}{dt} &= -\frac{\xi^4}{4} + \frac{\hat{\Theta}^2}{8}\xi^6 - \frac{\xi^3}{4}\partial_\xi \hat{V}(\xi, \hat{\theta}; \mu) \\ \frac{d\hat{\Theta}}{dt} &= \partial_{\hat{\theta}} \hat{V}(\xi, \hat{\theta}; \mu), \end{aligned} \tag{2.7}$$

where

$$\hat{V}(\xi, \hat{\theta}; \mu) = \frac{\xi^2}{2}\left(\frac{1-\mu}{\left(1 + \xi^2\mu\cos\hat{\theta} + \frac{\xi^4}{4}\mu^2\right)^{\frac{1}{2}}} + \frac{\mu}{\left(1 - \xi^2(1-\mu)\cos\hat{\theta} + \frac{\xi^4}{4}(1-\mu)^2\right)^{\frac{1}{2}}} - 1\right).$$

Note that the change of variables (2.5) is not symplectic, so the new vector field is no longer Hamiltonian. Nevertheless, the Hamiltonian (2.2) becomes a first integral of system (2.7) and is given by

$$\hat{H}(\xi, \hat{\theta}, \hat{R}, \hat{\Theta}) = \frac{1}{2}\left(\hat{R}^2 + \frac{\hat{\Theta}^2\xi^4}{4}\right) - \frac{\xi^2}{2} - \hat{\Theta} - \hat{V}(\xi, \theta; \mu).$$

From (2.7), one obtains that the manifold A in (2.6) is foliated by periodic orbits as $\mathrm{A} = \bigcup_{\hat{\Theta}_0 \in \mathbb{R}} \mathrm{A}_{\hat{\Theta}_0}$ with

$$\mathrm{A}_{\hat{\Theta}_0} = \left\{(\xi, \hat{\theta}, \hat{R}, \hat{\Theta}) \colon \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}^2 \colon \xi = 0, \hat{R} = 0, \hat{\Theta} = \hat{\Theta}_0\right\}. \tag{2.8}$$

In [37] it was proven that these periodic orbits have stable and unstable manifolds, which we denote by $W^s_\mu(\mathrm{A}_{\hat{\Theta}_0})$ and $W^u_\mu(\mathrm{A}_{\hat{\Theta}_0})$ respectively. Moreover, these manifolds depend analytically on $\hat{\Theta}_0$.

In [27] and [34] it is shown that $W^{s,u}_\mu(\mathrm{A}_{\hat{\Theta}_0})$ intersect transversally for any $\mu \in \left(0, \frac{1}{2}\right]$ if $\hat{\Theta}_0$ is big enough. Theorem 3.3 will guarantee this transversality for small values of $\hat{\Theta}_0$ and $\mu > 0$.

Note that the rates of convergence of the invariant manifolds $W^{s,u}_\mu(\mathrm{A}_{\hat{\Theta}_0})$ are polynomial in t and not exponential as in the case of normally hyperbolic invariant manifolds. For this reason, in [12] and [27], the set A in (2.6) is called a normally parabolic invariant manifold.

2.2 The McGehee coordinates at collision

To study the collision set $\mathcal{S}$ in (1.3), we express first the Hamiltonian (1.1) in synodical polar coordinates centered at the primary we want to regularize, i.e, $\mathcal{S}$.

$$
\begin{aligned}
q_1 &= -\mu + r\cos\theta \quad & p_1 &= R\cos\theta - \frac{\Theta}{r}\sin\theta, \\
q_2 &= r\sin\theta & p_2 &= R\sin\theta + \frac{\Theta}{r}\cos\theta.
\end{aligned}
\tag{2.9}
$$

In these new coordinates, the Hamiltonian $\mathcal{H}$ in (1.1) becomes

$$
\mathcal{H}(r,\theta,R,\Theta;\mu) = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{1}{r} - \Theta - V(r,\theta,R,\Theta;\mu), \tag{2.10}
$$

where

$$
V(r,\theta,R,\Theta;\mu) = -\mu\left(\frac{1}{r} + R\sin\theta + \frac{\Theta}{r}\cos\theta - \frac{1}{\sqrt{1+r^2-2r\cos\theta}}\right).
$$

Moreover, the symmetry (1.2) now reads

$$
(r,\theta,R,\Theta;t) \to (r,-\theta,-R,\Theta;-t). \tag{2.11}
$$

In these coordinates, the collision set $\mathcal{S}$ in (1.3) becomes $\{r=0\}$.

Following [39], we perform the transformation

$$
\begin{aligned}
\psi\colon \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}^2 &\to \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}^2 \\
(r,\theta,v,u) &\mapsto (r,\theta,R,\Theta) = \left(r,\theta, vr^{-\frac{1}{2}} - \mu\sin\theta, ur^{\frac{1}{2}} + r^2 - \mu r\cos\theta\right).
\end{aligned}
\tag{2.12}
$$

and the change of time

$$
dt = r^{\frac{3}{2}}d\tau,
$$

so that the equations of motion associated to the Hamiltonian $\mathcal{H}$ in (2.10) become

$$
\begin{aligned}
r' &= rv \\
\theta' &= u \\
v' &= \frac{v^2}{2} + u^2 + 2ur^{\frac{3}{2}} + r^3 - 1 + \mu\left[1 - r^2\left(\cos\theta + \frac{r-\cos\theta}{(1+r^2-2r\cos\theta)^{\frac{3}{2}}}\right)\right] \\
u' &= -\frac{uv}{2} - 2vr^{\frac{3}{2}} + \mu r^2\sin\theta\left[1 - \frac{1}{(1+r^2-2r\cos\theta)^{\frac{3}{2}}}\right],
\end{aligned}
\tag{2.13}
$$

where $'$ denotes $\frac{d}{d\tau}$. Observe that (2.13) is now regular at $r=0$.

The change of variables in (2.12) is not symplectic but the Hamiltonian $\mathcal{H}$ in (2.10) is still a first integral of (2.13). Moreover, the energy level $\{\mathcal{H}=h\}$ is now given by $(\mathcal{H}-h)\circ\psi = 0$, where

$$
(\mathcal{H}-h)\circ\psi(r,\theta,v,u) = -h + \frac{v^2+u^2}{2r} - \frac{r^2}{2} - \frac{1-\mu}{r} + \mu\left[-\frac{\mu}{2} + r\cos\theta - \frac{1}{\sqrt{1+r^2-2r\cos\theta}}\right].
$$

We now multiply by r to remove the singularity, obtaining

$$
\tilde{M}(r,\theta,v,u;\mu,h) = -\,rh + \frac{v^2+u^2}{2} - \frac{r^3}{2} - 1 + \mu + \mu r\left[-\frac{\mu}{2} + r\cos\theta - \frac{1}{\sqrt{1+r^2-2r\cos\theta}}\right].
$$

The orbits belonging to the hypersurface $\{\mathcal{H}(r,\theta,R,\Theta;\mu) = h\}$, including the ejection and collision ones, now lie in $\{\tilde{M}(r,\theta,v,u;\mu,h) = 0\}$. Therefore, we study (2.13) restricted to this manifold. It is convenient to introduce a last change of coordinates

$$\begin{aligned}\tilde{\psi}\colon \mathbb{R}^+ \times \mathbb{T}^2 \times \mathbb{R}^+ &\to \mathbb{R} \times \mathbb{T} \times \mathbb{R}^2 \\ (s,\theta,\alpha,\rho) &\mapsto (r,\theta,v,u) = \left(s^2, \theta, \sqrt{2(1-\mu)+\rho}\sin\alpha, \sqrt{2(1-\mu)+\rho}\cos\alpha\right),\end{aligned} \tag{2.14}$$

such that $\{\tilde{M}=0\}$ becomes

$$0 = M(s,\theta,\alpha,\rho;\mu,h) = -\rho + 2s^2 h + s^6 - 2\mu s^2\left[-\frac{\mu}{2} + s^2\cos\theta - \frac{1}{\sqrt{1+s^4-2s^2\cos\theta}}\right]. \tag{2.15}$$

Remark 2.1. *Note that we have taken $r = s^2$ so, from now on, it is enough to analyze $s \geq 0$.*

To study the motion in coordinates (s,θ,α,ρ), we define the 3-dimensional submanifold

$$\mathcal{M} = \{(s,\theta,\rho,\alpha) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R} \times \mathbb{T} \colon M(s,\theta,\rho,\alpha;\mu,h) = 0\}, \tag{2.16}$$

Therefore, using (s,θ,α) as coordinates in $\mathcal{M}$, the vector field (2.13) writes

$$\begin{aligned} s' &= \frac{s}{2}\sqrt{2(1-\mu)+\rho}\sin\alpha \\ \theta' &= \sqrt{2(1-\mu)+\rho}\cos\alpha \\ \alpha' &= \frac{\rho'}{2[2(1-\mu)+\rho]\tan\alpha} + \frac{\sqrt{2(1-\mu)+\rho}}{2}\cos\alpha + 2s^3 \\ &\quad - \frac{\mu s^4\sin\theta}{\sqrt{2(1-\mu)+\rho}\sin\alpha}\left[1 - \frac{1}{(1+s^4-2s^2\cos\theta)^{\frac{3}{2}}}\right], \end{aligned} \tag{2.17}$$

where ρ can be obtained from (2.15).

The collision manifold $\{r=0\}$ expressed in coordinates (s,θ,α) becomes the invariant torus

$$\Omega = \{(0,\theta,\alpha) \colon \theta \in \mathbb{T}, \alpha \in \mathbb{T}\} \subset \mathcal{M} \tag{2.18}$$

whose dynamics is given by

$$\begin{aligned} \theta' &= m_0\cos\alpha \\ \alpha' &= \frac{m_0}{2}\cos\alpha \end{aligned} \qquad \text{where } m_0 = \sqrt{2(1-\mu)}. \tag{2.19}$$

This system has two circles of critical points

$$S^+ = \left\{S^+_{\overline{\theta}} = \left(0,\overline{\theta},\frac{\pi}{2}\right) : \overline{\theta} \in \mathbb{T}\right\}, \qquad S^- = \left\{S^-_{\overline{\theta}} = \left(0,\overline{\theta},-\frac{\pi}{2}\right) : \overline{\theta} \in \mathbb{T}\right\}. \tag{2.20}$$

Next lemma analyzes the dynamics of system (2.17) close to these circles.

Lemma 2.2. *Consider the system* (2.17) *for $0 \leq \mu \leq \frac{1}{2}$. Then*

- *The invariant circles $S^{\pm}$ in* (2.20) *are normally hyperbolic. Moreover, they have* 2*-dimensional stable and unstable manifolds $W^{u,s}_{\mu}(S^{\pm}) = \bigcup_{\overline{\theta}\in\mathbb{T}} W^{u,s}_{\mu}(S^{\pm}_{\overline{\theta}})$.*

- *$W^s_\mu(S^+)$ and $W^u_\mu(S^-)$ are contained in Ω. Moreover, they coincide*

$$\Gamma = W^s_\mu(S^+) = W^u_\mu(S^-) \subset \Omega.$$

Therefore

$$\Omega = S^+ \cup S^- \cup \Gamma,$$

and Γ is foliated by a family of heteroclinic orbits between $S^-_{\overline{\theta}}$ and $S^+_{\overline{\theta}}$, for $\overline{\theta} \in \mathbb{T}$. The heteroclinic orbits in $\Gamma \cap \{\alpha \in (-\pi/2, \pi/2)\}$ can be parameterized as

$$\gamma_h\left(\tau;\overline{\theta}\right) = \left(0, \theta_h\left(\tau;\overline{\theta}\right), \alpha_h\left(\tau;\overline{\theta}\right)\right)$$

such that

$$\begin{aligned} \theta_h\left(\tau;\overline{\theta}\right) &= \overline{\theta} + \pi + 2\alpha_h\left(\tau;\overline{\theta}\right) \\ \alpha_h\left(\tau;\overline{\theta}\right) &= 2\tan^{-1}\left(\tanh\left(\frac{m_0\tau}{4}\right)\right). \end{aligned}$$

- *$W^s_\mu(S^-)$ and $W^u_\mu(S^+)$ belong to $\mathcal{M} \setminus \Omega$. When $\mu = 0$, in coordinates (r, θ, v, u), $W^u_0(S^+)$ can be parameterized by its trajectories as*

$$\left(\tilde{r}_h(\tau), \tilde{\theta}_h(\tau;\overline{\theta}), \tilde{v}_h(\tau), \tilde{u}_h(\tau)\right) = \left(\kappa e^{\sqrt{2}\tau}, \overline{\theta} - e^{\frac{3}{\sqrt{2}}\tau}, \sqrt{2}, -\kappa^{\frac{3}{2}} e^{\frac{3}{\sqrt{2}}\tau}\right) \tag{2.21}$$

with $\overline{\theta} \in \mathbb{T}$ and $\tau \in \mathbb{R}$ satisfying

$$\lim_{\tau\to-\infty}\left(\tilde{r}_h(\tau), \tilde{\theta}_h(\tau;\overline{\theta}), \tilde{v}_h(\tau), \tilde{u}_h(\tau)\right) = (0, \overline{\theta}, \sqrt{2}, 0) \in S^+_{\overline{\theta}}.$$

Symmetry (2.11) *gives us an analogous result for $W^s_0(S^-)$.*

Proof. The proof of the first item is a direct consequence of the expression of the differential of the vector field F_μ associated to (2.17) evaluated at the equilibrium points $S^\pm_{\overline{\theta}}$ in (2.20), which is given by

$$DF_\mu(S^\pm_{\overline{\theta}}) = \begin{pmatrix} \pm\frac{m_0}{2} & 0 & 0 \\ 0 & 0 & \mp m_0 \\ 0 & 0 & \mp\frac{m_0}{2} \end{pmatrix},$$

where we recall that $m_0 = \sqrt{2(1-\mu)}$, and whose corresponding eigenvalues are

$$\lambda_s = \pm\frac{m_0}{2}, \quad \lambda_\theta = 0, \quad \lambda_\alpha = \mp\frac{m_0}{2}.$$

The proof of the second item follows from integrating (2.19) and the third item from integrating (2.13) for $\mu = 0$. □

Remark 2.3. *The definitions of* $W^s_\mu(S^-)$ *and* $W^u_\mu(S^+)$ *can be translated to coordinates* (r,θ,R,Θ) *by means of the changes* (2.12) *and* (2.14)*. Abusing the notation, we denote the collision and ejection manifolds as*

$$
\begin{aligned}
W^s_\mu(S^-) =& \Big\{ (r,\theta,R,\Theta)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}^2 \colon \exists t_* = t_*(r,\theta,R,\Theta) > 0 \textit{ such that} \\
& \lim_{t\to t_*^-}\Phi_t^r(r,\theta,R,\Theta) = 0,\ \lim_{t\to t_*^-}\Phi_t^R(r,\theta,R,\Theta) = -\infty \Big\}, \\
W^u_\mu(S^+) =& \Big\{ (r,\theta,R,\Theta)\in\mathbb{R}^+\times\mathbb{T}\times\mathbb{R}^2 \colon \exists t_* = t_*(r,\theta,R,\Theta) < 0 \textit{ such that} \\
& \lim_{t\to t_*^+}\Phi_t^r(r,\theta,R,\Theta) = 0,\ \lim_{t\to t_*^+}\Phi_t^R(r,\theta,R,\Theta) = +\infty \Big\},
\end{aligned}
\tag{2.22}
$$

where Φ_t *refers to the flow of the equations of motion associated to the Hamiltonian* $\mathcal{H}$ *in* (2.10)*.*

We stress that, although invariant, they are not stable and unstable manifolds of any invariant objects since S^+ *and* S^- *collapse to the singular set* $\{r=0\}$*.*

2.3 The unperturbed case $\mu = 0$

As we will see in Sections 4 and 6, both proofs of Theorems 1.8 and 1.6 are based on the analysis of the invariant manifolds of infinity and collision respectively. To this end, the purpose of this section is to study them when $\mu = 0$. Since both synodical polar coordinates (2.1) and (2.9) are identical when $\mu = 0$, in this section we will use the notation for the synodical polar coordinates (r,θ,R,Θ) to study the dynamics.

When $\mu = 0$, the PCRTBP in synodical polar coordinates is defined by the integrable Hamiltonian

$$
\mathcal{H}(r,\theta,R,\Theta;0) = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{1}{r} - \Theta. \tag{2.23}
$$

The ejection and collision orbits of this Hamiltonian (see Definition 1.5) belong to $\{\Theta = 0\}$ and, at the energy level $\mathcal{H} = 0$, correspond to "heteroclinic connections" between $S^\pm$ in (2.20) and A_0 in (2.8).

Lemma 2.4. *The invariant manifolds* $W_0^{s,u}(\mathrm{A}_0)$ *of the system associated to Hamiltonian* (2.23) *can be written as*

$$
W_0^s(\mathrm{A}_0) = \left\{ \varphi_0^+(t,\overline{\theta}) \colon t>0, \overline{\theta}\in\mathbb{T} \right\} \quad \textit{and} \quad W_0^u(\mathrm{A}_0) = \left\{ \varphi_0^-(t,\overline{\theta}) \colon t<0, \overline{\theta}\in\mathbb{T} \right\}
$$

where the trajectories $\varphi_0^+(t;\overline{\theta})$ *and* $\varphi_0^-(t;\overline{\theta})$ *are given by*

$$
\begin{aligned}
\varphi_0^+(t;\overline{\theta}) &= \left(r_0^+(t), \theta_0^+(t;\overline{\theta}), R_0^+(t), \Theta_0^+(t)\right) = \left(\kappa t^{\frac{2}{3}}, \overline{\theta} - t, \sqrt{\frac{2}{\kappa}}\frac{1}{t^{\frac{1}{3}}}, 0\right) && \forall\, t>0, \\
\varphi_0^-(t;\overline{\theta}) &= \left(r_0^-(t), \theta_0^-(t;\overline{\theta}), R_0^-(t), \Theta_0^-(t)\right) = \left(\kappa t^{\frac{2}{3}}, \overline{\theta} - t, -\sqrt{\frac{2}{\kappa}}\frac{1}{|t|^{\frac{1}{3}}}, 0\right) && \forall\, t<0,
\end{aligned}
$$

where $\kappa = \frac{3^{\frac{2}{3}}}{2^{\frac{1}{3}}}$.

Therefore,

$$W_0^u(\mathrm{A}_0) = W_0^s(S^-) = \bigcup_{\overline{\theta}\in\mathbb{T}} W_0^s(S_{\overline{\theta}}^-),$$
$$W_0^s(\mathrm{A}_0) = W_0^u(S^+) = \bigcup_{\overline{\theta}\in\mathbb{T}} W_0^u(S_{\overline{\theta}}^+).$$

where $W_0^{s,u}(S^\pm)$ *are defined in* (2.22)*.*

2.4 The perturbed invariant manifolds of infinity

The next proposition gives a parameterization, in the synodical polar coordinates (2.1), of the invariant manifolds $W_\mu^{s,u}(\mathrm{A}_{\hat{\Theta}_0})$ close to the unperturbed ones obtained in Lemma 2.4, for $\mu > 0$ small enough and $\hat{\Theta}_0$ of order μ.

Note that for $\mu \neq 0$, the vector field associated to the Hamiltonian (2.2) becomes singular at $(\hat{r}, \hat{\theta}) = (\mu, \pi)$ and $(\hat{r}, \hat{\theta}) = (1-\mu, 0)$ – which correspond to the position of the primaries P_1 and P_2 respectively. Therefore, when extending the invariant manifolds one has to exclude neighborhoods of these points.

We provide the statement for the stable manifold. One can deduce an analogous result for the unstable one using that the system is reversible with respect to (2.4).

Proposition 2.5. *Fix* a, b *with* $a < b \in \mathbb{R}$, $\hat{D} \in (0, 1/2)$, $\hat{W} > 0$ *and* $\overline{\Theta}_0 \in [a, b]$*. Then, there exists* $\mu_0 > 0$ *small enough such that for* $0 < \mu < \mu_0$ *and* $\hat{\Theta}_0 = \mu\overline{\Theta}_0$*, a subset of the invariant manifold* $W_\mu^s(\mathrm{A}_{\hat{\Theta}_0})$*, which we denote by* $\tilde{W}_\mu^s(\mathrm{A}_{\hat{\Theta}_0})$*, can be written as*

$$\tilde{W}_\mu^s(\mathrm{A}_{\hat{\Theta}_0}) = \left\{(\hat{r}, \hat{\theta}, \hat{R}, \hat{\Theta}) = \hat{\Upsilon}^+(\hat{w}, \hat{\theta}, \hat{\Theta}_0) \colon (\hat{w}, \hat{\theta}) \in \hat{\mathcal{D}}^+_{\hat{W},\hat{D}}\right\}$$

where $\hat{\Upsilon}^+$ *is real-analytic and of the form*

$$\hat{\Upsilon}^+(\hat{w}, \hat{\theta}, \hat{\Theta}_0) = \left(\kappa\hat{w}^{\frac{2}{3}}, \hat{\theta}, \hat{R}^s_\infty(\hat{w}, \hat{\theta}, \hat{\Theta}_0), \hat{\Theta}^s_\infty(\hat{w}, \hat{\theta}, \hat{\Theta}_0)\right), \tag{2.24}$$

with

$$\begin{aligned}
\hat{R}^s_\infty(\hat{w}, \hat{\theta}, \hat{\Theta}_0) =& \sqrt{\frac{2}{\kappa}}\frac{1}{\hat{w}^{\frac{1}{3}}} + \mathcal{O}_1(\mu) \\
\hat{\Theta}^s_\infty(\hat{w}, \hat{\theta}, \hat{\Theta}_0) =& \hat{\Theta}_0 - \mu\kappa \int_{+\infty}^{\hat{w}} \left(\frac{\hat{s}^{\frac{2}{3}} \sin(\hat{\theta} + \hat{w} - \hat{s})}{\left(\kappa^2\hat{s}^{\frac{4}{3}} - 2\kappa\hat{s}^{\frac{2}{3}}\cos(\hat{\theta} + \hat{w} - \hat{s}) + 1\right)^{\frac{3}{2}}} \right. \\
&\left. - \frac{\sin(\hat{\theta} + \hat{w} - \hat{s})}{\kappa^3\hat{s}^{\frac{4}{3}}} \right) d\hat{s} + \mathcal{O}_2(\mu)
\end{aligned} \tag{2.25}$$

where $\kappa = \frac{3^{\frac{2}{3}}}{2^{\frac{1}{3}}}$*, and the domain* $\hat{\mathcal{D}}^+_{\hat{W},\hat{D}}$ *is defined as*

$$\hat{\mathcal{D}}^+_{\hat{W},\hat{D}} = \left[\hat{W}, +\infty\right) \times \hat{I}^+_{\hat{D}}(\hat{w})$$

with

$$\hat{I}^+_{\hat{D}}(\hat{w}) = \begin{cases} \mathbb{T} - \left(\frac{\sqrt{2}}{3} - \hat{w} - \hat{D}, \frac{\sqrt{2}}{3} - \hat{w} + \hat{D}\right) & \textit{if} \quad \hat{W} \leq \hat{w} \leq \frac{\sqrt{2}}{3}(1-\mu)^{\frac{3}{2}} + \hat{D}, \\ \mathbb{T} & \textit{if} \quad \hat{w} > \frac{\sqrt{2}}{3}(1-\mu)^{\frac{3}{2}} + \hat{D}. \end{cases} \tag{2.26}$$

Proof. We denote by $\hat{F}$ the vector field associated to the Hamiltonian (2.2), which can be written as

$$\hat{F}(\hat{r}, \hat{\theta}, \hat{R}, \hat{\Theta}) = \hat{F}_0(\hat{r}, \hat{\theta}, \hat{R}, \hat{\Theta}) + \hat{F}_1(\hat{r}, \hat{\theta}, \hat{R}, \hat{\Theta})$$

such that

$$\begin{aligned} \hat{F}_0(\hat{r}, \hat{R}, \hat{\Theta}) &= \left(\hat{R}, \frac{\hat{\Theta}}{\hat{r}^2} - 1, \frac{\hat{\Theta}^2}{\hat{r}^3} - \frac{1}{\hat{r}^2}, 0\right)^T \\ \hat{F}_1(\hat{r}, \hat{\theta}) &= \Big(0, 0, \partial_{\hat{r}} V(\hat{r}, \hat{\theta}; \mu), \partial_{\hat{\theta}} V(\hat{r}, \hat{\theta}; \mu)\Big)^T, \end{aligned} \tag{2.27}$$

where $V(\hat{r}, \hat{\theta}; \mu)$ is the potential in (2.3). The vector field $\hat{F}$ is regular for $\hat{r} \geq \hat{r}_0, \hat{\theta} \in \mathbb{T}$ for any $\hat{r}_0 > 1$.

Observe that for $\mu = 0$, the 2-dimensional invariant manifold $W^s_\mu(\mathrm{A}_{\hat{\Theta}_0})$ can be parameterized as a graph over $\hat{r}$, $\hat{\theta}$. For $\hat{r} > 1$, we can make use of the regularity with respect to parameters proven by McGehee in [37] to ensure that the perturbed manifold is μ-close to the unperturbed one given in Lemma 2.4, and therefore still a graph over the same variables. For convenience, we change the parameter $\hat{r}$ into $\hat{w}$ by

$$\hat{r} = \kappa \hat{w}^{\frac{2}{3}}, \quad \hat{w} > \hat{W}.$$

Then, the graph parameterization of the perturbed invariant manifold is given by

$$\hat{r}(\hat{w}) = \kappa \hat{w}^{\frac{2}{3}}, \quad \hat{\theta} = \hat{\theta}, \quad \hat{R}(\hat{w}, \hat{\theta}) = \sqrt{\frac{2}{\kappa}} \frac{1}{\hat{w}^{\frac{1}{3}}} + \mathcal{O}_1(\mu), \quad \hat{\Theta}(\hat{w}, \hat{\theta}) = \mathcal{O}_1(\mu). \tag{2.28}$$

Hence, the proof consists on extending the previous parameterization of $W^s_\mu(\mathrm{A}_{\hat{\Theta}_0})$ from $\{\hat{r} = \hat{r}_0 > 1\}$ to a section $\{\hat{r} = \hat{r}_F\}$ satisfying $\mu < \hat{r}_F < 1 - \mu$.

By (2.28), we can fix $\hat{w}_0 > \kappa^{-\frac{3}{2}}$ such that $\hat{r}_0 = \hat{r}(\hat{w}_0)$ and we consider the set of points of the form

$$W^s_\mu(\mathrm{A}_{\hat{\Theta}_0}) \bigcap \{\hat{w} = \hat{w}_0\} = \left\{ \left(\hat{r}_0, \hat{\theta}_0, \hat{R}(\hat{w}_0, \hat{\theta}_0), \hat{\Theta}(\hat{w}_0, \hat{\theta}_0)\right), \hat{\theta}_0 \in \mathbb{T} \right\}. \tag{2.29}$$

Recall that in Section 2.3, for $\mu = 0$ we have analyzed the invariant manifolds of infinity for the Hamiltonian $\hat{\mathcal{H}}$ in (2.2) restricted to the plane $\{\hat{\mathcal{H}} = \hat{\Theta} = 0\}$. Then, if we denote

$$z_0(t; \hat{\theta}_0) = \hat{\Phi}_0\left(t, \left(\hat{r}(\hat{w}_0), \hat{\theta}_0, \hat{R}(\hat{w}_0, \hat{\theta}_0), \hat{\Theta}(\hat{w}_0, \hat{\theta}_0)\right)\right) = \left(z_0^{\hat{r}}, z_0^{\hat{\theta}}, z_0^{\hat{R}}, z_0^{\hat{\Theta}}\right),$$

we have that

$$z_0^{\hat{r}}(t) = \left(\hat{r}(\hat{w}_0)^{\frac{3}{2}} + \frac{3t}{\sqrt{2}}\right)^{\frac{2}{3}}, \quad z_0^{\hat{\theta}}(t; \hat{\theta}_0) = \hat{\theta}_0 - t, \quad z_0^{\hat{R}}(t) = \sqrt{\frac{2}{z_0^{\hat{r}}(t)}}, \quad z_0^{\hat{\Theta}}(t; \hat{\theta}_0) = 0,$$

where $\hat{\Phi}_0$ is the flow of the vector field $\hat{F}_0$ in (2.27).

Then we can compute $(t_0, \hat{\theta}_0^0)$ such that

$$z_0^r(t_0; \hat{\theta}_0^0) = 1, \quad z_0^\theta(t_0; \hat{\theta}_0^0) = 0,$$

which corresponds to the position of the primary P_2 (when $\mu = 0$) and gives

$$\hat\theta_0^0 = t_0 = -\frac{\sqrt{2}}{3}\left(\hat r(\hat w_0)^{\frac{3}{2}} - 1\right) < 0.$$

Taking $0 < \hat d < \frac{1}{4}$, we define the following set of "bad" initial conditions close to $\hat\theta_0^0$

$$B_{\hat d}(\hat\theta_0^0) = \left\{\hat\theta_0 \in \mathbb{T}\colon |\hat\theta_0 - \hat\theta_0^0| < \hat d\right\}.$$

Any $\hat\theta_0 \in B_{\hat d}(\hat\theta_0^0)$ satisfies that

$$z_0^r(t_0;\hat\theta_0) = 1, \quad z_0^{\hat\theta}(t_0;\hat\theta_0) = \hat\theta_0 - t_0 \in \left(-\hat d, \hat d\right).$$

That is, the unperturbed flow sends $B_{\hat d}(\hat\theta_0^0)$ to a neighborhood of P_2 at the section $\hat r = 1$ of the form

$$B_{\hat d}(0) = \left\{\hat\theta \in \mathbb{T}\colon |\hat\theta| < \hat d\right\}.$$

Now we consider $\mu > 0$ small enough. In this case, for each point in (2.29), we denote its trajectory by

$$z_\mu(t;\hat\theta_0) = \hat\Phi_\mu\left(t, \left(\hat r(\hat w_0), \hat\theta_0, \hat R(\hat w_0, \hat\theta_0), \hat\Theta(\hat w_0, \hat\theta_0)\right)\right) = \left(z_\mu^{\hat r}, z_\mu^{\hat\theta}, z_\mu^{\hat R}, z_\mu^{\hat\Theta}\right),$$

where $\hat\Phi_\mu$ is the flow of the vector field $\hat F$ in (2.27).

Note that, for $\mu = 0$, any point in $W_\mu^s(\mathrm{A}_{\hat\Theta_0})$ has $\frac{d}{dt}\hat r = \hat R > 0$. Since for $\hat\theta \notin B_{\hat d}(0)$ the vector field $\hat F$ in (2.27) is regular with respect to μ, we have the same behaviour at $W_\mu^s(\mathrm{A}_{\hat\Theta_0})$ for $\mu > 0$ small enough.

Therefore, if we fix $\hat D = 2\hat d < \frac{1}{2}$ and we consider a set of initial conditions of the form

$$B_{\hat D}^{\mathrm{init}}(\hat\theta_0^0) = \left\{\hat\theta \in \mathbb{T}\colon |\hat\theta_0 - \hat\theta_0^0| < \hat D\right\}$$

then, for every $\hat\theta_0 \notin B_{\hat D}^{\mathrm{init}}(\hat\theta_0^0)$, there exists $t_F^\mu(\hat\theta_0) < 0$ such that $z_\mu^{\hat r}(t_F^\mu(\hat\theta_0), \hat\theta_0) = \hat r_F < 1 - \mu$. In particular, if we denote by $t_\mu^*(\hat\theta_0) \in \left[t_F^\mu(\hat\theta_0), 0\right]$ the time such that $z_\mu^{\hat r}(t_\mu^*(\hat\theta_0), \hat\theta_0) = 1 - \mu$, we know that $z_\mu^{\hat\theta}(t_\mu^*(\hat\theta_0), \hat\theta_0) = \hat\theta \notin B_{\hat d}(0)$.

Hence, the perturbed vector field $\hat F_1$ in (2.27) is uniformly bounded. Namely, there exist $C_1, C_2, C_3 > 0$ such that, for $t \in [t_F^\mu(\hat\theta_0), 0]$ and $\hat\theta_0 \notin B_{\hat D}(\hat\theta_0^0)$,

$$\left|\left|\hat F_1\left(z_\mu^{\hat r}(t;\hat\theta_0), z_\mu^{\hat\theta}(t;\hat\theta_0)\right)\right|\right| \le C_1 + \frac{C_2}{(\hat r(\hat w_0) - \mu)^3} + \mu\frac{C_3}{\left(1 - \cos\hat d\right)^{\frac{3}{2}}}.$$

This implies that, for $\hat\theta_0 \notin B_{\hat D}(\hat\theta_0^0)$ we have

$$z_\mu(t;\hat\theta_0) = z_0(t;\hat\theta_0) + \mathcal{O}_1(\mu). \tag{2.30}$$

Therefore, the parameterization (2.24) is well-defined for $\hat\theta$ defined in (2.26) until the section $\{\hat r = \hat r_F\}$.

The second part of the proof comes as a result of the fundamental theorem of calculus, along with the fact that

$$\lim_{t\to+\infty} z_\mu^{\hat\Theta}(t, \hat\theta_0) = \hat\Theta_0$$

by definition of $\tilde W_\mu^s(\mathrm{A}_{\hat\Theta_0})$. The μ-expansion for the equation of $\frac{d}{dt}\hat\Theta$, which corresponds to the fourth component of $\hat F_1$ in (2.27), the expression of the potential $\hat V$ in (2.3) and the approximation (2.30) allows us to write the μ-expansion of the $\hat\Theta$-component of the curve $\hat\Upsilon^+$ in (2.25). □

Although in Proposition 2.5 we give a parameterization of $\tilde{W}^s_\mu(\mathrm{A}_{\hat{\Theta}_0})$, in order to prove the main results stated in Section 1.1, we need to compare it with the invariant manifolds of collision (see Section 3) in a common set of coordinates. We do the comparison in polar coordinates centered at P_1 (2.9) and therefore we must reparameterize the invariant manifold of infinity $\tilde{W}^s_\mu(\mathrm{A}_{\hat{\Theta}_0})$ obtained in Proposition 2.5.

Proposition 2.6. *Fix $a, b \in \mathbb{R}$ with $a < b$, $D \in (0, 1/2)$, $W > 0$ and $\overline{\Theta}_0 \in [a, b]$. Then there exists $\mu_0 > 0$ small enough such that for $0 < \mu < \mu_0$ and $\hat{\Theta}_0 = \mu\overline{\Theta}_0$, a subset of the stable manifold $W^s_\mu(\mathrm{A}_{\hat{\Theta}_0})$, which we denote by $\tilde{W}^s_\mu(\mathrm{A}_{\hat{\Theta}_0})$, can be written as*

$$\tilde{W}^s_\mu(\mathrm{A}_{\hat{\Theta}_0}) = \left\{(r, \theta, R, \Theta) = \Upsilon^+(w, \theta, \hat{\Theta}_0) \colon (w, \theta) \in \mathcal{D}^+_{W,D}\right\},$$

where Υ^+ is a C^∞-function of the form

$$\Upsilon^+(w, \theta, \hat{\Theta}_0) = \left(\kappa w^{\frac{2}{3}}, \theta, R^s_\infty(w, \theta, \hat{\Theta}_0), \Theta^s_\infty(w, \theta, \hat{\Theta}_0)\right),$$

where

$$\begin{aligned} R^s_\infty(w, \theta, \hat{\Theta}_0) &= \sqrt{\frac{2}{\kappa}}\frac{1}{w^{\frac{1}{3}}} + \mathcal{O}_1(\mu), \\ \Theta^s_\infty(w, \theta, \hat{\Theta}_0) &= \hat{\Theta}_0 - \mu\kappa \int_{+\infty}^{w} \frac{s^{\frac{2}{3}} \sin(\theta + w - s)}{\left(1 + \kappa^2 s^{\frac{4}{3}} - 2\kappa s^{\frac{2}{3}} \cos(\theta + w - s)\right)^{\frac{3}{2}}} \, ds \\ &\quad - \mu\sqrt{\frac{2}{\kappa}} \int_{+\infty}^{w} \frac{\cos(\theta + w - s)}{s^{\frac{1}{3}}} \, ds + \mathcal{O}_2(\mu), \end{aligned} \tag{2.31}$$

with $\kappa = \frac{3^{\frac{2}{3}}}{2^{\frac{1}{3}}}$, and the domain $\mathcal{D}^+_{W,D}$ is defined as

$$\mathcal{D}^+_{W,D} = [W, +\infty) \times I^+_D(w)$$

with

$$I^+_D(w) = \begin{cases} \mathbb{T} - \left(\frac{\sqrt{2}}{3} - w - D, \frac{\sqrt{2}}{3} - w + D\right) & \text{if} \quad W \le w \le \frac{\sqrt{2}}{3} + D, \\ \mathbb{T} & \text{if } w > \frac{\sqrt{2}}{3} + D. \end{cases} \tag{2.32}$$

Analogously, due to the symmetry (2.11), a subset of the unstable manifold $W^u_\mu(\mathrm{A}_{\hat{\Theta}_0})$, which we denote by $\tilde{W}^u_\mu(\mathrm{A}_{\hat{\Theta}_0})$, can be written as

$$\tilde{W}^u_\mu(\mathrm{A}_{\hat{\Theta}_0}) = \left\{\Upsilon^-(w, \theta) \colon (w, \theta) \in \mathcal{D}^-_{W,D}\right\},$$

where

$$\Upsilon^-(w, \theta, \hat{\Theta}_0) = \Upsilon^+(-w, -\theta, \hat{\Theta}_0)$$

and the domain $\mathcal{D}^-_{W,D}$ is defined as

$$\mathcal{D}^-_{W,D} = (-\infty, -W] \times I^-_D(w)$$

with

$$I^-_D(w) = \begin{cases} \mathbb{T} - \left(-\frac{\sqrt{2}}{3} + w - D, -\frac{\sqrt{2}}{3} - w + D\right) & \text{if } -\frac{\sqrt{2}}{3} - D \le w \le -W \\ \mathbb{T} & \text{if } w < -\frac{\sqrt{2}}{3} - D \end{cases} \tag{2.33}$$

In particular, the Θ-component of Υ^- can be written as

$$\Theta^u_\infty(w, \theta, \hat{\Theta}_0) = \Theta^s_\infty(-w, -\theta, \hat{\Theta}_0). \tag{2.34}$$

Proof. We consider the transformation from rotating polar coordinates centered at P_1 (2.9) to the ones centered at the center of mass (2.1)

$$\begin{aligned}
r &= \sqrt{\hat{r}^2 + 2\hat{r}\mu\cos\hat{\theta} + \mu^2} \\
\theta &= \tan^{-1}\left(\frac{\hat{r}\sin\hat{\theta}}{\hat{r}\cos\hat{\theta} + \mu}\right) \\
R &= \hat{R}\frac{r - \mu\cos\theta}{\sqrt{r^2 - 2\mu r\cos\theta + \mu^2}} - \mu\hat{\Theta}\frac{r\sin\theta}{r^2 - 2\mu r\cos\theta + \mu^2}, \\
\Theta &= \mu\hat{R}\frac{r\sin\theta}{\sqrt{r^2 - 2\mu r\cos\theta + \mu^2}} + \hat{\Theta}\frac{r(r - \mu\cos\theta)}{r^2 - 2\mu r\cos\theta + \mu^2}.
\end{aligned} \tag{2.35}$$

This transformation satisfies

$$\begin{aligned}
r &= \hat{r} + \mu\cos\hat{\theta} + \mathcal{O}\left(\frac{\mu^2}{\hat{r}}\right), \\
\theta &= \hat{\theta} - \mu\frac{\sin\hat{\theta}}{\hat{r}} + \mathcal{O}\left(\frac{\mu^2}{\hat{r}^2}\right), \\
R &= \hat{R} + \mathcal{O}_1(\mu), \\
\Theta &= \hat{\Theta}\left(1 + \frac{\mu\cos\theta}{r}\right) + \mu\hat{R}\sin\theta + \mathcal{O}_2(\mu).
\end{aligned} \tag{2.36}$$

For any $\hat{W} > 0$, Proposition 2.5 gives us the parameterization of the invariant manifold in (2.24) such that $\hat{r} = \kappa\hat{w}^{\frac{2}{3}}$. Therefore, for $\hat{w} > \hat{W}$, equation (2.36) leads to

$$r = \kappa\hat{w}^{\frac{2}{3}} + \mathcal{O}_1(\mu).$$

To have a graph parameterization analogous to the one in Proposition 2.5, we define w such that $r = \kappa w^{\frac{2}{3}}$. Substituting in (2.35) we obtain

$$\hat{w}(w,\theta) = \frac{\left(\kappa^2 w^{\frac{4}{3}} - 2\kappa\mu w^{\frac{2}{3}}\cos\theta + \mu^2\right)^{\frac{3}{4}}}{\kappa^{\frac{3}{2}}} = w + \mathcal{O}_1(\mu). \tag{2.37}$$

Hence, given $W > 0$, one can find $\hat{W} = W + \mathcal{O}_1(\mu)$ so that $\hat{w} > \hat{W}$ when $w > W$. From this result and (2.35), we can also relate the parameter $\hat{\theta}$ as follows

$$\hat{\theta}(w,\theta) = \tan^{-1}\left(\frac{\kappa w^{\frac{2}{3}}\sin\theta}{\kappa w^{\frac{2}{3}}\cos\theta - \mu}\right) = \theta + \mathcal{O}_1(\mu). \tag{2.38}$$

Regarding now $\hat{R}, \hat{\Theta}$ as the parameterizations of the invariant manifold $W^s_\mu(\mathrm{A}_{\hat{\Theta}_0})$ in (2.25), substituting them in (2.36) and changing the parameters $(\hat{w}, \hat{\theta})$ to (w, θ) using the relations in (2.37) and (2.38), we obtain that

$$R^s_\infty(w,\theta,\hat{\Theta}_0) = \sqrt{\frac{2}{\kappa}}\frac{1}{w^{\frac{1}{3}}} + \mathcal{O}_1(\mu)$$

$$\begin{aligned}
\Theta^s_\infty(w,\theta,\hat{\Theta}_0) =& \hat{\Theta}^s_\infty\left(\hat{w}(w,\theta),\hat{\theta}(w,\theta),\hat{\Theta}_0\right)\left(1 + \frac{\mu\cos\theta}{\kappa w^{\frac{2}{3}}}\right) + \mu\hat{R}^s_\infty\left(\hat{w}(w,\theta),\hat{\theta}(w,\theta),\hat{\Theta}_0\right)\sin\theta \\
&+ \mathcal{O}_2(\mu) = \hat{\Theta}^s_\infty(w,\theta,\hat{\Theta}_0) + \mu\sqrt{\frac{2}{\kappa}}\frac{\sin\theta}{w^{\frac{1}{3}}} + \mathcal{O}_2(\mu),
\end{aligned}$$

which implies

$$\hat{\Theta}^s_\infty(w,\theta,\hat{\Theta}_0) = \hat{\Theta}_0 - \mu\kappa \int_{+\infty}^{w} \frac{s^{\frac{2}{3}}\sin(\theta+w-s)}{\left(1+\kappa^2 s^{\frac{4}{3}} - 2\kappa s^{\frac{2}{3}}\cos(\theta+w-s)\right)^{\frac{3}{2}}} ds + \mu\frac{1}{\kappa^2}\int_{+\infty}^{w} \frac{\sin(\theta+w-s)}{s^{\frac{4}{3}}} ds$$
$$+ \mathcal{O}_2(\mu).$$

Integrating by parts the second integral and using the fact that $\frac{3}{\kappa^2} = \sqrt{\frac{2}{\kappa}}$ lead to (2.31), completing the proof. □

2.5 The invariant manifolds of the collision

This section is devoted to prove the following proposition, which gives a parameterization, in polar coordinates centered at P_1 (see (2.9)), of the invariant manifolds of the collision, that is $W^u_\mu(S^+)$ and $W^s_\mu(S^-)$ (see Remark 2.3) perturbatively from those obtained in Lemma 2.4 for $\mu = 0$.

Proposition 2.7. *Fix $r^* \in (0,1)$. There exists $\mu_0 > 0$ such that for $0 < \mu < \mu_0$, the invariant manifolds $W^u_\mu(S^+)$, $W^s_\mu(S^-)$, written in polar coordinates centered at P_1 (see (2.9)), intersect the section $r = r^*$. Moreover, the intersections are graphs over θ of the form*

$$\begin{aligned}(r,\theta,R,\Theta) &= (r^*,\theta,R^u_{S^+}(\theta),\Theta^u_{S^+}(\theta))\\ (r,\theta,R,\Theta) &= (r^*,\theta,R^s_{S^-}(\theta),\Theta^s_{S^-}(\theta)),\end{aligned} \tag{2.39}$$

which depend smoothly on μ.

Moreover, $\Theta^u_{S^+}$ can be written as

$$\Theta^u_{S^+}(\theta) = \mu\int_{t(r^*)}^{0}\left(\frac{\kappa s^{\frac{2}{3}}\sin\left(\theta+t(r^*)-s\right)}{\left(1+\kappa^2 s^{\frac{4}{3}} - 2\kappa s^{\frac{2}{3}}\cos\left(\theta+t(r^*)-s\right)\right)^{\frac{3}{2}}} + \sqrt{\frac{2}{\kappa}}\frac{\cos\left(\theta+t(r^*)-s\right)}{s^{\frac{1}{3}}}\right) ds + \mathcal{O}_1(\mu^2), \tag{2.40}$$

where

$$t(r^*) = \left(\frac{r^*}{\kappa}\right)^{3/2} \qquad \text{and} \qquad \kappa = \frac{3^{\frac{2}{3}}}{2^{\frac{1}{3}}}.$$

The expression for $\Theta^s_{S^-}(\theta)$ comes from the symmetry (2.11) and is given by

$$\Theta^s_{S^-}(\theta) = \Theta^u_{S^+}(-\theta). \tag{2.41}$$

Proof. We provide the proof for the unstable manifold, since the statements for the stable manifold are just a consequence of the symmetry. We rely on McGehee coordinates to prove this proposition, and consider then the vector field (2.17) and the transverse section $\Sigma_{s^*} = \{s = s^*\}$ with $s^* = \sqrt{r^*} < 1$. Note that the vector field is regular for $0 < s < s^*$.

The first observation is that S^+ is a normally hyperbolic invariant manifold (by Lemma 2.2) and, by Fenichel's theory [17, 18, 19, 54], its unstable and stable invariant manifolds are regular and depend also smoothly on μ. As a consequence, since in the unperturbed case $\mu = 0$ (see (2.21)) the invariant manifold intersect the section $s = s^* = \sqrt{r^*} < 1$ and, at the section, it is a graph over θ, the same happens for μ small enough.

Returning to polar coordinates (2.9) yields the same result since they are regular away from $\mathcal{S}$. This gives the curve (2.39).

The second part of the proof is devoted to obtain the expression of Θ^u_{S+} in (2.40). We perform a Melnikov-like approach. That is, we consider a point of the form $z(\theta_0) = (r^*, \theta_0, R^u_{S+}(\theta_0), \Theta^u_{S+}(\theta_0))$ (see (2.39)) and describe its Θ-component by a backward integral.

Since polar coordinates become undefined at collision, we express Θ in McGehee coordinates (2.12) as

$$\Theta = ur^{1/2} + r^2 - \mu r\cos\theta.$$

Then, by (2.13), its evolution in these coordinates is described by the equation

$$\frac{d}{d\tau}\Theta = \mu\left(r^{\frac{5}{2}}\sin\theta - \frac{r^{\frac{5}{2}}\sin\theta}{(r^2-2r\cos\theta+1)^{\frac{3}{2}}} - rv\cos\theta + ru\sin\theta\right).$$

Now, we consider the trajectory departing from the point $z(\theta_0)$ expressed in McGehee coordinates,

$$\varphi_\mu(\tau;\theta_0) = (r(\tau;\theta_0), \theta(\tau;\theta_0), v(\tau;\theta_0), u(\tau;\theta_0)).$$

Since $\lim_{\tau\to-\infty} r(\tau;\theta_0) = 0$ exponentially and therefore $\lim_{\tau\to-\infty}\Theta(\tau;\theta_0) = 0$, one has

$$\begin{aligned}\Theta^u_{S+}(\theta_0) = \mu\int_{-\infty}^0 \Bigg(& r(\tau;\theta_0)^{\frac{5}{2}}\sin\theta(\tau;\theta_0) - \frac{r^{\frac{5}{2}}\sin\theta(\tau;\theta_0)}{(r(\tau;\theta_0)^2 - 2r(\tau;\theta_0)\cos\theta(\tau;\theta_0)+1)^{\frac{3}{2}}} \\ & - r(\tau;\theta_0)v(\tau;\theta_0)\cos\theta(\tau;\theta_0) + r(\tau;\theta_0)u(\tau;\theta_0)\sin\theta(\tau;\theta_0)\Bigg)d\tau.\end{aligned}$$

When $\mu = 0$, shifting time to fix the initial condition in (2.13), one has

$$\varphi_0(\tau;\theta_0) = \left(r^*e^{\sqrt{2}\tau}, \theta_0 + \left(\frac{r^*}{\kappa}\right)^{3/2}\left(1 - e^{\frac{3}{\sqrt{2}}\tau}\right), \sqrt{2}, -(r^*)^{3/2}e^{\frac{3}{\sqrt{2}}\tau}\right).$$

Moreover, since we are considering points in the invariant manifold, we have

$$\begin{aligned} r(\tau;\theta_0) &= r^*e^{\sqrt{2}\tau} + \mathcal{O}_1(\mu e^{C\tau}) \\ \theta(\tau;\theta_0) &= \theta_0 + \left(\frac{r^*}{\kappa}\right)^{3/2}\left(1 - e^{\frac{3}{\sqrt{2}}\tau}\right) + \mathcal{O}_1(\mu) \\ v(\tau;\theta_0) &= \sqrt{2} + \mathcal{O}_1(\mu) \\ u(\tau;\theta_0) &= -(r^*)^{3/2}e^{\frac{3}{\sqrt{2}}\tau} + \mathcal{O}_1(\mu),\end{aligned}$$

where $C > 0$ is an adequate constant.

Then, we obtain

$$\begin{aligned}\Theta^u_{S+}(\theta_0) = -\mu\int_{-\infty}^0 \Bigg[& \frac{(r^*)^{\frac{5}{2}}e^{\frac{5}{\sqrt{2}}\tau}\sin\left(\theta_0 + \left(\frac{r^*}{\kappa}\right)^{3/2}\left(1-e^{\frac{3}{\sqrt{2}}\tau}\right)\right)}{\left((r^*)^2e^{2\sqrt{2}\tau} - 2(r^*)e^{\sqrt{2}\tau}\cos\left(\theta_0 + \left(\frac{r^*}{\kappa}\right)^{3/2}\left(1-e^{\frac{3}{\sqrt{2}}\tau}\right)\right)+1\right)^{\frac{3}{2}}} \\ & + \sqrt{2}r^*e^{\sqrt{2}\tau}\cos\left(\theta_0 + \left(\frac{r^*}{\kappa}\right)^{3/2}\left(1-e^{\frac{3}{\sqrt{2}}\tau}\right)\right)\Bigg]d\tau \\ & + \mathcal{O}_2(\mu).\end{aligned}$$

To recover formula (2.40), it is enough to apply the change of variables $\kappa s^{2/3} = r^*e^{\sqrt{2}\tau}$. $\square$

3 The distance between the invariant manifolds

Once we have characterized the invariant manifolds of infinity and collision, we analyze their intersections for $\mu > 0$ small enough at a section $r = r^* = \delta^2$ for some small $0 < \delta < 1$ independent of μ. To this end we define, for a fixed energy $h = -\hat{\Theta}_0 = -\mu\overline{\Theta}_0$, the following curves

$$\begin{aligned}
\Delta_\infty^{s,u}(\mu) =& \tilde{W}_\mu^{s,u}(\mathrm{A}_{\hat{\Theta}_0}) \bigcap \Sigma_h = \{r = \delta^2, \mathcal{H}(\delta^2, \theta, R, \Theta; \mu) = h\},\\
\Delta_{S^+}^{u}(\mu) =& W_\mu^{u}(S^+) \bigcap \Sigma_h = \{r = \delta^2, \mathcal{H}(\delta^2, \theta, R, \Theta; \mu) = h\},\\
\Delta_{S^-}^{s}(\mu) =& W_\mu^{s}(S^-) \bigcap \Sigma_h = \{r = \delta^2, \mathcal{H}(\delta^2, \theta, R, \Theta; \mu) = h\},
\end{aligned} \tag{3.1}$$

where $\mathcal{H}$ is the Hamiltonian (2.10). This definition only involves the first intersection with the section Σ_h. Note that for a fixed $h = -\hat{\Theta}_0$, the surface $\mathcal{H}(\delta^2, \theta, R, \Theta; \mu) = h$ is 2-dimensional and can be locally defined by the coordinates (θ, Θ). Moreover, by Propositions 2.6 and 2.7, the curves $\Delta_\infty^{s,u}(\mu)$, $\Delta_{S^+}^{u}(\mu)$ and $\Delta_{S^-}^{s}(\mu)$ can be written as graphs with respect to θ. We characterize the distance between the curves using the Θ-component.

Hence, we can define $\Delta_{S^+}^{u}(\mu)$ (the definition of $\Delta_{S^-}^{s}(\mu)$ comes from the symmetry (2.11)) as

$$\Delta_{S^+}^{u}(\mu) = \{(\theta, \Theta_{S^+}^{u}(\theta)), \theta \in \mathbb{T}\}, \quad \Delta_{S^-}^{s}(\mu) = \{(\theta, \Theta_{S^-}^{s}(\theta)), \theta \in \mathbb{T}\}, \tag{3.2}$$

where $\Theta_{S^+}^{u}$, $\Theta_{S^-}^{s}$ are defined in Proposition 2.7 for $r^* = \delta^2$.

Note that the constant $t(r^*)$ introduced in Proposition 2.7 is given by

$$t(\delta^2) = \frac{\sqrt{2}}{3}\delta^3.$$

Then, we have

$$\Theta_{S^+}^{u}(\theta) = \mu I_{S^+}^{u}(\theta) + \mathcal{O}_2(\mu) \tag{3.3}$$

with

$$I_{S^+}^{u}(\theta) = \mathcal{I}_{S^+}^{u}\left(\theta + \frac{\sqrt{2}}{3}\delta^3\right)$$

where

$$\mathcal{I}_{S^+}^{u}(\alpha) = \kappa \int_{\frac{\sqrt{2}}{3}\delta^3}^{0} \frac{s^{\frac{2}{3}}\sin(\alpha - s)}{\left(1 + \kappa^2 s^{\frac{4}{3}} - 2\kappa s^{\frac{2}{3}}\cos(\alpha - s)\right)^{\frac{3}{2}}}\, ds + \sqrt{\frac{2}{\kappa}} \int_{\frac{\sqrt{2}}{3}\delta^3}^{0} \frac{\cos(\alpha - s)}{s^{\frac{1}{3}}}\, ds.$$

The expression for $\Theta_{S^-}^{s}(\theta)$ is given by the symmetry in (2.41).

On the other hand, by Proposition 2.6, the curves $\Delta_\infty^{s,u}(\mu)$ are defined as

$$\begin{aligned}
\Delta_\infty^{s}(\mu) =& \left\{(\theta, \Theta_\infty^{s}(w_\Sigma, \theta, \hat{\Theta}_0)), \theta \in I_D^{+}(w_\Sigma)\right\},\\
\Delta_\infty^{u}(\mu) =& \left\{(\theta, \Theta_\infty^{s}(w_\Sigma, \theta, \hat{\Theta}_0)), \theta \in I_D^{-}(w_\Sigma)\right\},
\end{aligned} \tag{3.4}$$

where $I_D^{\pm}(w)$ are defined in (2.32) and (2.33), $\Theta_\infty^{*}(w, \theta, \hat{\Theta}_0)$, $* = u, s$, are in (2.31) and (2.34), and w_Σ is the value such that $r^* = \kappa w^{\frac{2}{3}} = \delta^2$, that is

$$w_\Sigma = \frac{\sqrt{2}}{3}\delta^3. \tag{3.5}$$

Therefore we obtain

$$\Theta^s_\infty(w_\Sigma, \theta, \hat{\Theta}_0) = \hat{\Theta}_0 - \mu I^s_\infty(\theta) + \mathcal{O}_2(\mu) \tag{3.6}$$

with

$$I^s_\infty(\theta) = \mathcal{I}^s_\infty\left(\theta + \frac{\sqrt{2}}{3}\delta^3\right)$$

and

$$\mathcal{I}^s_\infty(\alpha) = \kappa \int_{+\infty}^{\frac{\sqrt{2}}{3}\delta^3} \frac{s^{\frac{2}{3}}\sin(\alpha - s)}{\left(1 + \kappa^2 s^{\frac{4}{3}} - 2\kappa s^{\frac{2}{3}}\cos(\alpha - s)\right)^{\frac{3}{2}}}\, ds + \sqrt{\frac{2}{\kappa}} \int_{+\infty}^{\frac{\sqrt{2}}{3}\delta^3} \frac{\cos(\alpha - s)}{s^{\frac{1}{3}}}\, ds.$$

Now we measure the distance between the invariant manifolds of infinity and collision. Note that the sign of the radial velocity R for the curves $\Delta^s_\infty(\mu)$ and $\Delta^u_{S^+}(\mu)$ is positive, and negative for the other two. Hence, we define the transverse sections

$$\begin{aligned} \Sigma_h^> &= \{(r, \theta, R, \Theta) \colon r = \delta^2, \mathcal{H}(\delta^2, \theta, R, \Theta; \mu) = h, R > 0\} \subset \Sigma_h, \\ \Sigma_h^< &= \{(r, \theta, R, \Theta) \colon r = \delta^2, \mathcal{H}(\delta^2, \theta, R, \Theta; \mu) = h, R < 0\} \subset \Sigma_h. \end{aligned} \tag{3.7}$$

In these sections, we provide an asymptotic formula for the distance between $\Delta^{s,u}_\infty(\mu)$ and $\Delta^{u,s}_{S^\pm}(\mu)$ at the sections $\Sigma_h^>$ and $\Sigma_h^<$ respectively for a given angle θ. That is,

$$d_+(\theta, \hat{\Theta}_0) = \Theta^s_\infty(\theta, \hat{\Theta}_0) - \Theta^u_{S^+}(\theta), \quad d_-(\theta, \hat{\Theta}_0) = \Theta^u_\infty(\theta, \hat{\Theta}_0) - \Theta^s_{S^-}(\theta). \tag{3.8}$$

The next theorem, whose proof is straightforward, gives an asymptotic formula for this distance.

Theorem 3.1. *Fix $\eta > 0$. There exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$ and $\hat{\Theta}_0 \in [-\eta\mu, \eta\mu]$, the distances d_+ and d_- in* (3.8) *are given by*

$$\begin{aligned} d_+(\theta, \hat{\Theta}_0) &= \hat{\Theta}_0 + \mu M_+(\theta) + \mathcal{O}_2(\mu) \quad \text{for } \theta \in I_D^+(w_\Sigma), \\ d_-(\theta; \hat{\Theta}_0) &= \hat{\Theta}_0 - \mu M_-(\theta) + \mathcal{O}_2(\mu) \quad \text{for } \theta \in I_D^-(w_\Sigma) \end{aligned} \tag{3.9}$$

where w_Σ is defined in (3.5) *and*

$$M_\pm(\theta) = \mathcal{M}_\pm(\theta \pm w_\Sigma) \tag{3.10}$$

where

$$\mathcal{M}_+(\alpha) = \kappa \int_0^{+\infty} \frac{s^{\frac{2}{3}}\sin(\alpha - s)}{\left(1 + \kappa^2 s^{\frac{4}{3}} - 2\kappa s^{\frac{2}{3}}\cos(\alpha - s)\right)^{\frac{3}{2}}}\, ds + \sqrt{\frac{2}{\kappa}} \int_0^{+\infty} \frac{\cos(\alpha - s)}{s^{\frac{1}{3}}}\, ds, \tag{3.11}$$

with $\kappa = \frac{3^{\frac{2}{3}}}{2^{\frac{1}{3}}}$. The functions $M_-(\theta)$ and $\mathcal{M}_-(\alpha)$ are given by the symmetry

$$M_-(\theta) = -M_+(-\theta), \qquad \mathcal{M}_-(\alpha) = -\mathcal{M}_+(-\alpha). \tag{3.12}$$

Once we have an asymptotic formula for the distance between the invariant manifolds of infinity and collision, we analyze their first order to prove that they intersect transversally.

The next lemma is proven in Appendix A and it is computer assisted.

Lemma 3.2. *For every $\theta_+ \in B_+ = [-1.72851, -0.583065] \cup [-0.407155, 0.0578054] \cup [0.921743, 4.15633]$*

$$\frac{d}{d\theta}\mathcal{M}_+(\theta_+) \neq 0.$$

Moreover,

$$\frac{d}{d\theta}\mathcal{M}_+(0) < 0.$$

The result $\mathcal{M}'_-(\theta_-) \neq 0$ for $\theta_- \in B_-$, where $B_- = 2\pi - B_+$, comes as a consequence of (3.10) *and* (3.12)*.*

Note that $B^\pm \subset I_D^\pm(w_\Sigma)$ in (2.32) and (2.33) (with w_Σ given in (3.5)) for $D = \frac{3}{10}$. Smaller values of D will lead to larger intervals of $B_\pm$, at the cost of increased computational time to obtain the derivatives.

The following theorem, which is a direct consequence of Theorem 3.1, Lemma 3.2 and the Implicit Function Theorem, gives the transversality of the intersection between the invariant manifolds $W_\mu^{u,s}(S^\pm)$ and $W_\mu^{s,u}(\mathrm{A}_{\hat{\Theta}_0})$ for some values of $\hat{\Theta}_0$.

Theorem 3.3. *Consider $U_+ \subset B_+ \subset I^+_{\frac{3}{10}}(w_\Sigma)$ an open set satisfying that, for any $\theta^+ \in U_+$, $\theta^+ - w_\Sigma \in I^+_{\frac{3}{10}}(w_\Sigma)$ and $-\theta^+ + w_\Sigma \in I^-_{\frac{3}{10}}(w_\Sigma)$. Then, for any $\theta^+ \in U_+$, there exists $\mu_0 > 0$ such that, for any $\mu \in (0, \mu_0)$ and energy $h = -\hat{\Theta}_0 = -\mu\overline{\Theta}_0 = \mu M_+(\theta^+ - w_\Sigma)$,*

- *$\tilde{W}_\mu^s(\mathrm{A}_{\hat{\Theta}_0}) \pitchfork W_\mu^u(S^+)$ at a point*

  $$p_>(\theta^+,\mu) = (\theta_>(\theta^+,\mu), \Theta^s_\infty(\theta_>(\theta^+,\mu), \hat{\Theta}_0)) = (\theta_>(\theta^+,\mu), \Theta^u_{S^+}(\theta_>(\theta^+,\mu))),$$

  where $\theta_>(\theta^+,\mu) = \theta^+ - w_\Sigma + \mathcal{O}_1(\mu)$.

- *$\tilde{W}_\mu^u(\mathrm{A}_{\hat{\Theta}_0}) \pitchfork W_\mu^s(S^-)$ at a point*

  $$p_<(\theta^+,\mu) = (\theta_<(\theta^+,\mu), \Theta^u_\infty(\theta_<(\theta^+,\mu), \hat{\Theta}_0)) = (\theta_<(\theta^+,\mu), \Theta^s_{S^-}(\theta_<(\theta^+,\mu))),$$

  where $\theta_<(\theta^+,\mu) = -\theta^+ + w_\Sigma + \mathcal{O}_1(\mu)$.

- *Both points $p_>(\theta^+,\mu)$ and $p_<(\theta^+,\mu)$ belong to the surface $\mathcal{H}(\delta^2,\theta,R,\Theta;\mu) = h = \mu M_+(\theta^+ - w_\Sigma)$.*

Proof. Finding a transverse intersection between $\tilde{W}_\mu^s(\mathrm{A}_{\hat{\Theta}_0})$ and $W_\mu^u(S^+)$ is equivalent to find a non-degenerate zero of $d_+(\theta, \hat{\Theta}_0)$ (see (3.9)). To this end, consider $\theta^+ \in U_+$, denote $\hat{\Theta}_0 = \mu\overline{\Theta}_0$ with $\overline{\Theta}_0 = -M_+(\theta^+ - w_\Sigma)$ and define

$$\mathcal{F}_+(\theta,\mu) = \mu^{-1} \cdot d_+(\theta, \hat{\Theta}_0) = -M_+(\theta^+ - w_\Sigma) + M_+(\theta) + \mathcal{O}(\mu),$$

which satisfies

- $\mathcal{F}_+(\theta^+ - w_\Sigma, 0) = 0$.
- $\frac{d}{d\theta}\mathcal{F}_+(\theta^+ - w_\Sigma, 0) = M'_+(\theta^+ - w_\Sigma) = \mathcal{M}'_+(\theta^+) \neq 0$ due to (3.10) and Lemma 3.2.

Then, the Implicit Function Theorem ensures that there exists $\mu_0 > 0$ such that, for any $\mu \in (0, \mu_0)$, there is $\theta_>(\mu)$ with $\theta_>(0) = \theta^+ - w_\Sigma$ such that $d_+(\theta_>(\mu), \mu\overline{\Theta}_0) = 0$.

The procedure to obtain a transverse intersection between $\tilde{W}_\mu^u(\mathrm{A}_{\hat{\Theta}_0})$ and $W_\mu^s(S^-)$ at the same energy level is completely analogous. □

4 Proof of Theorem 1.8

To prove Theorem 1.8, we show that there exist ejection-collision orbits "close" to the invariant manifolds of infinity. To this end, we rely on a suitable return map.

We fix $\mu \in (0, \mu_0)$ so that Theorem 3.3 holds and we denote by $p_> = (\theta_>, \Theta_>)$ the point of intersection of $\Delta^s_\infty(\mu)$ and $\Delta^u_{S^+}(\mu)$ at $\Sigma^>_h$ (see (3.1) and (3.7)). Since $p_> \in \Delta^s_\infty(\mu)$, then $\Theta_> = \Theta^s_\infty(\theta_>)$, where $\Theta^s_\infty(\theta)$ is defined in (3.6). We consider a sufficiently small "rectangle" $U \subset \Sigma^>_h$ defined as

$$U = \left\{(\theta, \Theta) \in \Sigma^>_h \colon |\theta - \theta_>| < \varepsilon, \Theta^s_\infty(\theta) - \varepsilon < \Theta < \Theta^s_\infty(\theta)\right\},$$

for some $\varepsilon > 0$ small enough. Then, we define the Poincaré map

$$\begin{aligned} \mathcal{P} \colon U \subset \Sigma^>_h &\to \Sigma^<_h \\ p = (\theta, \Theta) &\mapsto \mathcal{P}(p) = \Phi_\mu(t_\mu(p), (\delta^2, \theta, R(\theta, \Theta; h, \mu), \Theta)), \end{aligned}$$

where Φ_μ is the flow of the equations of motion associated to the Hamiltonian $\mathcal{H}$ in (2.10); $R(\theta, \Theta; h, \mu)$ is the radial velocity, which can also be computed from (2.10); $\Sigma^>_h$ and $\Sigma^<_h$ are the sections defined in (3.7) and $t_\mu(p)$ is the time needed for the orbit with initial condition at $p \in U$ to reach $\Sigma^<_h$. By construction, $t_\mu(p)$ is well-defined and finite for $p \in U$ (but becomes unbounded as p gets closer to the invariant manifold $\Delta^s_\infty(\mu)$).

Now we consider the C^1- curve

$$\gamma_{u,>} = U \cap \Delta^u_{S^+}(\mu),$$

that, by Theorem 3.3, intersects transversally $\Delta^s_\infty(\mu)$ at $p_>$.

Since the points of the curve $\gamma_{u,>}$ are close enough to the point $p_>$, they are close to $\Delta^s_\infty(\hat{\Theta}_0, \mu)$. Hence, the study of the image $\mathcal{P}(\gamma_{u,>})$ is reduced to the analysis of the dynamics "close" to $\mathrm{A}_{\hat{\Theta}_0}$. Thus, one can easily adapt the approach done by Moser for the Sitnikov problem in [43] (obtained from [27]) to this case, and prove that the image $\mathcal{P}(\gamma_{u,>})$ "spirals" towards $\Delta^u_\infty(\mu)$ in (3.1) due to the fact that $\dot{\theta} = \frac{\Theta}{r^2} - 1 + \frac{\mu}{r}\cos\theta < 0$ (see (2.10)) in this region of the phase space. In particular, we have that

$$\Delta^u_\infty(\mu) \subset \overline{\mathcal{P}(\gamma_{u,>})}.$$

Theorem 3.3 ensures that $\Delta^s_{S^-}(\mu)$ and $\Delta^u_\infty(\mu)$ intersect transversally in $\Sigma^<_h$ at a point that we denote by $q_< \in W^s_\mu(S^-) \pitchfork W^u_\mu(\mathrm{A}_{\hat{\Theta}_0})$.

Therefore, if we consider a C^1-curve $\gamma_{s,<}$ defined as follows

$$\gamma_{s,<} = V \cap \Delta^s_{S^-}(\mu),$$

where V is a sufficiently small neighborhood of $q_<$, there exists a sequence $q_k \in \gamma_{s,<} \cap \mathcal{P}(\gamma_{u,>})$ such that $\lim_{k\to+\infty} q_k = q_<$ (see Figure 4.1). In particular, the closer is q_k to $\Delta^u_\infty(\mu)$, the larger is the time $t_\mu(p_k)$, where p_k is such that $\mathcal{P}(p_k) = q_k$. Since $\mathcal{P}(\gamma_{u,>}) \subset W^u_\mu(S^+)$, the points $q_k \in W^u_\mu(S^+) \cap W^s_\mu(S^-)$ and, therefore, give rise to ejection-collision orbits.

5 Dynamics close to collision

To prove Theorem 1.6, it is necessary to study first the dynamics close to collision, which is the purpose of this section. To this end, we work with the coordinates (s, θ, α) in (2.14) and in a neighborhood of the collision manifold (2.18) (defined by $s = 0$) of the form

$$\mathcal{B}_\delta = \{(s, \alpha, \theta) \in \mathcal{M} \colon s \in (0, 2\delta)\},$$

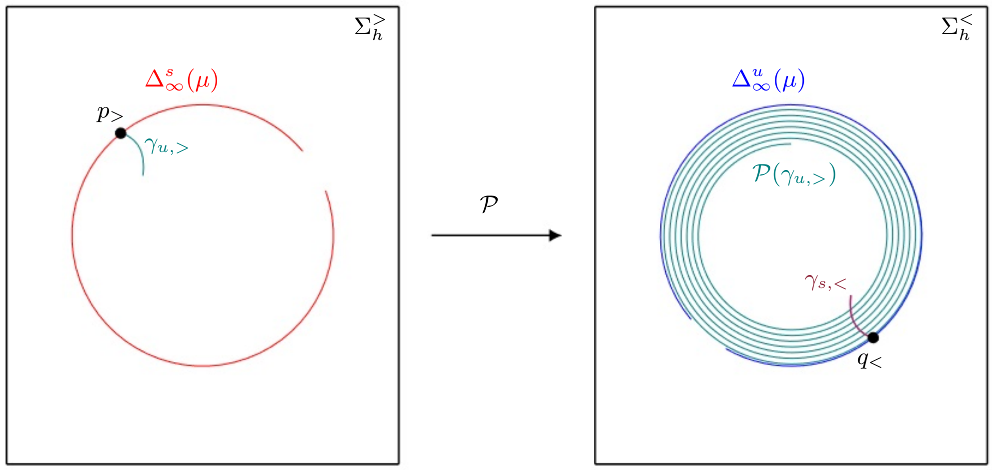

Figure 4.1: Representation of the "spiraling effect" of the transition map $\mathcal{P}$ on the curve $\gamma_{u,>} \subset \Delta^u_{S^+}(\mu) \subset \Sigma_h^>$ and its transverse intersections with the curve $\gamma_{s,<} \subset \Delta^s_{S^-}(\mu) \subset \Sigma_h^<$.

where $\mathcal{M}$ is the 3 dimensional manifold (2.16) and $\delta > 0$ is small enough and defined in (3.1).

A first step towards a proper description of the dynamics close to collision consists on "symplifying" equations (2.17) in a neighborhood of the circles of equilibrium points $S^\pm$ in (2.20) by means of suitable changes of coordinates. This is done in Section 5.1, where we perform two changes of coordinates, each of them defined around the circles $S^\pm$ respectively, which straighten the invariant manifolds of $S^\pm$ (see Lemma 2.2). In Section 5.2, we define a transition map between $\Sigma_h^<$ and $\Sigma_h^>$ and show that it sends transverse curves to $W^s_\mu(S^-)$ to transverse curves to $W^u_\mu(S^+)$.

5.1 Local straightening of the invariant manifolds

We consider the following neighborhoods of the circles $S^\pm$ in (2.20) respectively

$$\begin{aligned} \mathcal{B}_\delta^+ &= \left\{(s,\alpha,\theta) \in \mathcal{B}_\delta \colon \alpha \in \left(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta\right)\right\}, \\ \mathcal{B}_\delta^- &= \left\{(s,\alpha,\theta) \in \mathcal{B}_\delta \colon \alpha \in \left(-\frac{\pi}{2} - \delta, -\frac{\pi}{2} + \delta\right)\right\}. \end{aligned} \tag{5.1}$$

The following proposition ensures the existence of two sets of coordinates defined in these neighborhoods straightening the invariant manifolds $W^{u,s}_\mu(S^\pm)$.

Proposition 5.1. *Fix any $\mu_0 \in (0, 1/2]$ and take $\delta > 0$ small enough. Then, for any $\mu \in [0, \mu_0]$, there exists a change of variables*

$$\begin{aligned} &\Gamma_- \colon \mathcal{B}_\delta^- \to (0, 2\delta) \times (-2\delta, 2\delta) \times \mathbb{T} \\ &(s,\alpha,\theta) \mapsto \Gamma_-(s,\alpha,\theta) = (s, \tilde{\beta}, z) \end{aligned} \tag{5.2}$$

which transforms the system (2.17) *into*

$$\begin{aligned} s' &= -\frac{m_0}{2} s \left(1 + \tilde{g}_1(s,\tilde{\beta},z)\right) \\ \tilde{\beta}' &= \frac{m_0}{2} \tilde{\beta} \left(1 + \tilde{g}_2(s,\tilde{\beta},z)\right) \\ z' &= s\tilde{\beta}\left(4\lambda(\mu,h)s + \tilde{g}_3(s,\tilde{\beta},z)\right) \end{aligned} \tag{5.3}$$

with $m_0 = \sqrt{2(1-\mu)}$, $\lambda(\mu,h) = \frac{\mu^2+2h+2\mu}{4m_0}$ *and*

$$\tilde{g_1}(s,\tilde{\beta},z) = \mathcal{O}_2(s,\tilde{\beta}); \quad \tilde{g_2}(s,\tilde{\beta},z) = \mathcal{O}_1(s^2,\tilde{\beta}); \quad \tilde{g_3}(s,\tilde{\beta},z) = \mathcal{O}_3(s,\tilde{\beta}).$$

Moreover, in these coordinates, $S^- = \{(0,0,z), z \in \mathbb{T}\}$ *and its invariant manifolds become*

$$\begin{aligned} W^s_\mu(S^-) &= \{\tilde{\beta} = 0\}, \\ W^u_\mu(S^-) &= \{s = 0\}. \end{aligned} \tag{5.4}$$

Analogously, there exists a diffeomorphism

$$\begin{aligned} &\Gamma_+ \colon \mathcal{B}^+_\delta \to (0,2\delta) \times (-2\delta,2\delta) \times \mathbb{T} \\ &(s,\alpha,\theta) \mapsto \Gamma_+(s,\alpha,\theta) = (s,\tilde{\iota},w) \end{aligned} \tag{5.5}$$

so that in the coordinates $(s,\tilde{\iota},w)$*, system* (2.17) *becomes*

$$\begin{aligned} s' &= \frac{m_0}{2} s \left(1 + \tilde{j_1}(s,\tilde{\iota},w)\right) \\ \tilde{\iota}' &= -\frac{m_0}{2} \tilde{\iota} \left(1 + \tilde{j_2}(s,\tilde{\iota},w)\right) \\ w' &= s\tilde{\iota}\left(-4\lambda(\mu,h)s + \tilde{j_3}(s,\tilde{\iota},w)\right), \end{aligned} \tag{5.6}$$

where

$$\tilde{j_1}(s,\tilde{\iota},w) = \mathcal{O}_2(s,\tilde{\iota}); \quad \tilde{j_2}(s,\tilde{\iota},w) = \mathcal{O}_1(s,\tilde{\iota}); \quad \tilde{j_3}(s,\tilde{\iota},w) = \mathcal{O}_3(s,\tilde{\iota}).$$

In these coordinates, $S^+ = \{(0,0,w), w \in \mathbb{T}\}$ *and its invariant manifolds become*

$$\begin{aligned} W^u_\mu(S^+) &= \{\tilde{\iota} = 0\}, \\ W^s_\mu(S^+) &= \{s = 0\}. \end{aligned} \tag{5.7}$$

Proof. We start by straightening the local invariant manifolds of S^-. Lemma 2.2 implies that $W^u_\mu(S^-)$ is already straightened in the coordinates (s,θ,α), since it lies in the hyperplane $\{s = 0\}$. Hence, we only need to straighten $W^s_\mu(S^-)$ and the fibers $W^u_\mu(S^-_\theta)$ and $W^s_\mu(S^-_\theta)$. We define the following change of coordinates

$$\begin{aligned} \beta &= \alpha + \frac{\pi}{2} \\ y &= \theta - 2\left(\alpha + \frac{\pi}{2}\right), \end{aligned} \tag{5.8}$$

to straighten the fibers $W^u_\mu(S^-_\theta)$. In coordinates (s,β,y), system (2.17) becomes

$$\begin{aligned} s' &= -\frac{m_0}{2}\cos\beta s - \lambda(\mu,h)\cos\beta s^3 + \mathcal{O}_5(s) \\ \beta' &= \frac{m_0}{2}\sin\beta - 3\lambda(\mu,h)\sin\beta s^2 + 2s^3 + \mathcal{O}_4(s) \\ y' &= 4\lambda(\mu,h)\sin\beta s^2 - 4s^3 + \mathcal{O}_4(s), \end{aligned} \tag{5.9}$$

where $m_0 = \sqrt{2(1-\mu)}$ and $\lambda(\mu,h) = \frac{\mu^2+2h+2\mu}{4m_0}$.

Moreover, the circle S^- in (2.20) and $\mathcal{B}^-_\delta$ in (5.1) become

$$\begin{aligned} S^- &= \left\{S^-_\theta = (0,0,y) : y \in \mathbb{T}\right\}, \\ \mathcal{B}^-_\delta &= \{(s,\beta,y) \colon s \in (0,2\delta), \beta \in (-\delta,\delta), y \in \mathbb{T}\}. \end{aligned}$$

Expanding around $s=0, \beta=0$, we write system (5.9) as

$$\begin{aligned} s' &= -\frac{m_0}{2} s\left(1+g_1(s,\beta,y)\right) \\ \beta' &= \frac{m_0}{2}\beta + g_2(s,\beta,y) \\ y' &= s g_3(s,\beta,y), \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} g_1(s,\beta,y) &= \frac{2\lambda(\mu,h)}{m_0}s^2 - \frac{1}{2}\beta^2 + \mathcal{O}_4(s,\beta), \\ g_2(s,\beta,y) &= \mathcal{O}_1\left(s^3, \beta s^2, \beta^3\right), \\ g_3(s,\beta,y) &= 4\lambda(\mu,h)s\beta - 4s^2 + \mathcal{O}_4(s,\beta). \end{aligned} \tag{5.11}$$

Note that in these coordinates $W^u_\mu(S^-_{y_0}) = \{s=0, y=y_0\in\mathbb{T}\}$.

On the other hand, to straighten $W^s_\mu(S^-)$, we use the fact that it is tangent to the plane $\{\beta = 0\}$. Therefore, by the stable manifold theorem, $W^s_\mu(S^-)$ can be written as the graph of a function as

$$W^s_\mu(S^-) = \left\{(s,\beta,y)\in\mathcal{B}^-_\delta : \beta = \psi_-(s,y)\right\},$$

where ψ_- satisfies

$$\psi_-(s,y) = \mathcal{O}_2(s). \tag{5.12}$$

Hence if we perform the change of coordinates

$$(s,\tilde\beta,y) = (s, \beta - \psi_-(s,y), y), \tag{5.13}$$

then $W^s_\mu(S^-)$ becomes $\{\tilde\beta = 0\}$ and, using the expressions of g_1, g_2, g_3 in (5.11), (5.10) becomes

$$\begin{aligned} s' &= -\frac{m_0}{2} s\left(1+\overline{g}_1(s,\tilde\beta,y)\right) \\ \tilde\beta' &= \frac{m_0}{2}\tilde\beta\left(1+\overline{g}_2(s,\tilde\beta,y)\right) \\ y' &= s\overline{g}_3(s,\tilde\beta,y), \end{aligned} \tag{5.14}$$

with

$$\begin{aligned} \overline{g}_1(s,\tilde\beta,y) &= g_1(s,\tilde\beta+\psi_-(s,y),y) = \frac{2\lambda(\mu,h)}{m_0}s^2 - \frac{1}{2}\tilde\beta^2 + \mathcal{O}_4(s,\tilde\beta), \\ \overline{g}_2(s,\tilde\beta,y) &= \partial_{\tilde\beta} g_2(s,\tilde\beta+\psi_-(s,y),y)\big|_{\tilde\beta=0} + \mathcal{O}_1(\tilde\beta) = \mathcal{O}_1(s^2,\tilde\beta), \\ \overline{g}_3(s,\tilde\beta,y) &= g_3(s,\tilde\beta+\psi_-(s,y),y) = 4\lambda(\mu,h)s\tilde\beta - 4s^2 + \mathcal{O}_4(s,\tilde\beta). \end{aligned}$$

Finally, we straighten the fibers $W^s_\mu(S^-_{y_0})$. Indeed, Lemma 2.2 allows us to define $W^s_\mu(S^-)$ as the union of fibers $W^s_\mu(S^-_{y_0})$, that is

$$W^s_\mu(S^-) = \bigcup_{y_0\in\mathbb{T}} W^s_\mu(S^-_{y_0}).$$

Moreover, since $W^s_\mu(S^-)$ corresponds to the plane $\{\tilde\beta = 0\}$, the stable manifold theorem ensures that $W^s_\mu(S^-_{y_0})$ can be described by the graph of a function as

$$W^s_\mu(S^-_{y_0}) = \left\{(s,0,y)\in\mathcal{B}^-_\delta : y = \chi_-(s,y_0)\right\},$$

for some function χ_- satisfying

$$\chi_-(s,y_0) = y_0 + a_0(y_0)s^2 + a_1(y_0)s^3 + \mathcal{O}_4(s). \tag{5.15}$$

Now we consider the following change of coordinates

$$(s, \tilde{\beta}, y) = \left(s, \tilde{\beta}, \chi_-(s, z)\right), \tag{5.16}$$

so that system (5.14) becomes

$$\begin{aligned} s' &= -\frac{m_0}{2} s \left(1 + \tilde{g}_1(s, \tilde{\beta}, z)\right) \\ \tilde{\beta}' &= \frac{m_0}{2} \tilde{\beta} \left(1 + \tilde{g}_2(s, \tilde{\beta}, z)\right) \\ z' &= \frac{s}{\partial_z \chi_-(s, z)} \left(\tilde{g}_3(s, \tilde{\beta}, z) + \frac{m_0}{2} \partial_s \chi_-(s, z) \left(1 + \tilde{g}_1(s, \tilde{\beta}, z)\right)\right) = s \cdot Z(s, \tilde{\beta}, z), \end{aligned} \tag{5.17}$$

with

$$\begin{aligned} \tilde{g}_1(s, \tilde{\beta}, z) &= \overline{g}_1(s, \tilde{\beta}, \chi_-(s, z)) = \frac{2\lambda(\mu, h)}{m_0} s^2 - \frac{1}{2}\tilde{\beta}^2 + \mathcal{O}_4(s, \tilde{\beta}) \\ \tilde{g}_2(s, \tilde{\beta}, z) &= \overline{g}_2(s, \tilde{\beta}, \chi_-(s, z)) = \mathcal{O}_1(s^2, \tilde{\beta}) \\ \tilde{g}_3(s, \tilde{\beta}, z) &= \overline{g}_3(s, \tilde{\beta}, \chi_-(s, z)) = 4\lambda(\mu, h) s\tilde{\beta} - 4s^2 + \mathcal{O}_4(s, \tilde{\beta}). \end{aligned} \tag{5.18}$$

The invariance property ensures that $Z(s, 0, z) = 0$. Therefore

$$\tilde{g}_3(s, 0, z) + \frac{m_0}{2} \partial_s \chi_-(s, z) \left(1 + \tilde{g}_1(s, 0, z)\right) = 0.$$

Using (5.15), (5.18) and comparing coefficients, we have

$$m_0 a_0(z) \cdot s + \left(-4 + \frac{3m_0}{2} a_1(z)\right) s^2 + \mathcal{O}_3(s) = 0.$$

Hence

$$a_0(z) = 0, \quad a_1(z) = \frac{8}{3m_0}. \tag{5.19}$$

Moreover, the fact that $Z(s, 0, z) = 0$ implies that $Z(s, \tilde{\beta}, z)$ cannot have independent terms in neither s nor z. Therefore, $Z(s, \tilde{\beta}, z)$ in (5.17) can be computed ignoring the independent terms in s for $\tilde{g}_1$ and $\tilde{g}_3$ from (5.18). Namely

$$\begin{aligned} Z(s, \tilde{\beta}, z) &= \frac{1}{1 + \mathcal{O}_4(s)} \left(4\lambda(\mu, h) s\tilde{\beta} + \mathcal{O}_4(s, \tilde{\beta}) + \left(4s^2 + \mathcal{O}_3(s)\right)\left(-\frac{1}{2}\tilde{\beta}^2 + \mathcal{O}_4(s, \tilde{\beta})\right)\right) \\ &= \tilde{\beta} \left(4\lambda(\mu, h) s + \mathcal{O}_3(s, \tilde{\beta})\right), \end{aligned}$$

which implies that we can rewrite system (5.17) as in (5.3), completing the proof.

The procedure to straighten the invariant manifolds for S^+ is completely analogous to the one explained for S^-, involving the following changes instead:

- The changes to translate S^+ and to straighten the fibers $W^u_\mu(S^+_{x_0})$

$$\begin{aligned} \iota &= \alpha - \frac{\pi}{2}, \\ x &= \theta - 2\left(\alpha - \frac{\pi}{2}\right). \end{aligned} \tag{5.20}$$

- The change to straighten the unstable manifold $W^u_\mu(S^+)$

$$\tilde{\iota} = \iota - \psi_+(s, x), \tag{5.21}$$

  where $\psi_+(s, x)$ has analogous properties as $\psi_-(s, y)$ in (5.12), that is

$$\psi_+(s, x) = \mathcal{O}_2(s). \tag{5.22}$$

- The change to straighten the fibers $W^u_\mu(S^+_{x_0})$

$$x = \chi_+(s,w), \tag{5.23}$$

where $\chi_+(s,w)$ has similar properties as $\chi_-(s,z)$ from (5.15), implying that

$$\chi_+(s,w) = w + \mathcal{O}_3(s). \tag{5.24}$$

□

Remark 5.2. *We want to stress the relation between coordinates $(s,\tilde\beta,z)$ and $(s,\tilde\iota,w)$, which will be of major importance for the computations in Section 5.2. It follows from (5.8) and (5.20) that*

$$\begin{aligned} \iota &= \beta + \pi, \\ x &= y \mod 2\pi. \end{aligned} \tag{5.25}$$

Therefore, one can use (5.13) and (5.25), along with equations (5.20) and (5.21), to relate $\tilde\iota$ and $\tilde\beta$ as

$$\tilde\iota = \tilde\beta + \pi + (\psi_-(s,y) - \psi_+(s,x)),$$

where we recall

$$\psi_-(s,y) = \mathcal{O}_2(s), \quad \psi_+(s,x) = \mathcal{O}_2(s).$$

Additionally, one can relate the variables w and z using (5.16) and (5.23), to obtain

$$w = z + \mathcal{O}_3(s).$$

5.2 Transition map close to collision

The main purpose of this section is to define a transition map from $\Sigma^<_h$ to $\Sigma^>_h$ and prove that it sends transverse curves to $W^s_\mu(S^-)$ to transverse curves to $W^u_\mu(S^+)$. To this end, we consider the submanifold $\mathcal{M}$ (see (2.16)) for a fixed h and we define, in the straightened coordinates $(s,\tilde\beta,z)$ and $(s,\tilde\iota,w)$, the following sections

$$\begin{aligned} \tilde\Sigma^<_h &= \{(s,\tilde\beta,z)\colon s=\delta, -\delta<\tilde\beta<\delta, z\in\mathbb{T}\} = \tilde\Sigma^{<,+}_h \cup \tilde\Sigma^{<,-}_h, \\ \tilde\Sigma^>_h &= \{(s,\tilde\iota,w)\colon s=\delta, -\delta<\tilde\iota<\delta, w\in\mathbb{T}\} = \tilde\Sigma^{>,+}_h \cup \tilde\Sigma^{>,-}_h, \end{aligned}$$

which are transverse to the invariant manifolds. We also define

$$\begin{aligned} \tilde\Sigma^{<,+}_h &= \{(s,\tilde\beta,z)\colon s=\delta, 0<\tilde\beta<\delta, z\in\mathbb{T}\}, \\ \tilde\Sigma^{<,-}_h &= \{(s,\tilde\beta,z)\colon s=\delta, -\delta<\tilde\beta<0, z\in\mathbb{T}\}, \\ \tilde\Sigma^{>,+}_h &= \{(s,\tilde\iota,w)\colon s=\delta, 0<\tilde\iota<\delta, w\in\mathbb{T}\}, \\ \tilde\Sigma^{>,-}_h &= \{(s,\tilde\iota,w)\colon s=\delta, -\delta<\tilde\iota<0, w\in\mathbb{T}\}. \end{aligned} \tag{5.26}$$

This section is devoted to prove the following theorem.

Theorem 5.3. *Let $\delta,\sigma>0$ be small enough with $0<\sigma\ll\delta$. Consider a curve $\tilde\gamma_{\rm in}$ of the form*

$$\left(s,\tilde\beta,z\right) = \tilde\gamma_{\rm in}(\nu) = (\delta,\nu,z_{\rm in}(\nu)), \qquad \nu\in(-\sigma,\sigma) \tag{5.27}$$

where $z_{\rm in}$ is a C^1 function, which is transverse to $W^s_\mu(S^-)\cap\tilde\Sigma^<_h$ at

$$\tilde p_s = \tilde\gamma_{\rm in}(0) = (\delta,0,z_{\rm in}(0)).$$

Then, if we restrict the curve $\tilde\gamma_{\rm in}$ for $\nu\in(0,\sigma)$, we have

- *The transition map* $\tilde{f}\colon \tilde{\Sigma}_h^{<,+} \to \tilde{\Sigma}_h^{>,-}$ *maps the curve* $\tilde{\gamma}_{\rm in}$ *to a curve* $\tilde{\gamma}_{\rm out} \subset \tilde{\Sigma}_h^{>}$*, parameterized as*

$$\tilde{\gamma}_{\rm out}(\nu) = \tilde{f}(\tilde{\gamma}_{\rm in}(\nu)) = (s, \tilde{\imath}, w) = \big(\delta, -\nu + \mathcal{O}_1(\delta\nu), z_{\rm in}(\nu) + \mathcal{O}_1(\delta^2\nu, \nu^2)\big) \qquad \nu \in (0, \sigma).$$

Moreover, it has a well-defined limit

$$\tilde{p}_u = \lim_{\nu \to 0^+} \tilde{\gamma}_{\rm out}(\nu) = (\delta, 0, z_{\rm in}(0)) \in W^u_\mu(S^+) \cap \tilde{\Sigma}_h^{>}. \tag{5.28}$$

- *The tangent vector* $\tilde{\gamma}'_{\rm out}(\nu)$ *has a well-defined limit as* $\nu \to 0$*, which is of the form*

$$\tilde{\gamma}'_{\rm out}(0) = \lim_{\nu \to 0^+} \tilde{\gamma}'_{\rm out}(\nu) = \big(0, -1 + \mathcal{O}_1(\delta), z'_{\rm in}(0) + \mathcal{O}_1(\delta)\big)$$

and, therefore, $\tilde{\gamma}_{\rm out}$ *is transverse to* $W^u_\mu(S^+) \cap \tilde{\Sigma}_h^{>}$ *at* $\tilde{p}_u$*.*

Remark 5.4. *In view of Theorem 5.3, and in particular of* (5.28)*, we can extend continuously the map* $\tilde{f}$ *to points in* $(\delta, 0, z) \in W^s_\mu(S^-) \cap \tilde{\Sigma}_h^{>}$ *as*

$$\tilde{f}(\delta, 0, z) = (\delta, 0, z) \in W^u_\mu(S^+) \cap \tilde{\Sigma}_h^{>}.$$

To prove Theorem 5.3, we are going to consider the following intermediate sections

$$\begin{aligned} \tilde{\Sigma}_1^{\pm} &= \{(s, \tilde{\beta}, z) \colon 0 < s < \delta, \tilde{\beta} = \pm\delta, z \in \mathbb{T}\}, \\ \tilde{\Sigma}_2^{\pm} &= \{(s, \tilde{\imath}, w) \colon 0 < s < \delta, \tilde{\beta}(s, \tilde{\imath}, w) = \pi \pm \delta, w \in \mathbb{T}\}, \end{aligned} \tag{5.29}$$

where $\tilde{\beta}(s, \tilde{\imath}, w)$ is obtained from Remark 5.2 (see Figure 5.1). We split the transition map into three maps: from $\tilde{\Sigma}_h^{<,+}$ to $\tilde{\Sigma}_1^{+}$, from $\tilde{\Sigma}_1^{+}$ to $\tilde{\Sigma}_2^{-}$ and from $\tilde{\Sigma}_2^{-}$ to $\tilde{\Sigma}_h^{>,-}$. The other case, i.e., the transition map from $\tilde{\Sigma}_h^{<,-}$ to $\tilde{\Sigma}_h^{>,+}$, is built analogously.

The following lemma (whose proof is done in Appendix B), provides information regarding the first transition map.

Lemma 5.5. *The transition map*

$$\begin{aligned} \mathcal{T}^+_{s,1} \colon \tilde{\Sigma}_h^{<,+} &\to \tilde{\Sigma}_1^{+} \\ (\delta, \tilde{\beta}_0, z_0) &\mapsto (s_1, \delta, z_1) \end{aligned}$$

takes the curve $\tilde{\gamma}_{\rm in}(\nu)$ *defined in* (5.27) *with* $\nu \in (0, \sigma)$ *to a curve* $\tilde{\gamma}_1(\nu)$ *defined as*

$$\tilde{\gamma}_1(\nu) = (s_1(\nu), \delta, z_1(\nu)), \tag{5.30}$$

with

$$\begin{aligned} s_1(\nu) &= \nu\,(1 + \mathcal{O}_1(\delta)) \\ z_1(\nu) &= z_{\rm in}(\nu) + \mathcal{O}_1(\delta^2\nu), \end{aligned}$$

where $z_{\rm in}$ *is the* z*-component of the curve* $\tilde{\gamma}_{\rm in}(\nu)$ *described in* (5.27)*.*

Moreover, for $\nu \in (0, \sigma)$*, its tangent vector is of the form*

$$\tilde{\gamma}'_1(\nu) = \big(s'_1(\nu), 0, z'_1(\nu)\big), \tag{5.31}$$

where

$$s_1'(\nu) = 1 + \mathcal{O}_1(\delta)$$
$$z_1'(\nu) = z_{\text{in}}'(\nu) + \mathcal{O}_1(\delta)$$

and it has a well-defined limit as $\nu \to 0$. *Thus,* $\tilde{\gamma}_1$ *is transverse to* $W^u_\mu(S^-) \cap \tilde{\Sigma}_1^+$ *at* $\lim_{\nu\to 0^+} \tilde{\gamma}_1(\nu) = (0, \delta, z_{\text{in}}(0))$.

The next step is to define a map from $\tilde{\Sigma}_1^+$ to $\tilde{\Sigma}_2^-$ (see (5.29)), which we denote by $\mathcal{T}^+_{1,2}$. To this end, we consider $\varepsilon > 0$ small enough and the following domain

$$U = \left\{(s, \tilde{\beta}, z)\colon \delta \leq \tilde{\beta} \leq \pi - \delta, 0 < s < \varepsilon, |z - z_1(\nu)| < \varepsilon,\ \nu \in (0, \sigma)\right\},$$

where z_1 is given in (5.30).

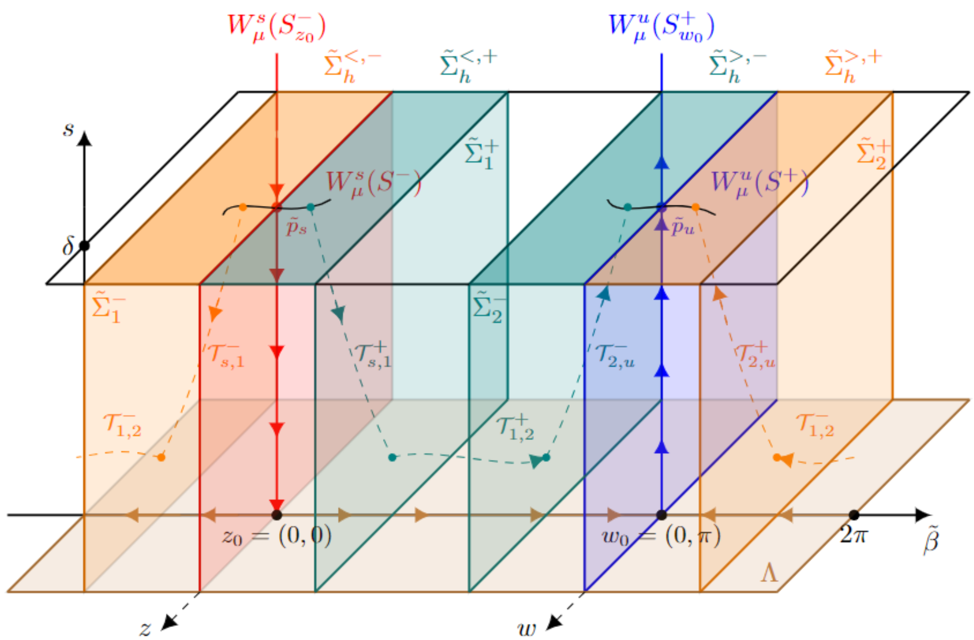

Figure 5.1: Representation of the dynamics near the collision manifold. The line $\tilde{\beta} = 2\pi$ is identified with $\tilde{\beta} = 0$. The transition maps are given by the composition of the maps $\mathcal{T}^{\mp}_{2,u} \circ \mathcal{T}^{\pm}_{1,2} \circ \mathcal{T}^{\pm}_{s,1}$. Recall that we are considering different systems coordinates in neighborhoods of the two circles $S^{\pm}$.

Note that the values of the variable $\tilde{\beta}$ in U are no longer small. However, since s is close to 0, the diffeomorphism Γ_- in Proposition 5.1, which is linear in $\tilde{\beta}$ and straightens the invariant manifolds of S^-, is still well-defined. Performing the changes (5.13) and (5.16) to system (5.9), as we did in the proof of Proposition 5.1, but without Taylor expanding in the $\tilde{\beta}$ coordinate, we obtain the system

$$\begin{aligned} s' &= -\frac{m_0}{2} s \left(\cos\tilde{\beta} + \tilde{G}_1(s, \tilde{\beta}, z)\right) \\ \tilde{\beta}' &= \frac{m_0}{2} \sin\tilde{\beta} + \tilde{G}_2(s, \tilde{\beta}, z) \\ z' &= s\left(\lambda(\mu, h) s \sin\tilde{\beta} + \tilde{G}_3(s, \tilde{\beta}, z)\right), \end{aligned} \tag{5.32}$$

where

$$\tilde{G}_1(s, \tilde{\beta}, z) = \mathcal{O}_2(s), \quad \tilde{G}_2(s, \tilde{\beta}, z) = \mathcal{O}_2(s), \quad \tilde{G}_3(s, \tilde{\beta}, z) = \mathcal{O}_3(s).$$

The following lemma (whose proof is done in Appendix C), analyzes the transition map $\mathcal{T}_{1,2}^{+}$.

Lemma 5.6. *The transition map $\mathcal{T}_{1,2}^{+}$ from $\tilde{\Sigma}_1^{+}$ to $\tilde{\Sigma}_2^{-}$, defined in* (5.29) *and expressed in coordinates* $(s, \tilde{\beta}, z)$,

$$\begin{aligned} \mathcal{T}_{1,2}^{+}\colon \tilde{\Sigma}_1^{+} &\to \tilde{\Sigma}_2^{-} \\ (s_1, \delta, z_1) &\mapsto (s_2, \pi-\delta, z_2), \end{aligned} \tag{5.33}$$

maps the curve $\tilde{\gamma}_1$ in (5.30) *to a curve $\tilde{\gamma}_2$ given by*

$$\tilde{\gamma}_2(\nu) = (s_2(\nu), \pi-\delta, z_2(\nu)), \quad \nu \in (0, \sigma) \tag{5.34}$$

with

$$\begin{aligned} s_2(\nu) &= \nu\left(1 + \mathcal{O}_1(\delta)\right), \\ z_2(\nu) &= z_{\rm in}(\nu) + \mathcal{O}_1(\delta^2\nu, \nu^2), \end{aligned}$$

where $z_{\rm in}$ is to the z-component of the curve $\tilde{\gamma}_{\rm in}$ in (5.27).

Moreover, for $\nu \in (0, \sigma)$, its tangent vector is of the form

$$\tilde{\gamma}_2'(\nu) = \left(s_2'(\nu), 0, z_2'(\nu)\right) \tag{5.35}$$

and

$$\begin{aligned} s_2'(\nu) &= 1 + \mathcal{O}_1(\delta), \\ z_2'(\nu) &= z_{\rm in}'(\nu) + \mathcal{O}_2(\delta). \end{aligned}$$

and it has a well-defined limit as $\nu \to 0$. Thus, the curve $\tilde{\gamma}_2$ is transverse to $W^s_\mu(S^+) \cap \tilde{\Sigma}_2^-$ at $\lim_{\nu\to 0^+} \tilde{\gamma}_2(\nu) = (0, \pi-\delta, z_{\rm in}(0))$.

Note that $\tilde{\gamma}_2$ is expressed in coordinates $(s, \tilde{\beta}, z)$, whereas the section $\tilde{\Sigma}_2^-$ is defined in coordinates $(s, \tilde{\iota}, w)$. The following corollary, whose proof is straightforward from Remark 5.2, gives an expression of the curve $\tilde{\gamma}_2$ of Lemma 5.6 in the latter variables.

Corollary 5.7. *The curve $\tilde{\gamma}_2$ given in* (5.34) *for $\nu \in (0, \sigma)$ can be expressed in coordinates $(s, \tilde{\iota}, w)$ as*

$$\tilde{\gamma}_2(\nu) = (s_2(\nu), \tilde{\iota}_2(\nu), w_2(\nu)) \tag{5.36}$$

with

$$\begin{aligned} s_2(\nu) &= \nu\left(1 + \mathcal{O}_1(\delta)\right), \\ \tilde{\iota}_2(\nu) &= -\delta + \mathcal{O}_2(\nu), \\ w_2(\nu) &= z_{\rm in}(\nu) + \mathcal{O}_1(\delta^2\nu, \nu^2), \end{aligned}$$

and $z_{\rm in}$ is to the z-component of the curve $\tilde{\gamma}_{\rm in}$ in (5.27).

Moreover, its tangent vector is of the form

$$\tilde{\gamma}_2'(\nu) = \left(s_2'(\nu), 0, w_2'(\nu)\right)$$

where

$$\begin{aligned} s_2'(\nu) &= 1 + \mathcal{O}_1(\delta) \\ w_2'(\nu) &= z_{\rm in}'(\nu) + \mathcal{O}_1(\delta) \end{aligned}$$

and has a well-defined limit as $\nu \to 0$. Thus, the curve $\tilde{\gamma}_2$ is transverse to $W^s_\mu(S^+) \cap \tilde{\Sigma}_2^-$ at $\lim_{\nu\to 0^+} \tilde{\gamma}_2(\nu) = (0, -\delta, z_{\rm in}(0))$.

Finally, the following lemma (whose proof is done in Appendix B), defines the transition map from $\tilde{\Sigma}_2^-$ to $\tilde{\Sigma}_h^{>,-}$ in an analogous way as in Lemma 5.5.

Lemma 5.8. *The transition map*

$$\mathcal{T}_{2,u}^-\colon \tilde{\Sigma}_2^- \to \tilde{\Sigma}_h^{>,-}$$
$$(s_2, \tilde{\iota}_2, w_2) \mapsto (\delta, \tilde{\iota}_u, w_u)$$

(see (5.29) *and* (5.26)*) takes the curve* $\tilde{\gamma}_2$ *in* (5.36) *with* $\nu \in (0, \sigma)$ *to a curve* $\tilde{\gamma}_{\rm out}$ *defined as*

$$\tilde{\gamma}_{\rm out}(\nu) = (\delta, \tilde{\iota}_{\rm out}(\nu), w_{\rm out}(\nu))$$

such that

$$\tilde{\iota}_{\rm out}(\nu) = -\nu + \mathcal{O}_1(\delta\nu)$$
$$w_{\rm out}(\nu) = z_{\rm in}(\nu) + \mathcal{O}_1(\delta^2\nu, \nu^2).$$

Moreover, its tangent vector is of the form

$$\tilde{\gamma}'_{\rm out}(\nu) = \big(0, \tilde{\iota}'_{\rm out}(\nu), w'_{\rm out}(\nu)\big)$$

where

$$\tilde{\iota}'_{\rm out}(\nu) = -1 + \mathcal{O}_1(\delta)$$
$$w'_{\rm out}(\nu) = z'_{\rm in}(\nu) + \mathcal{O}_1(\delta)$$

and its limit as $\nu \to 0$ *is well-defined and, therefore,* $\tilde{\gamma}_{\rm out}$ *is transverse to* $W^u_\mu(S^+)$ *at* $\lim_{\nu\to 0}\tilde{\gamma}_{\rm out}(\nu) = (\delta, 0, z_{\rm in}(0)) = \tilde{p}_u$*.*

To complete the proof of Theorem 5.3, we observe that the transition map $\tilde{f}$ is obtained as the composition of the maps from Lemmas 5.5, 5.6 and 5.8 respectively, i.e.

$$\tilde{f} = \mathcal{T}_{2,u}^- \circ \mathcal{T}_{1,2}^+ \circ \mathcal{T}_{s,1}^+.$$

6 Proof of Theorems 1.6 and 1.7

The purpose of this section is to prove Theorem 1.6. To this end, we recall the following notation which is used along the section:

- The curves $\Delta^{s,u}_\infty(\mu)$, $\Delta^u_{S^+}(\mu)$ and $\Delta^s_{S^-}(\mu)$ defined in (3.1) are the intersection of the corresponding invariant manifolds with Σ_h. They admit a graph parameterization in polar coordinates (θ, Θ), as shown in (3.2) and (3.4).
- The sets $I_D^\pm(w_\Sigma)$, defined in (2.32) and (2.33) (with w_Σ as in (3.5)), where the graphs of the curves $\Delta^{s,u}_\infty(\mu)$ are defined.
- The graph parameterizations $\Theta^s_\infty(\theta, \hat{\Theta}_0)$, $\Theta^u_{S^+}(\theta)$ (defined in (3.6) and (3.3) respectively) and $\Theta^u_\infty(\theta, \hat{\Theta}_0)$, $\Theta^s_{S^-}(\theta)$ (obtained from the symmetries in (2.34) and (2.41)).

The key step in the proof of Theorem 1.6 is Proposition 6.2 below. To state it, we introduce the terminology *triple intersection* between invariant manifolds. Certainly, this is an abuse of language, since it is well known that two stable (or unstable) manifolds cannot intersect. Therefore, let us explain what we mean by that.

Definition 6.1. *We say that the invariant manifolds* $W^u_\mu(S^+)$*,* $\tilde{W}^u_\mu(\mathrm{A}_{\hat{\Theta}_0^*})$ *and* $\tilde{W}^s_\mu(\mathrm{A}_{\hat{\Theta}_0^*})$ *have a triple intersection at* $p^*_> = (\theta_>, \Theta_>) \in \Sigma^>_{h^*}$ *(see* (3.7)*) if*

- *$\Delta^u_{S^+}(\mu)$ and $\Delta^s_\infty(\mu)$ intersect at $p^*_> = (\theta_>, \Theta_>) \in \Sigma^>_{h^*}$,*

- *$\Delta^s_{S^-}(\mu)$ and $\Delta^u_\infty(\mu)$ intersect at $p^*_< = (\theta_<, \Theta_<) \in \Sigma^<_{h^*}$ (in the usual sense),*

- *The point $p^*_>$ belongs to the unstable fiber within $W^u_\mu(S^+)$ of a point $(0, \bar\theta, \pi/2) \in S^+$ and the point $p^*_<$ belongs to the stable fiber within $W^s_\mu(S^-)$ of a point $(0, \bar\theta, -\pi/2) \in S^-$.*

Note that this can be phrased as saying that the continuous extension of the local map $\tilde f$ given in Theorem 5.3 (see Remark 5.4) maps $p^*_<$ to $p^*_>$. Moreover, note that $\tilde f$ maps $\Delta^u_\infty(\mu) \setminus \{p^*_<\}$ to a curve in $\Sigma^>_{h^*}$ which as $p^*_>$ as endpoint. Then, we say that the triple intersection is transverse if this curve and the curves $\Delta^s_\infty(\mu), \Delta^u_{S^+}(\mu)$ intersect pairwise transversally at $p^*_>$.

Next proposition proves the existence of a transverse triple intersection at a suitable energy level. Note that, since we parameterize the curves as graphs, the angles are taken in $[-\pi/2, \pi/2]$.

Proposition 6.2. *There exist $\mu_0, \delta_0 > 0$ such that, for any $\mu \in (0, \mu_0)$, $\delta \in (0, \delta_0)$, there is an angular momentum $\hat\Theta^*_0$ satisfying*

$$\hat\Theta^*_0 = \hat\Theta^*_0(\mu, \delta) = \mu\left(\mathcal{M}_+(0) + \mathcal{O}_1(\mu, \delta^2)\right)$$

(where $\mathcal{M}_+$ is defined in (3.11)) such that, for $h = h^*(\mu, \delta) = -\hat\Theta^*_0$

- *The invariant manifold $\tilde W^u_\mu(\mathrm{A}_{\hat\Theta^*_0})$ intersects the section $\Sigma^>_{h^*}$ and this intersection can be written locally as graph as $(\theta, \Theta^{u,>}_\infty(\theta))$ in a neighborhood of the image of the point $p^*_< \in \Delta^u_\infty(\mu) \pitchfork \Delta^s_{S^-}(\mu)$ under the (continuous extension of the) local map provided by Theorem 5.3.*

- *The invariant manifolds $W^u_\mu(S^+)$, $\tilde W^u_\mu(\mathrm{A}_{\hat\Theta^*_0})$ and $\tilde W^s_\mu(\mathrm{A}_{\hat\Theta^*_0})$ intersect transversally at $p^*_> = (\theta_>, \Theta_>) \in \Sigma^>_{h^*}$ (see (3.7)) where*

$$\theta_> = \mathcal{O}_1(\mu, \delta^2), \quad \Theta_> = \Theta^s_\infty(\theta_>, \hat\Theta^*_0) = \Theta^u_{S^+}(\theta_>) = \Theta^{u,>}_\infty(\theta_>).$$

- *For a fixed $\varepsilon > 0$ small enough independent of μ, denote by $B_\varepsilon(p^*_>) \subset \Sigma^>_{h^*}$ a ε-neighborhood of $p^*_>$, and define the following C^1-curves parameterized by θ*

$$\begin{aligned}\gamma^{s,>}_\infty(\theta) =& \tilde W^s_\mu(\mathrm{A}_{\hat\Theta^*_0}) \cap B_\varepsilon(p^*_>),\\ \gamma^{u,>}_\infty(\theta) =& \tilde W^u_\mu(\mathrm{A}_{\hat\Theta^*_0}) \cap B_\varepsilon(p^*_>),\\ \gamma^{u,>}_{S^+}(\theta) =& W^u_\mu(S^+) \cap B_\varepsilon(p^*_>).\end{aligned} \tag{6.1}$$

  Hence, the angles

$$A = \measuredangle(\gamma^{u,>}_\infty(\theta_>)', \gamma^{s,>}_\infty(\theta_>)'), \quad B = \measuredangle(\gamma^{u,>}_\infty(\theta_>)', \gamma^{u,>}_{S^+}(\theta_>)') \tag{6.2}$$

  (taken in $[-\pi/2, \pi/2]$) satisfy

$$-\frac{\pi}{2} < A < B < 0, \tag{6.3}$$

  leading to the configuration depicted in Figure 6.1.

The proof of this proposition is divided into two parts: in Section 6.1 we prove that the triple intersection $p^*_>$ (in the sense of Definition 6.1) exists and in Section 6.2 we establish the transversality between the curves in (6.1) in $p^*_>$ as well as the ordering given by (6.3). Now we prove Theorems 1.6 and 1.7 relying on Proposition 6.2.

Proof of Theorems 1.6 and Theorem 1.7. At $h = h^*$, there exist transverse heteroclinic orbits between the stable invariant manifold of infinity and the unstable manifold of the collision set $\mathcal{S}$ and between the unstable manifold of infinity and the stable manifold of $\mathcal{S}$ which guarantees the existence of a parabolic ejection/collision orbit. Moreover, this transversality ensures that, for $h = h^*$, the manifolds $\tilde{W}^s_\mu(\mathrm{A}_{\hat{\Theta}_0}), \tilde{W}^u_\mu(\mathrm{A}_{\hat{\Theta}_0})$ also intersect transversally in the sense of Definition 6.1.

Then, one can adapt the construction done by Moser in [43] to our setting (see also the work of Simó and Llibre in [34]; and the work of Guardia, Martin and Seara in [27]) to prove the existence of a hyperbolic set (at a suitable Poincaré section transverse to the flow) whose dynamics is conjugated to the shift of infinite symbols and which accumulates to the invariant manifolds of infinity. This leads to the existence of any combination of hyperbolic, parabolic, periodic and oscillatory orbits in forward and backward time at the energy level h^*.

The classical Moser approach requires a transverse homoclinic point of the invariant manifolds of infinity. This does not exist in the present setting since we only have a triple intersection point in the sense of Definition 6.1. However, the Moser approach can be easily adapted to this setting. Let us explain how.

First, recall that at the section $\Sigma^>_{h^*}$ (within the energy level $h = h^*$), the curves $\gamma^{s,>}_\infty$, $\gamma^{u,>}_\infty$ and $\gamma^{u,>}_{S+}$ intersect transversally at $p^*_>$ (in the sense of Definition 6.1) and have the ordering depicted in Figure 6.1. We define a return map from a suitable open domain in this section, which contains the triple intersection point at its closure.

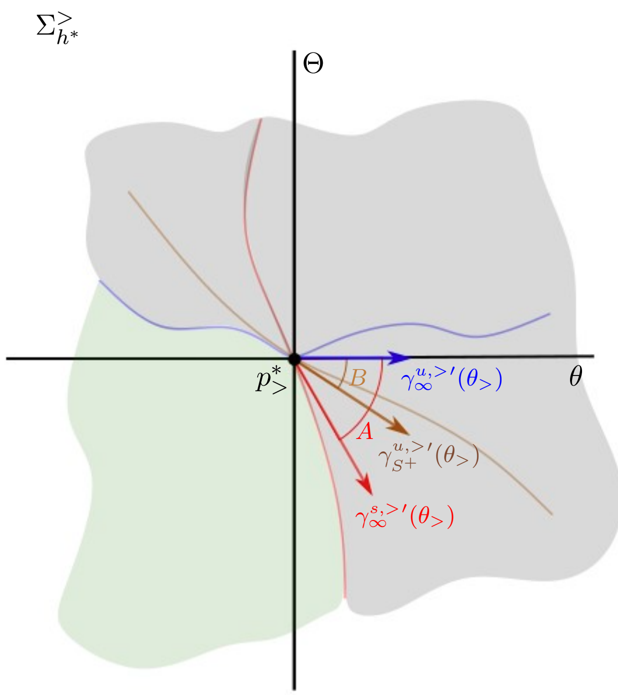

Figure 6.1: Representation of the "ordering of the invariant manifolds" given by the angles A and B in (6.2). The region shaded in green represents the set $\mathcal{D}$ where the forward and backward return maps to the section $\Sigma^>_{h^*}$ are well defined. That is, the region where the construction made by Moser in [43] is performed.

To this end, we consider first the following domains. Let us fix $\varepsilon > 0$ and consider the points $p^*_>$, $p^*_<$

introduced in the Definition 6.1. Then, we define

- $D^>$ as the points in $\Sigma^>_{h^*}$, ε-close to $p^*_>$, whose forward orbit hits $\Sigma^<_{h^*}$.
- $D^<$ as the points in $\Sigma^<_{h^*}$, ε-close to $p^*_<$, whose backward orbit hits $\Sigma^>_{h^*}$.

To characterize the domains $D^>$ and $D^<$, consider first two ε-neighborhoods of the intersection points $p^*_>$ and $p_<$, which we denote by $B_\varepsilon(p^*_>) \subset \Sigma^>_{h^*}$ and $B_\varepsilon(p_<) \subset \Sigma^<_{h^*}$ respectively.

The curve $\Delta^s_\infty(\mu)$ in (3.1) intersects $B_\varepsilon(p^*_>)$, dividing it into two connected open regions within $B_\varepsilon(p^*_>)$. By definition, $\Delta^s_\infty(\mu)$ corresponds to points whose forward orbits are parabolic. As a result, one of these regions contains points whose forward orbits escape infinity, classifying them as hyperbolic, while the other region contains points whose forward orbits return and intersect the section $\Sigma^<_{h^*}$. The latter region, corresponding to the domain $D^>$, is defined by points with smaller radial momentum than those on the curve $\Delta^s_\infty(\mu)$. Namely

$$D^> = \left\{(\theta,\Theta) \in B_\varepsilon(p^*_>) \colon R(\delta^2,\theta,\Theta;h^*) < R^s_\infty(\theta)\right\} \subset \Sigma^>_{h^*}$$

where

$$R^s_\infty(\theta) = R(\delta^2,\theta,\Theta^s_\infty(\theta),h^*) > 0$$

is obtained from the Hamiltonian (2.10).

Similarly, the curve $\Delta^u_\infty(\mu)$ in (3.1) separates $B_\varepsilon(p^*_<)$ into two connected components within $B_\varepsilon(p_<)$. In this case, the domain $D^<$ corresponds to the points with smaller radial momentum (in absolute value) than those on $\Delta^u_\infty(\mu)$. Therefore

$$D^< = \left\{(\theta,\Theta) \in B_\varepsilon(p^*_<) \colon R(\delta^2,\theta,\Theta;h^*) > R^u_\infty(\theta)\right\} \subset \Sigma^<_{h^*}$$

where

$$R^u_\infty(\theta) = R(\delta^2,\theta,\Theta^u_\infty(\theta),h^*) < 0 \tag{6.4}$$

is also obtained from the Hamiltonian (2.10).

As the points in $D^<$ are close to $p_<$, one can map this domain to $\Sigma^>_{h^*}$ by means of the (continuous extension of the) local map analyzed in Theorem 5.3.

To this end, let (θ_0, R_0) denote an arbitrary point in $D^<$. Then, we apply the changes of coordinates ψ and $\tilde{\psi}$ introduced in (2.12) and (2.14) to translate the point into coordinates (θ,α) obtaining

$$\alpha_0 = -\arcsin\left(\frac{\delta\,(R_0 + \mu\sin\theta_0)}{\sqrt{2(1-\mu)+\rho(\theta_0)}}\right)$$

where $\rho(\theta_0) = \rho(\delta,\theta_0)$ is defined in (2.15) and satisfies

$$\rho(\theta) = \mathcal{O}_1(\delta^6, \mu\delta^2).$$

Once in coordinates (θ,α), we apply the diffeomorphism Γ_- from Proposition 5.1 to express the point in coordinates $(\tilde{\beta}, z)$ such that

$$\tilde{\beta}_0 = \alpha_0 + \frac{\pi}{2} - \psi_-(\delta, y_0), \quad z_0 = \chi^{-1}(\delta, y_0)$$

where

$$y_0 = \theta_0 - 2\left(\alpha_0 + \frac{\pi}{2}\right)$$

and ψ_- and χ_- are defined in (5.12) and (5.15) respectively. Therefore, they are of the form

$$\tilde{\beta}_0 = \alpha_0 + \frac{\pi}{2} + \mathcal{O}_2(\delta), \quad z_0 = y_0 + \mathcal{O}_3(\delta).$$

Now we are ready to apply the transition map $\tilde{f}$ from Theorem 5.3 to the point $(\tilde{\beta}_0, z_0)$, which leads to the following point in $\Sigma^{>}_{h^*}$ expressed in coordinates $(\tilde{\iota}, w)$

$$(\tilde{\iota}_1, w_1) = \left(-\tilde{\beta}_0 \left(1 + \mathcal{O}_1(\delta)\right), z_0 + \tilde{\beta}_0 \mathcal{O}_1(\delta^2, \tilde{\beta}_0)\right).$$

Finally, we apply the inverse of the diffeomorphism Γ^+ from Proposition 5.1 and the inverse of the changes (2.12) and (2.14) to express it back to polar coordinates (θ, R) yielding

$$\begin{aligned} \theta_1 &= \chi_+(\delta, w_1) - 2\left(\tilde{\iota}_1 - \psi_+(\delta, \chi_+(\delta, w_1))\right) \\ R_1 &= \sqrt{2(1-\mu) + \rho(\theta_1)} \sin\left(\tilde{\iota}_1 + \psi_+(\delta, \chi_+(\delta, w_1)) + \frac{\pi}{2}\right) - \mu \sin\theta_1 \end{aligned}$$

where ψ^+ and χ^+ are defined in (5.22) and (5.24) respectively. They satisfy

$$\theta_1 = \theta_0 + \mathcal{O}_2(\delta), \quad R_1 = -R_0 + \mathcal{O}_2(\delta).$$

This result gives the domain $\tilde{D}^<$, defined as

$$\tilde{D}^< = \left\{(\theta, R) \in B_\varepsilon(p^*_>) \colon R < R^{u,>}_\infty(\theta)\right\} \subset \Sigma^{>}_{h^*},$$

where $R^{u,>}_\infty(\theta)$ corresponds to the image through the continuous extension of the transition map of $R^u_\infty(\theta)$ in (6.4).

Alternatively, by the conservation of the Hamiltonian (2.10), the domains $D^>$ and $\tilde{D}^<$ can be also defined as

$$\begin{aligned} D^> &= \left\{(\theta, \Theta) \in B_\varepsilon(p^*_>) \colon \Theta < \Theta^s_\infty(\theta)\right\} \subset \Sigma^{>}_{h^*}, \\ \tilde{D}^< &= \left\{(\theta, \Theta) \in B_\varepsilon(p^*_>) \colon \Theta < \Theta^{u,>}_\infty(\theta)\right\} \subset \Sigma^{>}_{h^*}, \end{aligned}$$

where $\Theta^{u,>}_\infty(\theta)$ is introduced in Proposition 6.2.

Recall that Proposition 6.2 gives the "ordering" of the invariant manifolds at the triple intersection. It implies that the transition of the points from $D^<$ to $\tilde{D}^<$ has not gone through collision. Therefore, the domain $\mathcal{D} = D^> \cap \tilde{D}^<$ contains points in $\Sigma^{>}_{h^*}$, ε-close to $p^*_>$, whose backward and forward orbit hit $\Sigma^{>}_{h^*}$. Hence, proceeding as Moser in [43], one can analyze the return map in the domain $\mathcal{D}$ and prove that

- There exists a hyperbolic set which accumulates to the triple intersection point $p^*_>$. Note that in the domain $\mathcal{D}$ the return map is well-defined. The only difference between the classical setting and the present one is the passage close to collision. However, by Theorem 5.3 we have a very precise description of local map close to collision, which in good coordinates is close to the identity (in $\mathcal{C}^1$ regularity). This allows to apply the approach in [43], which leads to a hyperbolic set homeomorphic to $\mathbb{N}^{\mathbb{Z}}$ whose dynamics is conjugated to the shift of infinite symbols. Moreover, this set contains the triple intersection in its closure. Indeed, the symbols in the sequence $\mathbb{N}^{\mathbb{Z}}$ labels the closeness of the corresponding orbit to the invariant manifolds. The higher the symbol, the closer is the point to the invariant manifold.

  Then, unbounded (forward and backward) sequences in $\mathbb{N}^{\mathbb{Z}}$ correspond to oscillatory orbits which accumulate also to collision. Periodic sequences correspond to periodic orbits which by taking high enough symbols can be as large and as close to collision as prescribed.

- The complement of $\mathcal{D}$ within $B_\varepsilon(p_>^*)$ contains points whose orbits are either parabolic or hyperbolic. As shown in [43], the image of $\mathcal{D}$ under the return map hits both $\partial\mathcal{D}$ and $B_\varepsilon(p_>^*)\setminus\mathcal{D}$ (both forward and backward in time). Then, one can construct parabolic and hyperbolic excursions after any given number of returns to the section. Moreover, since this analysis can be done both for the forward and backward return maps, one can combine any future and past final motions.

□

6.1 Existence of the triple intersection

Fix $\mu > 0$ small enough. Then, for any $\theta^+ \in U_+$, fix an energy level $h = -\hat{\Theta}_0 = -\mu\bar{\Theta}_0 = \mu M_+(\theta^+ - w_\Sigma)$ so that Theorem 3.3 holds and consider the point

$$p_< = p_<(\theta^+,\mu) = (\theta_<(\theta^+,\mu), \Theta_<(\theta^+,\mu)) \in \Delta^s_{S^-}(\mu) \pitchfork \Delta^u_\infty(\mu), \tag{6.5}$$

where $\Theta_<(\theta^+,\mu) = \Theta^u_\infty(\theta_<(\theta^+,\mu),\hat{\Theta}_0) = \Theta^s_{S^-}(\theta_<(\theta^+,\mu))$.

Since the changes of coordinates $\psi, \tilde{\psi}$ (defined in (2.12) and (2.14)) and Γ_- (defined in (5.2)) are diffeomorphisms, we can apply Theorem 5.3 and Remark 5.4 so that the (continuous extension of the) transition map $\tilde{f}$ sends $\tilde{p}_s = \left(\Gamma_- \circ \tilde{\psi} \circ \psi\right)(p_<)$ to a point $\tilde{p}_u \in \tilde{\Sigma}_h^> \cap W^u_\mu(S^+)$ defined in (5.28). Then, applying the change of coordinates Γ_+ in (5.5), this point can be written as $p_>^u = (\theta_>^u, \Theta_>^u) = \left(\psi^{-1} \circ \tilde{\psi}^{-1} \circ \Gamma_+^{-1}\right)(\tilde{p}_u)$, and therefore $\Theta_>^u = \Theta^u_{S^+}(\theta_>^*)$. Abusing notation (in the same sense as in Definition 6.1) we say that $p_>^u \in \Delta^u_{S^+}(\mu) \pitchfork \tilde{W}^u_\mu(\mathrm{A}_{\hat{\Theta}_0})$. In particular, the θ-component of both points can be related as

$$\theta_>^u = \theta_>^u(\theta^+,\mu,\delta) = \theta_<(\theta^+,\mu) + \mathcal{O}_2(\delta).$$

To guarantee the triple intersection we need that $p_>^u = p_>(\theta^+,\mu) \in \Delta^u_{S^+}(\mu) \pitchfork \Delta^s_\infty(\mu)$ from Theorem 3.3. Namely, we look for θ^+ such that

$$\theta_>^u(\theta^+,\mu,\delta) = \theta_>(\theta^+,\mu) \tag{6.6}$$

which is equivalent to solve

$$\theta^+ + \mathcal{O}_1(\mu,\delta^2) = 0, \tag{6.7}$$

given that w_Σ in (3.5) is of order δ^3.

Hence, the Implicit Function Theorem ensures that there exist $\mu_0, \delta_0 > 0$ such that, for all $(\mu,\delta) \in (0,\mu_0)\times(0,\delta_0)$, there exists a unique $\theta^+(\mu,\delta)$ with $\theta^+(0,0) = 0$ satisfying (6.7) and therefore (6.6). Since $\theta^+(\mu,\delta) \in U^+$ for $\mu,\delta > 0$ small enough (see Theorem 3.3), the point

$$p_>^* = (\theta_>(\mu,\delta), \Theta_>(\mu,\delta)), \quad \Theta_>(\mu,\delta) = \Theta^u_{S^+}(\theta_>(\mu,\delta)) \tag{6.8}$$

belongs to $\tilde{W}^s_\mu(\mathrm{A}_{\hat{\Theta}_0^*}) \cap \tilde{W}^u_\mu(\mathrm{A}_{\hat{\Theta}_0^*}) \cap W^u_\mu(S^+)$ at

$$h^* = h(\mu,\delta) = -\hat{\Theta}_0^*(\mu,\delta) = \mu M_+(\theta^+(\mu,\delta) - w_\Sigma) = \mu(\mathcal{M}_+(0) + \mathcal{O}_1(\mu,\delta^2)). \tag{6.9}$$

6.2 Transversality and ordering at the triple intersection

From now on, we consider h^* as in (6.9), and use the expression for $\theta_>$ given in Theorem 3.3 with the value of θ^+ obtained in (6.7) and denote by

$$\begin{aligned}
&\theta_> = \theta_>(\mu,\delta) = \theta^+(\mu,\delta) - w_\Sigma + \mathcal{O}_1(\mu) = \mathcal{O}_1(\mu,\delta^2) \in I^+_{\frac{3}{10}}(w_\Sigma) \cap I^-_{\frac{3}{10}}(w_\Sigma),\\
&\theta_< = \theta_<(\mu,\delta) = \theta_>(\mu,\delta) + \mathcal{O}_2(\delta) = \theta_> + \mathcal{O}_2(\delta) = \mathcal{O}_1(\mu,\delta^2) \in I^+_{\frac{3}{10}}(w_\Sigma) \cap I^-_{\frac{3}{10}}(w_\Sigma),\\
&d_\pm(\theta) = d_\pm(\theta,\hat{\Theta}_0^*), \quad \Theta^{u,s}_\infty(\theta) = \Theta^{u,s}_\infty(\theta,\hat{\Theta}_0^*),
\end{aligned} \tag{6.10}$$

the θ-components of $p_<$ and $p^*_>$ in (6.5) and (6.8) respectively, the distances in (3.9) and the values of the angular momenta of the points in $\Delta^{u,s}_\infty(\mu)$. For a fixed $\varepsilon > 0$ small enough, we consider an ε-neighborhood of $\theta_>$ containing $\theta_<$ (see (6.10)) which we denote $B_\varepsilon(\theta_>)$, and the following C^1-curves

$$\begin{aligned}
\gamma^{u,<}_\infty &= \Big\{ (\theta, \Theta^u_\infty(\theta)) : \theta \in B_\varepsilon(\theta_>) \Big\} \subset \Delta^u_\infty(\mu) \subset \Sigma^<_{h^*}, \\
\gamma^{s,<}_{S^-} &= \Big\{ (\theta, \Theta^s_{S^-}(\theta)) : \theta \in B_\varepsilon(\theta_>) \Big\} \subset \Delta^s_{S^-}(\mu) \subset \Sigma^<_{h^*}, \\
\gamma^{s,>}_\infty &= \Big\{ (\theta, \Theta^s_\infty(\theta)) : \theta \in B_\varepsilon(\theta_>) \Big\} \subset \Delta^s_\infty(\mu) \subset \Sigma^>_{h^*}, \\
\gamma^{u,>}_{S^+} &= \Big\{ (\theta, \Theta^u_{S^+}(\theta)) : \theta \in B_\varepsilon(\theta_>) \Big\} \subset \Delta^u_{S^+}(\mu) \subset \Sigma^>_{h^*}.
\end{aligned} \tag{6.11}$$

Observe that $\gamma^{u,<}_\infty$, $\gamma^{s,>}_\infty$ and $\gamma^{u,>}_{S^+}$ are the curves in (6.1).

Note that, by Theorem 3.3, these curves satisfy

$$p_<(\mu,\delta) \in \gamma^{u,<}_\infty \pitchfork \gamma^{s,<}_{S^-}, \quad p_>(\mu,\delta) \in \gamma^{s,>}_\infty \pitchfork \gamma^{u,>}_{S^+}.$$

Moreover, in Section 6.1 we have seen that

$$p_>(\mu,\delta) = p^*_> = (\theta_>, \Theta^u_{S^+}(\theta_>)) \in \Delta^u_{S^+}(\mu) \cap \Delta^s_\infty(\mu) \cap \tilde{W}^u_\mu(\mathrm{A}_{\hat{\Theta}^*_0}) \in \Sigma^>_{h^*}. \tag{6.12}$$

Then, the following lemma (whose proof is done in Appendix D) gives us information regarding the image of $\gamma^{u,<}_\infty$ through the transition map $\tilde{f}$ obtained in Theorem 5.3, translated into coordinates (θ, Θ).

Lemma 6.3. *Denote by*

$$\gamma^{u,>}_\infty = \Big(\psi^{-1} \circ \tilde{\psi}^{-1} \circ \Gamma^{-1}_+ \circ \tilde{f} \circ \Gamma_- \circ \tilde{\psi} \circ \psi\Big) \circ \gamma^{u,<}_\infty \subset \Sigma^>_{h^*}$$

where $\gamma^{u,<}_\infty$ is the curve in (6.11) and

- *The diffeomorphisms ψ and $\tilde{\psi}$ are defined in (2.12) and (2.14) respectively.*
- *The diffeomorphisms Γ_- and Γ_+ are given in Proposition 5.1.*
- *The transition map $\tilde{f}$ from $\Sigma^<_{h^*}$ to $\Sigma^>_{h^*}$ is given in Theorem 5.3.*

Then, the curve $\gamma^{u,>}_\infty$ can be written as a graph as

$$\gamma^{u,>}_\infty = \Big\{ \big(\theta, \Theta^{u,>}_\infty(\theta)\big) : \theta \in B_\varepsilon(\theta_>) \Big\},$$

where the function $\Theta^{u,>}_\infty(\theta)$ satisfies

$$\begin{aligned}
\Theta^{u,>}_\infty(\theta) &= \Theta^u_{S^+}(\theta) + \mathcal{O}_1(\mu\delta), \\
\Theta^{u,>}_\infty{}'(\theta_>) &= \Theta^u_{S^+}{}'(\theta_>) + d'_-(\theta_>) + \mathcal{O}_1(\mu\delta^2),
\end{aligned} \tag{6.13}$$

where $d_-(\theta)$ is defined in (6.10).

As a result, the transversality condition of the triple intersection is guaranteed once we prove that

$$\begin{aligned}
\Theta^u_{S^+}{}'(\theta_>) - \Theta^s_\infty{}'(\theta_>) &\neq 0 \\
\Theta^s_\infty{}'(\theta_>) - \Theta^{u,>}_\infty{}'(\theta_>) &\neq 0 \\
\Theta^u_{S^+}{}'(\theta_>) - \Theta^{u,>}_\infty{}'(\theta_>) &\neq 0.
\end{aligned}$$

By (3.8) and (6.13), these inequalities can be written as

$$
\begin{aligned}
{\Theta^u_{S+}}'(\theta_>) - {\Theta^s_\infty}'(\theta_>) &= -d'_+(\theta_>) \neq 0, \\
{\Theta^u_{S+}}'(\theta_>) - \Theta^{u,>\prime}_\infty(\theta_>) &= -d'_-(\theta_>) + \mathcal{O}_1(\mu\delta^2) \neq 0, \\
{\Theta^s_\infty}'(\theta_>) - \Theta^{u,>\prime}_\infty(\theta_>) &= d'_+(\theta_>) - d'_-(\theta_>) + \mathcal{O}_1(\mu\delta^2) \neq 0.
\end{aligned} \tag{6.14}
$$

Using (6.10), the first and second inequalities are given by the first and second items of Theorem 3.3 respectively. Finally; by (3.9), (3.10) and (3.12) we have

$$
d'_+(\theta_>) = -d'_-(\theta_>) = \mu\left(\mathcal{M}'_+(0) + \mathcal{O}_1(\mu, \delta^2)\right), \tag{6.15}
$$

and therefore the third inequality can be written as

$$
\mu\left(2{\mathcal{M}_+}'(0) + \mathcal{O}_1(\mu, \delta^2)\right) \neq 0.
$$

Lemma 3.2 implies this inequality for μ, δ small enough.

Following (6.14) and (6.15), we compute the angles in (6.2) as follows

$$
\begin{aligned}
\sin A =& \frac{{\Theta^s_\infty}'(\theta_>) - \Theta^{u,>\prime}_\infty(\theta_>)}{\sqrt{\left(1 + ({\Theta^s_\infty}'(\theta_>))^2\right)\left(1 + \left(\Theta^{u,>\prime}_\infty(\theta_>)\right)^2\right)}} = \frac{\mu\left(2\mathcal{M}'_+(0) + \mathcal{O}_1(\mu, \delta^2)\right)}{\sqrt{\left(1 + ({\Theta^s_\infty}'(\theta_>))^2\right)\left(1 + \left(\Theta^{u,>\prime}_\infty(\theta_>)\right)^2\right)}} \\
\sin B =& \frac{{\Theta^u_{S+}}'(\theta_>) - \Theta^{u,>\prime}_\infty(\theta_>)}{\sqrt{\left(1 + \left({\Theta^u_{S+}}'(\theta_>)\right)^2\right)\left(1 + \left(\Theta^{u,>\prime}_\infty(\theta_>)\right)^2\right)}} = \frac{\mu\left(\mathcal{M}'_+(0) + \mathcal{O}_1(\mu, \delta^2)\right)}{\sqrt{\left(1 + \left({\Theta^u_{S+}}'(\theta_>)\right)^2\right)\left(1 + \left(\Theta^{u,>\prime}_\infty(\theta_>)\right)^2\right)}}.
\end{aligned}
$$

Lemma 3.2 ensures that $\mathcal{M}'_+(0) < 0$, leading to (6.3). See Figure 6.1 to see the relative position of the curves.

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- **Data availability** : The authors declare that all the data supporting the findings of this study are available within the paper.
- **Code availability** : The authors declare that all the code supporting the findings of this study are available within the paper.

A Proof of Lemma 3.2

The proof of Lemma 3.2 is computer-assisted. We rely on the CAPD libraries developed in [30].

We compute the integrals involved in the derivative of the Melnikov function $\mathcal{M}_+(\theta)$ in (3.11) (the result for $\mathcal{M}_-(\theta)$ follows from (3.12)), which can be written as

$$\frac{d}{d\theta}\mathcal{M}_+(\theta) = \mathcal{I}_1(\theta) - \mathcal{I}_2(\theta), \tag{A.1}$$

with

$$\begin{aligned}\mathcal{I}_1(\theta) &= \kappa \int_0^{+\infty} \left(\frac{s^{\frac{2}{3}}\cos(\theta - s)}{\left(1+\kappa^2 s^{\frac{4}{3}} - 2\kappa s^{\frac{2}{3}}\cos(\theta-s)\right)^{\frac{3}{2}}} - \frac{3\kappa s^{\frac{4}{3}}\sin^2(\theta-s)}{\left(1+\kappa^2 s^{\frac{4}{3}} - 2\kappa s^{\frac{2}{3}}\cos(\theta-s)\right)^{\frac{5}{2}}} \right) ds,\\ \mathcal{I}_2(\theta) &= \sqrt{\frac{2}{\kappa}} \int_0^{+\infty} \frac{\sin(\theta-s)}{s^{\frac{1}{3}}}\, ds,\end{aligned} \tag{A.2}$$

where $\kappa = \frac{3^{\frac{2}{3}}}{2^{\frac{1}{3}}}$.

Note that $\mathcal{I}_2(\theta)$ can be computed explicitly (by means of a Laplace transform)

$$\mathcal{I}_2(\theta) = \sqrt{\frac{2}{\kappa}}\Gamma\left(\frac{2}{3}\right)\left(\frac{\sin\theta}{2} - \frac{\sqrt{3}}{2}\cos\theta\right). \tag{A.3}$$

To compute $\mathcal{I}_1(\theta)$ we proceed as follows. We consider two constants $c, C > 0$ and we split it as

$$\kappa^{-1}\mathcal{I}_1(\theta) = i(0,c;\theta) + i(c,C;\theta) + i(C,+\infty;\theta)$$

with

$$i(0,c;\theta)=\int_0^c\left(\frac{s^{\frac{2}{3}}\cos(\theta-s)}{\left(1+\kappa^2s^{\frac{4}{3}}-2\kappa s^{\frac{2}{3}}\cos(\theta-s)\right)^{\frac{3}{2}}}-\frac{3\kappa s^{\frac{4}{3}}\sin^2(\theta-s)}{\left(1+\kappa^2s^{\frac{4}{3}}-2\kappa s^{\frac{2}{3}}\cos(\theta-s)\right)^{\frac{5}{2}}}\right)ds,$$

$$i(c,C;\theta)=\int_c^C\left(\frac{s^{\frac{2}{3}}\cos(\theta-s)}{\left(1+\kappa^2s^{\frac{4}{3}}-2\kappa s^{\frac{2}{3}}\cos(\theta-s)\right)^{\frac{3}{2}}}-\frac{3\kappa s^{\frac{4}{3}}\sin^2(\theta-s)}{\left(1+\kappa^2s^{\frac{4}{3}}-2\kappa s^{\frac{2}{3}}\cos(\theta-s)\right)^{\frac{5}{2}}}\right)ds, \quad \text{(A.4)}$$

$$i(C,+\infty;\theta)=\int_C^{+\infty}\left(\frac{s^{\frac{2}{3}}\cos(\theta-s)}{\left(1+\kappa^2s^{\frac{4}{3}}-2\kappa s^{\frac{2}{3}}\cos(\theta-s)\right)^{\frac{3}{2}}}-\frac{3\kappa s^{\frac{4}{3}}\sin^2(\theta-s)}{\left(1+\kappa^2s^{\frac{4}{3}}-2\kappa s^{\frac{2}{3}}\cos(\theta-s)\right)^{\frac{5}{2}}}\right)ds.$$

Taking $c<\left(\frac{1}{\kappa}\right)^{\frac{3}{2}}<C$, we have the following uniform bounds for $i(0,c;\theta)$ and $i(C,+\infty;\theta)$

$$|i(0,c;\theta)|\le\int_0^c\left(\frac{s^{\frac{2}{3}}}{\left(1-\kappa s^{\frac{2}{3}}\right)^3}+\frac{3\kappa s^{\frac{4}{3}}}{\left(1-\kappa s^{\frac{2}{3}}\right)^5}\right)ds\le\frac{3}{5}\frac{c^{\frac{5}{3}}}{\left(1-\kappa c^{\frac{2}{3}}\right)^3}+\frac{9\kappa}{7}\frac{c^{\frac{7}{3}}}{\left(1-\kappa c^{\frac{2}{3}}\right)^5},$$

$$|i(C,+\infty;\theta)|\le\int_C^{+\infty}\left(\frac{1}{s^{\frac{4}{3}}\left(\kappa-s^{-\frac{2}{3}}\right)^3}+\frac{3\kappa}{s^2\left(\kappa-s^{-\frac{2}{3}}\right)^5}\right)ds \quad \text{(A.5)}$$

$$\le\frac{3}{C^{\frac{1}{3}}\left(\kappa-C^{-\frac{2}{3}}\right)^3}+\frac{3\kappa}{C\left(\kappa-C^{-\frac{2}{3}}\right)^5}.$$

From these estimates, we can obtain the intervals where $\mathcal{I}_1$ and $\mathcal{I}_2$ in (A.2) belong using computer computations[1].

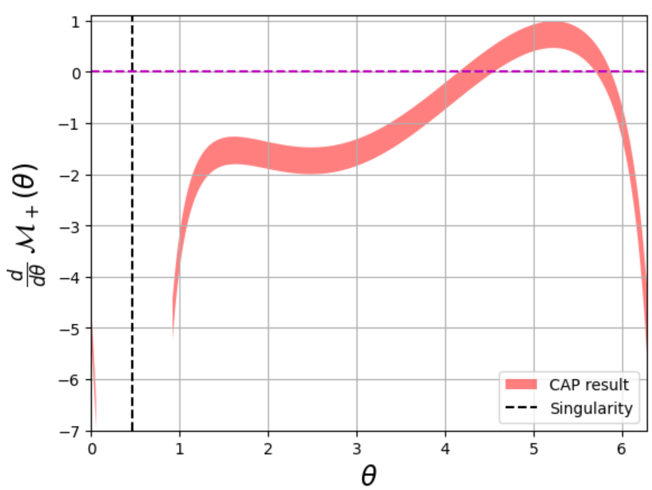

Figure A.1: Representation of $\mathcal{M}_+'(\theta)$ from the computer assisted algorithm.

To this end note that, to have a well-defined formula for $\mathcal{M}_+(\theta)$, the conditions of Theorem 3.1 must be satisfied. In particular, we have to consider values of $\theta\in I_D^+(w_\Sigma)$ defined in (2.32) (with w_Σ given in (3.5)). In our case, we have taken $\theta\in\mathbb{T}-\left(\frac{\sqrt{2}}{3}-0.45,\frac{\sqrt{2}}{3}+0.45\right)\subset I_D^+(w_\Sigma)$ for $D=0.3$.

[1]The code to compute the expressions in (A.4) and the data plotted in Figure A.1 can be found at https://github.com/JoseLamasRodriguez/CAPD-Code.git.

For the estimates in (A.5), we have considered $c = 0.001$ and $C = 100$. Smaller values of c or higher values of C may result in better approximations for the derivative $\mathcal{M}'_+(\theta)$ in (A.1) at the expense of increased computational time to evaluate $i(c, C; \theta)$, defined in (A.4). Finally, the integral $i(c, C; \theta)$ is estimated using a computer-assisted algorithm.

The provided implementation iterates through a range of angles, dividing it into $N = 10000$ equidistant intervals of size $\frac{2\pi}{N} + \text{eps}$, where $\text{eps} = 10^{-5}$ is introduced to ensure that these intervals intersect with one another. For each angle within the range, the code ensures that it belongs to $\mathbb{T} - \left(\frac{\sqrt{2}}{3} - 0.45, \frac{\sqrt{2}}{3} + 0.45\right)$ where, as previously stated, the Melnikov function can be computed. It then estimates the integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ defined in (A.2), where $\mathcal{I}_1$ is estimated using the previous results and $\mathcal{I}_2$ is estimated from (A.3). Finally, the difference between these integrals, and therefore the derivative $\mathcal{M}'_+(\theta)$, is computed and written in the corresponding files.

As a result, we obtain the representation of $\mathcal{M}'_+(\theta)$ depicted in Figure A.1. Moreover,

$$\mathcal{M}'_+(0) \in [-5.15341, -4.56572].$$

B Proof of Lemma 5.5 and Lemma 5.8

The first part of this section is devoted to prove Lemma 5.5. The proof is divided into two parts. The first part consists on finding the curve $\tilde{\gamma}_1$ in (5.30). Then, we show that the tangent vector $\tilde{\gamma}'_1$ is of the form (5.31). Both proofs rely on a fixed point argument.

To study the evolution of the curve $\tilde{\gamma}_{\text{in}}$ defined in (5.27), instead of using equation (5.3), we consider the equations of the orbits

$$\begin{aligned} \frac{ds}{d\tilde{\beta}} &= -\frac{s}{\tilde{\beta}}\left(1 + \tilde{G}_s(s, \tilde{\beta}, z)\right) \\ \frac{dz}{d\tilde{\beta}} &= \frac{2}{m_0} s\tilde{G}_z(s, \tilde{\beta}, z), \end{aligned} \tag{B.1}$$

where $\tilde{G}_s$ and $\tilde{G}_z$ satisfy

$$\begin{aligned} \tilde{G}_s(s, \tilde{\beta}, z) &= \mathcal{O}_1(s, \tilde{\beta}), \quad \tilde{G}_z(s, \tilde{\beta}, z) = 4\lambda(\mu, h)s + \mathcal{O}_3(s, \tilde{\beta}), \\ \partial_s\tilde{G}_s(s, \tilde{\beta}, z) &= \mathcal{O}_0(s), \quad \partial_s\tilde{G}_z(s, \tilde{\beta}, z) = 4\lambda(\mu, h) + \mathcal{O}_2(s, \tilde{\beta}), \\ \partial_z\tilde{G}_s(s, \tilde{\beta}, z) &= \mathcal{O}_1(s, \tilde{\beta}), \quad \partial_z\tilde{G}_z(s, \tilde{\beta}, z) = \mathcal{O}_3(s, \tilde{\beta}), \end{aligned} \tag{B.2}$$

and $\lambda(\mu, h) = \frac{\mu^2 + 2h + 2\mu}{4m_0}$. For simplicity we denote $\lambda = \lambda(\mu, h)$.

We denote by $\Phi^{\tilde{\beta}}(s_0, \tilde{\beta}_0, z_0)$ the general solution of (B.1). Then,

$$\tilde{\gamma}_1(\nu) = (s_1(\nu), \delta, z_1(\nu)) = \Phi^\delta(\delta, \nu, z_{\text{in}}(\nu)).$$

To compute the functions s_1 and z_1, we analyze the evolution of $\Phi^{\tilde{\beta}}$ from $\tilde{\beta} = \nu$ to $\tilde{\beta} = \delta$ by a fixed point argument.

To this end, for fixed $\nu \in (0, \sigma]$, we consider the set of continuous functions $f\colon [\nu, \delta] \to \mathbb{R}$ and two different norms: the classical supremum norm, which we denote by $\|\cdot\|$, and

$$||f||_{\mathcal{Z}_s} = \sup_{\tilde{\beta} \in [\nu, \delta]} \left(\frac{\tilde{\beta}}{\delta\nu}\left|f(\tilde{\beta}; \nu)\right|\right),$$

We denote by $\mathcal{C}^0$ and $\mathcal{Z}_s$ the associated Banach spaces.

Hence, the functional space $\mathcal{Z}_s \times \mathcal{C}^0$ is also a Banach space under the norm

$$||(f_s, f_z)||_{\mathcal{Z}_s \times \mathcal{C}^0} = \max\left(||f_s||_{\mathcal{Z}_s}, ||f_z||\right).$$

Now we define the following operator acting on $\mathcal{Z}_s \times \mathcal{C}^0$

$$\begin{aligned}\mathcal{F} \colon \mathcal{Z}_s \times \mathcal{C}^0 &\to \mathcal{Z}_s \times \mathcal{C}^0 \\ (f_s, f_z) &\mapsto \mathcal{F}(f_s, f_z) = \left(\mathcal{F}^1(f_s, f_z), \mathcal{F}^2(f_s, f_z)\right),\end{aligned} \tag{B.3}$$

with

$$\begin{aligned}\mathcal{F}^1(f_s, f_z) &= \delta \frac{\nu}{\tilde{\beta}} \exp\left(-\int_\nu^{\tilde{\beta}} \frac{\tilde{G}_s(f_s(\alpha;\nu), \alpha, f_z(\alpha;\nu))}{\alpha}\, d\alpha\right) \\ \mathcal{F}^2(f_s, f_z) &= z_{\text{in}}(\nu) + \frac{2}{m_0}\int_\nu^{\tilde{\beta}} f_s(\alpha;\nu)\cdot \tilde{G}_z(f_s(\alpha;\nu), \alpha, f_z(\alpha;\nu))\, d\alpha,\end{aligned} \tag{B.4}$$

which corresponds to the solution $\Phi^{\tilde{\beta}}(\delta, \nu, z_{\text{in}}(\nu))$ of the equations of the orbits of (B.1) such that at the initial time $\tilde{\beta} = \nu$ satisfy $s = \delta$ and $z = z_{\text{in}}(\nu)$.

Once we have defined the operator, a fixed point theorem argument allows us to estimate the components $s_1(\delta;\nu)$ and $z_1(\delta;\nu)$ in (5.30) and prove the first part of the lemma.

Relying on the estimates in (B.2), we can estimate $\mathcal{F}(0,0)$ by

$$\begin{aligned}||\mathcal{F}^1(0,0)||_{\mathcal{Z}_s} &\le \sup_{\tilde{\beta}\in[\nu,\delta]} \exp\left(-\int_\nu^{\tilde{\beta}} \frac{\tilde{G}_s(0,\alpha,0)}{\alpha}\right) \le e^{C(\delta-\nu)} \le e^{C\delta} \\ ||\mathcal{F}^2(0,0)|| &= |z_{\text{in}}(\nu)|.\end{aligned}$$

for some constant $C > 0$. Therefore,

$$||\mathcal{F}(0,0)||_{\mathcal{Z}_s \times \mathcal{C}^0} \le \max\left(e^{C\delta}, \max_{\nu\in(0,\sigma]} |z_{\text{in}}(\nu)|\right) = M.$$

We define now the ball

$$B_{2M} = \{(f_s, f_z) \in \mathcal{Z}_s \times \mathcal{C}^0 \colon ||(f_s, f_z)||_{\mathcal{Z}_s\times\mathcal{C}^0} < 2M\},$$

and we prove that the operator $\mathcal{F}$ is Lipschitz in B_{2M}.

Consider $(f_s, f_z), (g_s, g_z) \in B_{2M}$. The norm $||\mathcal{F}^1(f_s, f_z) - \mathcal{F}^1(g_s, g_z)||_{\mathcal{Z}_s}$ can be estimated as follows

$$\begin{aligned}&||\mathcal{F}^1(f_s, f_z) - \mathcal{F}^1(g_s, g_z)||_{\mathcal{Z}_s} \\ &\le \left|\exp\left(-\int_\nu^{\tilde{\beta}} \frac{\tilde{G}_s(f_s,\alpha,f_z)}{\alpha}\, d\alpha\right) - \exp\left(-\int_\nu^{\tilde{\beta}} \frac{\tilde{G}_s(g_s,\alpha,g_z)}{\alpha}; d\alpha\right)\right| \\ &\le \left|\exp\left(-\int_\nu^{\tilde{\beta}} \frac{\tilde{G}_s(g_s,\alpha,g_z)}{\alpha}\, d\alpha\right)\right| \cdot \left|\exp\left(\int_\nu^{\tilde{\beta}} \frac{\tilde{G}_s(g_s,\alpha,g_z) - \tilde{G}_s(f_s,\alpha,f_z)}{\alpha}\, d\alpha\right) - 1\right| \\ &\le \left|\exp\left(-\int_\nu^{\tilde{\beta}} \frac{\tilde{G}_s(g_s,\alpha,g_z)}{\alpha}\, d\alpha\right)\right| \\ &\quad \cdot \left|\int_{\tilde{\beta}}^\nu \frac{\tilde{G}_s(f_s,\alpha,f_z) - \tilde{G}_s(g_s,\alpha,g_z)}{\alpha}\, d\alpha\right| \cdot \exp\left(\left|\int_{\tilde{\beta}}^\nu \frac{\tilde{G}_s(f_s,\alpha,f_z) - \tilde{G}_s(g_s,\alpha,g_z)}{\alpha}\, d\alpha\right|\right).\end{aligned} \tag{B.5}$$

We now use the following properties, satisfied for any $(f_s, g_s) \in \mathcal{Z}_s$ and any $\alpha \in [\nu, \tilde{\beta}]$ (we recall that $\tilde{\beta} \in [\nu, \delta]$)

$$\begin{aligned} |g_s(\alpha;\nu)| &\le \frac{\delta\nu}{\alpha}||g_s||_{\mathcal{Z}_s} \le 2M\frac{\delta\nu}{\alpha} \le 2M\delta, \\ |f_s(\alpha;\nu) - g_s(\alpha;\nu)| &\le \frac{\delta\nu}{\alpha}||f_s(\alpha;\nu) - g_s(\alpha;\nu)||_{\mathcal{Z}_s}. \end{aligned} \tag{B.6}$$

The last inequality of $g_s(\alpha;\nu)$ is written to stress the fact that any function in $\mathcal{Z}_s$ is small (of size δ) in terms of $|\cdot|$. Clearly, the function f_s satisfies the same bound.

Therefore, by (B.2), the first factor in the last term in (B.5) can be estimated as

$$\begin{aligned} &\left|\exp\left(-\int_\nu^{\tilde{\beta}} \frac{\tilde{G}_s(g_s(\alpha;\nu),\alpha,g_z(\alpha;\nu))}{\alpha}\, d\alpha\right)\right| \le \exp\left(C\int_{\tilde{\beta}}^{\nu}\left(\frac{|g_s(\alpha;\nu)|}{\alpha}+1\right)\, d\alpha\right) \\ &\le \exp\left(C\int_{\tilde{\beta}}^{\nu}\left(\frac{2M\delta\nu}{\alpha^2}+1\right)\, d\alpha\right) \le \exp\left(2CM\delta\nu\left[\frac{1}{\nu}-\frac{1}{\tilde{\beta}}\right]+C|\nu-\tilde{\beta}|\right) \le e^{C(2M+1)\delta}, \end{aligned} \tag{B.7}$$

for some constant $C > 0$.

For the rest of the estimates we use the following result, which is a consequence of (B.2) and (B.6). The function

$$h^t(\alpha;\nu) = (h_s^t(\alpha;\nu),\alpha,h_z^t(\alpha;\nu)) = (tf_s(\alpha;\nu)+(1-t)g_s(\alpha;\nu),\alpha,tf_z(\alpha;\nu)+(1-t)g_z(\alpha;\nu)),$$

satisfies

$$|h_s^t| \le 2M\frac{\delta\nu}{\alpha} \le 2M\delta, \quad |h_z^t| \le 2M.$$

Then, there exists a constant $C > 0$ such that

$$\begin{aligned} \left|\tilde{G}_s(f_s,\alpha,f_z) - \tilde{G}_s(g_s,\alpha,g_z)\right| &\le \int_0^1 \left(\left|\partial_s\tilde{G}_s(h^t)\right|\cdot|f_s-g_s| + \left|\partial_z\tilde{G}_s(h^t)\right|\cdot|f_z-g_z|\right) dt \\ &\le C\frac{\delta\nu}{\alpha}||f_s-g_s||_{\mathcal{Z}_s} + C\left(\frac{2M\delta\nu}{\alpha}+\alpha\right)||f_z-g_z||. \end{aligned}$$

Then, the second factor can be bounded as follows

$$\begin{aligned} &\left|\int_{\tilde{\beta}}^{\nu} \frac{\tilde{G}_s(f_s,\alpha,f_z)-\tilde{G}_s(g_s,\alpha,g_z)}{\alpha}\, d\alpha\right| \le \int_{\tilde{\beta}}^{\nu} \frac{|\tilde{G}_s(f_s,\alpha,f_z)-\tilde{G}_s(g_s,\alpha,g_z)|}{\alpha}\, d\alpha \\ &\le C\delta\nu||f_s-g_s||_{\mathcal{Z}_s}\int_{\tilde{\beta}}^{\nu}\frac{1}{\alpha^2}\, d\alpha + C||f_z-g_z||\int_{\tilde{\beta}}^{\nu}\left(1+\frac{2M\delta\nu}{\alpha^2}\right)\, d\alpha \\ &\le C(2M+1)\delta\left(||f_s-g_s||_{\mathcal{Z}_s}+||f_z-g_z||\right). \end{aligned} \tag{B.8}$$

Finally, the last factor can be bounded as

$$\exp\left(\left|\int_{\tilde{\beta}}^{\nu} \frac{\tilde{G}_s(f_s,\alpha,f_z)-\tilde{G}_s(g_s,\alpha,g_z)}{\alpha}\, d\alpha\right|\right) \le e^{8CM(2M+1)\delta}, \tag{B.9}$$

which follows from (B.8).

Estimates (B.7), (B.8) and (B.9) yield the following bound for the norm $||\mathcal{F}^1(f_s,f_z)-\mathcal{F}^1(g_s,g_z)||_{\mathcal{Z}_s}$

$$||\mathcal{F}^1(f_s,f_z)-\mathcal{F}^1(g_s,g_z)||_{\mathcal{Z}_s} \le \delta K_1\left(||f_s-g_s||_{\mathcal{Z}_s}+||f_z-g_z||\right),$$

with $K_1 = C(2M+1)e^{(8M+1)(2M+1)C\delta}$.

On the other hand, the norm $||\mathcal{F}^2(f_s,f_z) - \mathcal{F}^2(g_s,g_z)||$ can be bounded as follows.

First note that proceeding as for the other component,

$$\begin{aligned}
&\left|\tilde{G}_z(f_s,\alpha,f_z) - \tilde{G}_z(g_s,\alpha,g_z)\right| \\
&\leq \frac{\delta\nu}{\alpha} C\left(4\lambda + (|f_s|+|g_s|)^2 + \alpha^2\right)||f_s-g_s||_{\mathcal{Z}_s} + C\left((|f_s|+|g_s|)^3+\alpha^3\right)||f_z-g_z|| \\
&\leq C\frac{\delta\nu}{\alpha}\left(4\lambda + 16M^2\frac{\delta^2\nu^2}{\alpha^2}+\alpha^2\right)||f_s-g_s||_{\mathcal{Z}_s} + C\left(64M^3\frac{\delta^3\nu^3}{\alpha^3}+\alpha^3\right)||f_z-g_z||.
\end{aligned} \tag{B.10}$$

Therefore,

$$\begin{aligned}
||\mathcal{F}^2(f_s,f_z) - \mathcal{F}^2(g_s,g_z)|| &\leq \frac{2}{m_0}\left|\int_\nu^{\tilde{\beta}}\left(f_s\cdot\tilde{G}_z(f_s,\alpha,f_z) - g_s\cdot\tilde{G}_z(g_s,\alpha,g_z)\right)d\alpha\right| \\
\leq \frac{2}{m_0}\int_\nu^{\tilde{\beta}}|f_s-g_s|\cdot|\tilde{G}_z(f_s,\alpha,f_z)|\,d\alpha &+ \frac{2}{m_0}\int_\nu^{\tilde{\beta}}|g_s|\cdot\left|\tilde{G}_z(f_s,\alpha,f_z)-\tilde{G}_z(g_s,\alpha,g_z)\right|\,d\alpha.
\end{aligned}$$

The first term can be estimated as

$$\begin{aligned}
&\frac{2}{m_0}\int_\nu^{\tilde{\beta}}|f_s-g_s|\cdot|\tilde{G}_z(f_s,\alpha,f_z)|\,d\alpha \leq \frac{2}{m_0}\delta\nu||f_s-g_s||_{\mathcal{Z}_s}\int_\nu^{\tilde{\beta}}\frac{4\lambda|f_s|+C\left(|f_s|^3+\alpha^3\right)}{\alpha}d\alpha \\
&\leq \frac{2}{m_0}\delta^2\nu\left(8M\lambda + C\left(8M^3+1\right)\delta^2\right)||f_s-g_s||_{\mathcal{Z}_s} \leq \delta k_1||f_s-f_g||_{\mathcal{Z}_s}
\end{aligned}$$

and the second as

$$\begin{aligned}
&\frac{2}{m_0}\int_\nu^{\tilde{\beta}}|g_s|\cdot\left|\tilde{G}_z(f_s,\alpha,f_z)-\tilde{G}_z(g_s,\alpha,g_z)\right|\,d\alpha \\
&\leq\left(\frac{16MC}{m_0}\lambda\delta^2\nu^2\int_\nu^{\tilde{\beta}}\frac{d\alpha}{\alpha^2} + \frac{64M^3C}{m_0}\delta^4\nu^4\int_0^{\tilde{\beta}}\frac{d\alpha}{\alpha^4} + \frac{4MC}{m_0}\delta^2\nu^2\right)||f_s-g_s||_{\mathcal{Z}_s} \\
&\quad+\left(\frac{256M^4C}{m_0}\delta^4\nu^4\int_\nu^{\tilde{\beta}}\frac{d\alpha}{\alpha^4} + \frac{4MC}{m_0}\delta^4\nu\right)||f_z-g_z|| \leq \delta\left(k_2||f_s-g_s||_{\mathcal{Z}_s} + k_3||f_z-g_z||\right),
\end{aligned}$$

where we have used (B.2), (B.6) and (B.10), and k_1, k_2, k_3 are defined as

$$\begin{aligned}
k_1 &= \frac{2}{m_0}\delta\nu\left(8M\lambda + C\left(8M^3+1\right)\delta^2\right), \\
k_2 &= \frac{4MC}{m_0}\delta\nu\left(4\lambda + 16M^2\delta^2 + \nu\right), \\
k_3 &= \frac{4MC}{m_0}\delta^3\nu\left(64M^3+1\right).
\end{aligned}$$

Denote by $K = \max\left(K_1, \max(k_1,k_2,k_3)\right)$. Then

$$\begin{aligned}
||\mathcal{F}(f_s,f_z) - \mathcal{F}(g_s,g_z)||_{\mathcal{Z}_s\times\mathcal{C}^0} &\leq \delta K\left(||f_s-g_s||_{\mathcal{Z}_s} + ||f_z-g_z||\right) \\
&\leq 2\delta K||(f_s,f_z)-(g_s,g_z)||_{\mathcal{Z}_s\times\mathcal{C}^0}.
\end{aligned}$$

That is, for $\delta > 0$ such that $2K\delta < 1/2$, the operator $\mathcal{F}$ is contractive and satisfies $\mathcal{F}(B_{2M}) \subset B_{2M}$.

Then, the Banach fixed point theorem ensures that there exists a unique fixed point in B_{2M} for the operator $\mathcal{F}$ in (B.3), which we denote by $(s(\tilde{\beta},\nu), z(\tilde{\beta},\nu))$ and satisfies

$$\left|s(\tilde{\beta},\nu)\right| \leq 2M\frac{\delta\nu}{\tilde{\beta}} \qquad \text{and} \qquad \left|z(\tilde{\beta},\nu)\right| \leq 2M. \tag{B.11}$$

Moreover

$$||(s,z)-\mathcal{F}(0,0)||_{\mathcal{Z}_s\times\mathcal{C}^0} = ||\mathcal{F}(s,z)-\mathcal{F}(0,0)||_{\mathcal{Z}_s\times\mathcal{C}^0} \leq 4MK\delta.$$

Therefore, for all $\tilde{\beta}\in[\nu,\delta]$,

$$|s(\tilde{\beta};\nu)-\mathcal{F}^1(0,0)(\tilde{\beta};\nu)| \leq \frac{\delta\nu}{\tilde{\beta}}||(s,z)-\mathcal{F}^1(0,0)||_{\mathcal{Z}_s} \leq 4MK\delta^2\frac{\nu}{\tilde{\beta}}$$

so, using that (B.2), we can write $s(\tilde{\beta},\nu)$ as

$$\begin{aligned} s(\tilde{\beta};\nu) &= \mathcal{F}^1(0,0)+\mathcal{O}\left(\delta^2\frac{\nu}{\tilde{\beta}}\right) = \delta\frac{\nu}{\tilde{\beta}}\exp\left(-\int_\nu^{\tilde{\beta}}\frac{\tilde{G}_s(0,\alpha,0)}{\alpha}\,d\alpha\right)+\mathcal{O}\left(\delta^2\frac{\nu}{\tilde{\beta}}\right) \\ &= \frac{\nu}{\tilde{\beta}}\delta\left(1+\mathcal{O}\left(\delta\right)\right). \end{aligned} \tag{B.12}$$

To obtain an expression for $z(\tilde{\beta};\nu)$, we can replace (B.12) in the integral equation for $z(\tilde{\beta};\nu)$ in (B.4) obtaining

$$\begin{aligned} &|z(\tilde{\beta};\nu)-z_{\text{in}}(\nu)| = \frac{2}{m_0}\left|\int_\nu^{\tilde{\beta}} s(\alpha;\nu)\cdot\tilde{G}_z(s(\alpha;\nu),\alpha,z(\alpha;\nu))\,d\alpha\right| \\ &\leq \frac{2}{m_0}\int_\nu^{\tilde{\beta}}\left(2M\frac{\delta\nu}{\alpha}\left(4\lambda|s(\alpha;\nu)|+C\left(|s(\alpha;\nu)|^3+\alpha^3\right)\right)\right)d\alpha \leq \frac{4M\delta\nu}{m_0}\int_\nu^{\tilde{\beta}}\frac{8M\lambda\frac{\delta\nu}{\alpha}+C\left(8M^3\frac{\delta^3\nu^3}{\alpha^3}+\alpha^3\right)}{\alpha}\,d\alpha \\ &\leq \frac{32M^2}{m_0}\lambda\delta^2\nu+\frac{32M^4C}{m_0}\delta^4\nu+\frac{4MC}{m_0}\delta^5\nu, \end{aligned}$$

where we used (B.2) and the fact that $s(\alpha;\nu)\in\mathcal{Z}_s$ (and therefore satisfies (B.6)), leading to (5.30) and finishing the first part of the proof.

For the second part of the proof concerning the existence and expression of the tangent vector $\tilde{\gamma}_1'(\nu)$ in (5.31) at the point $\tilde{\gamma}_1(\nu)$, we rake formally the corresponding derivative on the integral equations for $s(\tilde{\beta};\nu)$ and $z(\tilde{\beta};\nu)$ in (B.4). We denote by $s_\partial(\tilde{\beta};\nu)=\frac{d}{d\nu}s(\tilde{\beta};\nu)$ and $z_\partial(\tilde{\beta};\nu)=\frac{d}{d\nu}z(\tilde{\beta};\nu)$ respectively, and they have the following expressions

$$\begin{aligned} s_\partial(\tilde{\beta};\nu) =& \frac{s(\tilde{\beta};\nu)}{\nu}\left(1+\tilde{G}_s(\delta,\nu,z_{\text{in}}(\nu))-\nu\int_\nu^{\tilde{\beta}}\frac{\partial_s\tilde{G}_s(s,\alpha,z)\cdot s_\partial(\alpha;\nu)+\partial_z\tilde{G}_s(s,\alpha,z)\cdot z_\partial(\alpha;\nu)}{\alpha}\,d\alpha\right) \\ z_\partial(\tilde{\beta};\nu) =& z_{\text{in}}'(\nu)-\frac{2\delta}{m_0}\tilde{G}_z(\delta,\nu,z_{\text{in}}(\nu))+\frac{2}{m_0}\int_\nu^{\tilde{\beta}}\Bigg(s_\partial(\alpha;\nu)\cdot\tilde{G}_z(s,\alpha,z)+s(\alpha;\nu)\Big(\partial_s\tilde{G}_z(s,\alpha,z)\cdot s_\partial(\alpha;\nu) \\ &+\partial_z\tilde{G}_z(s,\alpha,z)\cdot z_\partial(\alpha;\nu)\Big)\Bigg)\,d\alpha. \end{aligned}$$

where s,z refer to $s(\alpha;\nu), z(\alpha;\nu)$ respectively.

To estimate the solutions, we proceed as before. To this end, for fixed $\nu \in (0, \sigma]$, we consider the set of continuous functions $f\colon [\nu, \delta] \to \mathbb{R}$ and two different norms: the classical supremum norm and

$$||f||_{\mathcal{Z}_\partial} = \sup_{\tilde{\beta}\in[\nu,\delta]} \left(\frac{\tilde{\beta}}{\delta} \left| f(\tilde{\beta};\nu) \right| \right),$$

We denote by $\mathcal{C}^0$ and $\mathcal{Z}_\partial$ the associated Banach spaces. Hence, the functional space $\mathcal{Z}_\partial \times \mathcal{C}^0$ is also a Banach space under the norm

$$||(f_s, f_z)||_{\mathcal{Z}_\partial\times\mathcal{C}^0} = \max\left(||f_s||_{\mathcal{Z}_\partial}, ||f_z||\right).$$

Now we define the following affine operator acting on $\mathcal{Z}_\partial \times \mathcal{C}^0$

$$\begin{aligned} \mathcal{F}_\partial\colon \mathcal{Z}_\partial \times \mathcal{C}^0 &\to \mathcal{Z}_\partial \times \mathcal{C}^0 \\ (f_s, f_z) &\mapsto \mathcal{F}_\partial(f_s, f_z) = \left(\mathcal{F}_\partial^1(f_s, f_z), \mathcal{F}_\partial^2(f_s, f_z)\right), \end{aligned}$$

with

$$\begin{aligned} \mathcal{F}_\partial^1(f_s, f_z) &= \frac{s(\tilde{\beta};\nu)}{\nu} \left(1 + \tilde{G}_s(\delta, \nu, z_{\text{in}}(\nu)) - \nu \int_\nu^{\tilde{\beta}} \frac{\partial_s \tilde{G}_s(s,\alpha,z)\cdot f_s + \partial_z \tilde{G}_s(s,\alpha,z)\cdot f_z}{\alpha}\, d\alpha \right) \\ \mathcal{F}_\partial^2(f_s, f_z) &= z_{\text{in}}'(\nu) - \frac{2\delta}{m_0} \tilde{G}_z(\delta,\nu,z_{\text{in}}(\nu)) \\ &\quad + \frac{2}{m_0} \int_\nu^{\tilde{\beta}} \left(f_s \cdot \tilde{G}_z(s,\alpha,z) + s(\alpha;\nu) \Big(\partial_s \tilde{G}_z(s,\alpha,z)\cdot f_s + \partial_z \tilde{G}_z(s,\alpha,z)\cdot f_z \Big) \right) d\alpha, \end{aligned}$$

which is well-defined and continuous.

Once we have defined the operator $\mathcal{F}_\partial$, we prove it has a fixed point. We start estimating $\mathcal{F}_\partial(0,0)$. Each component satisfies

$$\begin{aligned} ||\mathcal{F}_\partial^1(0,0)||_{\mathcal{Z}_\partial} &= \sup_{\tilde{\beta}\in[\nu,\delta]} \left(\frac{\tilde{\beta}}{\delta} \left| \frac{s(\tilde{\beta};\nu)}{\nu} \left(1 + \tilde{G}_s(\delta,\nu,z_{\text{in}}(\nu)) \right) \right| \right) \le 2M\left(1 + \mathcal{O}_1(\delta,\nu)\right) \le 4M \\ ||\mathcal{F}_\partial^2(0,0)|| &\le \left| z_{\text{in}}'(\nu) - \frac{2\delta}{m_0} \tilde{G}_z(\delta,\nu,z_{\text{in}}(\nu)) \right| \le |z_{\text{in}}'(\nu)| + \frac{2}{m_0}\delta^2\left(4\lambda + 2C\delta^2\right), \end{aligned}$$

obtained from (B.2) and the estimate of $s(\tilde{\beta};\nu)$ in (B.11). Therefore

$$||\mathcal{F}_\partial(0,0)||_{\mathcal{Z}_\partial\times\mathcal{C}^0} \le \max\left(4M, \max_{\nu\in(0,\sigma]} |z_{\text{in}}'(\nu)| + \frac{2}{m_0}\delta^2\left(4\lambda + 2C\delta^2\right) \right) = M_\partial.$$

We define then the following ball

$$B_{2M^\partial} = \{(f_s, f_z) \in \mathcal{Z}_\partial \times \mathcal{C}^0 \colon ||(f_s, f_z)||_{\mathcal{Z}_\partial\times\mathcal{C}^0} < 2M_\partial\},$$

and we prove that the operator $\mathcal{F}_\partial$ is Lipschitz in B_{2M^∂}.

Take $(f_s, f_z), (g_s, g_z) \in B_{2M_\partial}$. Then, the norm $||\mathcal{F}^1_\partial(f_s, f_z) - \mathcal{F}^1_\partial(g_s, g_z)||_{\mathcal{Z}_\partial}$ can be bounded as follows

$$
\begin{aligned}
&||\mathcal{F}^1_\partial(f_s, f_z) - \mathcal{F}^1_\partial(g_s, g_z)||_{\mathcal{Z}_\partial} \\
&\leq \sup_{\tilde{\beta}\in[\nu,\delta]} \left| \frac{\tilde{\beta}}{\delta} \cdot s(\tilde{\beta};\nu) \int_\nu^{\tilde{\beta}} \frac{\partial_s \tilde{G}_s(s,\alpha,z)\cdot(f_s - g_s) - \partial_z \tilde{G}_s(f_s,\alpha,f_z)\cdot(f_s-g_s)}{\alpha} d\alpha \right| \\
&\leq 2M\nu \int_\nu^{\tilde{\beta}} \frac{|\partial_s \tilde{G}_s(s,\alpha,z)|\cdot|f_s-g_s| + |\partial_z \tilde{G}_s(s,\alpha,z)|\cdot|f_z-g_z|}{\alpha}\, d\alpha \\
&\leq 2CM\nu \int_\nu^{\tilde{\beta}} \frac{|f_s-g_s|}{\alpha}\, d\alpha \\
&\quad + 2CM\nu \int_\nu^{\tilde{\beta}} \frac{(|s(\alpha;\nu)|+\alpha)|f_z-g_z|}{\alpha}\, d\alpha \leq 2CM\delta\left(||f_s-g_s||_{\mathcal{Z}_s} + \nu\,(2M+1)\,||f_z-g_z||\right),
\end{aligned}
$$

where we have used (B.2), that $s(\alpha;\nu) \in \mathcal{Z}_s$ and therefore satisfies (B.11) and also that, since $f_s, g_s \in \mathcal{Z}_\partial$, for all $\alpha \in [\nu, \tilde{\beta}]$

$$
|f_s(\alpha;\nu) - g_s(\alpha;\nu)| \leq \frac{\delta}{\alpha}||f_s - g_s||_{\mathcal{Z}_\partial}. \tag{B.13}
$$

Therefore, if we define $K_1 = 2CM\max(1, \nu(2M+1))$, we obtain the following estimate

$$
||\mathcal{F}^1_\partial(f_s, f_z) - \mathcal{F}^1_\partial(g_s, g_z)||_{\mathcal{Z}_\partial} \leq \delta K_1\left(||f_s-g_s||_{\mathcal{Z}_\partial} + ||f_z-g_z||\right).
$$

On the other hand, the norm $||\mathcal{F}^2_\partial(f_s, f_z) - \mathcal{F}^2_\partial(g_s, g_z)||$ is bounded as follows

$$
\begin{aligned}
&||\mathcal{F}^2_\partial(f_s, f_z) - \mathcal{F}^2_\partial(g_s, g_z)|| \\
&\leq \frac{2}{m_0}\int_\nu^{\tilde{\beta}} \left(|\tilde{G}_z(s,\alpha,z)|\cdot|f_s-g_s| + |s(\alpha;\nu)| \Big(|\partial_s \tilde{G}_z(s,\alpha,z)|\cdot|f_s-g_s| + |\partial_z \tilde{G}_z(s,\alpha,z)|\cdot|f_z-g_z| \Big) \right) d\alpha \\
&\leq \frac{2}{m_0}\delta||f_s-g_s||_{\mathcal{Z}_\partial}\int_\nu^{\tilde{\beta}} \frac{|\tilde{G}_z(s,\alpha,z)|}{\alpha}\, d\alpha \\
&\quad + \frac{2}{m_0}\delta||f_s-g_s||_{\mathcal{Z}_\partial}\int_\nu^{\tilde{\beta}} \frac{|s(\alpha;\nu)|}{\alpha}\cdot|\partial_s\tilde{G}_z(s,\alpha,z)|\, d\alpha + ||f_z-g_z||\int_\nu^{\tilde{\beta}} |s(\alpha;\nu)|\cdot|\partial_z\tilde{G}_z(s,\alpha,z)|\, d\alpha.
\end{aligned}
$$

Then, for the first term, we use that

$$
\int_\nu^{\tilde{\beta}} \frac{|\tilde{G}_z(s,\alpha,z)|}{\alpha}\, d\alpha \leq \int_\nu^{\tilde{\beta}} \frac{4\lambda|s(\alpha;\nu)| + C(|s(\alpha;\nu)|^3+\alpha^3)}{\alpha} d\alpha \leq k_1,
$$

with $k_1 = 2M\delta^2\left(8M\lambda + C\delta^2\left(8M^3+\nu\delta\right)\right)$. For the second term,

$$
\int_\nu^{\tilde{\beta}} \frac{|s(\alpha;\nu)|}{\alpha}\cdot|\partial_s\tilde{G}_z(s,\alpha,z)| d\alpha \leq 2M\delta\nu\int_\nu^{\tilde{\beta}} \frac{4\lambda + C\left(|s(\alpha;\nu)|^2+\alpha^2\right)}{\alpha^2} d\alpha \leq k_2,
$$

with $k_2 = 2M\delta\left(4\lambda + C\delta\left(4M^2\delta+\nu\right)\right)$ and, for the third,

$$
\int_\nu^{\tilde{\beta}} |s(\alpha;\nu)|\cdot|\partial_z\tilde{G}_z(s,\alpha,z)| d\alpha \leq k_3\delta,
$$

with $k_3 = 2MC\delta^3\nu\left(8M^3+1\right)$.

To obtain these estimates, we have used the results in (B.2) and the fact that $s(\alpha;\nu)$ satisfies (B.11) and (B.13). Hence, if we denote by $K_2 = \max\left(\frac{2}{m_0}k_1, \frac{2}{m_0}k_2, k_3\right)$, we have

$$
||\mathcal{F}^2_\partial(f_s, f_z) - \mathcal{F}^2_\partial(g_s, g_z)|| \leq \delta K_2\left(||f_s-g_s||_{\mathcal{Z}_\partial} + ||f_z-g_z||\right),
$$

leading to the following estimate of $||\mathcal{F}_\partial(f_s, f_z) - \mathcal{F}_\partial(g_s, g_z)||_{\mathcal{Z}_\partial \times \mathcal{C}^0}$

$$||\mathcal{F}_\partial(f_s, f_z) - \mathcal{F}_\partial(g_s, g_z)||_{\mathcal{Z}_\partial \times \mathcal{C}^0} \leq 2\delta K ||(f_s, f_z) - (g_s, g_z)||_{\mathcal{Z}_\partial \times \mathcal{C}^0},$$

with $K = \max(K_1, K_2)$, proving that, taking δ small enough, it is contractive and satisfies $\mathcal{F}_\partial(B_{2M^\partial}) \subset B_{2M^\partial}$.

Hence, there exists a unique fixed point in B_{2M^∂} of the operator $\mathcal{F}_\partial$, which we denote by $(s_\partial(\tilde{\beta};\nu), z_\partial(\tilde{\beta};\nu))$. We can estimate its value as follows

$$||(s_\partial, z_\partial) - \mathcal{F}_\partial(0,0)||_{\mathcal{Z}_\partial \times \mathcal{C}^0} = ||\mathcal{F}_\partial(s_\partial, z_\partial) - \mathcal{F}_\partial(0,0)||_{\mathcal{Z}_\partial \times \mathcal{C}^0} \leq 2\delta K ||(s_\partial, z_\partial)||_{\mathcal{Z}_\partial \times \mathcal{C}^0} \leq 4\delta M_\partial K.$$

Therefore, for all $\tilde{\beta} \in [\nu, \delta]$

$$|s_\partial(\tilde{\beta};\nu) - \mathcal{F}^1_\partial(0,0)| \leq 4M_\partial K \frac{\delta^2}{\tilde{\beta}}$$
$$|z_\partial(\tilde{\beta};\nu) - \mathcal{F}^2_\partial(0,0)| \leq 4M_\partial K \delta.$$

The expression of the tangent vector $\tilde{\gamma}_1^\partial$ in (5.31) is obtained by taking $\tilde{\beta} = \delta$,

$$\begin{aligned} s_\partial(\delta;\nu) =& \mathcal{F}^1_\partial(0,0) + \mathcal{O}_1(\delta) = \frac{s(\delta;\nu)}{\nu}\left(1 + \tilde{G}_s(\delta, \nu, z_{\text{in}}(\nu))\right) + \mathcal{O}_1(\delta) \\ z_\partial(\delta;\nu) =& \mathcal{F}^2_\partial(0,0) + \mathcal{O}_2(\delta) = z'_{\text{in}}(\nu) + \frac{2\delta}{m_0}\tilde{G}_z(\delta, \nu, z_{\text{in}}(\nu)) + \mathcal{O}_1(\delta). \end{aligned}$$

Using (B.12), this leads to

$$\begin{aligned} s_\partial(\delta;\nu) =& 1 + \mathcal{O}_1(\delta) \\ z_\partial(\delta;\nu) =& z'_{\text{in}}(\nu) + \mathcal{O}_1(\delta), \end{aligned}$$

which implies that the curve $\tilde{\gamma}_1$ in (5.30) is transverse with $W^u_\mu(S^-) \cap \tilde{\Sigma}^+_1$, which is given by $\{s = 0\}$, at the point $\lim_{\nu\to 0} \tilde{\gamma}_1(\nu) = (0, \delta, z_{\text{in}}(0))$.

The proof of Lemma 5.8 is completely analogous, using instead the equations of the orbits associated to system (5.6) obtained in Proposition 5.1, given by

$$\begin{aligned} \frac{d\tilde{\iota}}{ds} &= -\frac{\tilde{\iota}}{s}\left(1 + J_{\tilde{\iota}}(s, \tilde{\iota}, w)\right) \\ \frac{dw}{ds} &= \frac{2}{m_0}\tilde{\iota} J_w(s, \tilde{\iota}, w) \end{aligned}$$

where

$$J_{\tilde{\iota}}(s, \tilde{\iota}, w) = \mathcal{O}_1(s, \tilde{\iota}), \quad J_w(s, \tilde{\iota}, w) = -4\lambda(\mu, h)s + s\mathcal{O}_1(s, \tilde{\iota}).$$

C Proof of Lemma 5.6

To prove Lemma 5.6, note that, for $s = 0$, the equations of motion (5.32) are reduced to

$$\begin{aligned} \tilde{\beta}' &= \frac{m_0}{2}\sin\tilde{\beta} \\ z' &= 0, \end{aligned}$$

which describe the heteroclinic connections between $W^u_\mu(S^-)$ and $W^s_\mu(S^+)$ given in Lemma 2.2. They can be parameterized as

$$\begin{aligned} \tilde{\beta}_h(\tau) &= 2\tan^{-1}\left(e^{\frac{m_0}{2}\tau}\tan\left(\frac{\delta}{2}\right)\right) \\ z_h(\tau) &= z, \end{aligned} \tag{C.1}$$

by fixing the initial condition at $\tau = 0$ as $(0, \delta, z) \in \tilde{\Sigma}_1^+$.

Denote by $\tau_h > 0$ the time such that $\tilde{\beta}_h(\tau_h) = \pi - \delta$. Then, if we consider a point $(s, \delta, z) \in \tilde{\Sigma}_1^+$, one can see that there exist a time $\bar{\tau}(s, z)$ so that the orbit hits $\tilde{\Sigma}_2^-$ by an Implicit Function Theorem argument. Indeed, define,

$$\mathcal{F}(\tau, s; z) = \Phi_{\tilde{\beta}}(\tau; s, \delta, z) - (\pi - \delta),$$

where $\Phi_{\tilde{\beta}}$ refers to the $\tilde{\beta}$-component of the flow Φ, then

- $\mathcal{F}(\tau_h, 0; z) = 0$.
- $\frac{d}{d\tau}\mathcal{F}(\tau_h, 0; z) = \frac{m_0}{2}\sin(\tilde{\beta}_h(\tau_h)) = \frac{m_0}{2}\sin\delta \neq 0$.

Therefore, we can apply the Implicit Function Theorem to ensure that for $0 < s < \delta$ and $\delta > 0$ small enough, there exists a unique $\bar{\tau} = \bar{\tau}(s; z)$ with $\bar{\tau}(0; z) = \tau_h$ satisfying

$$\mathcal{F}(\bar{\tau}(s; z), s; z) = 0.$$

As a result, for any initial condition $(s, \delta, z) \in \tilde{\Sigma}_1^+$, there exists a time $\bar{\tau}(s; z)$ satisfying $\Phi_{\tilde{\beta}}(\bar{\tau}(s; z); s, \delta, z) = \pi - \delta$ for the flow Φ. Therefore, we define the map $\mathcal{T}_{1,2}^+$ in (5.33) as

$$\mathcal{T}_{1,2}^+(s, \delta, z) = \Phi(\bar{\tau}(s; z); s, \delta, z) = (\Phi_s(\bar{\tau}(s; z); s, \delta, z), \delta, \Phi_z(\bar{\tau}(s; z); s, \delta, z)) \in \tilde{\Sigma}_2^-. \tag{C.2}$$

The remaining task is then to compute the expansion with respect to s of the map. Equivalently, s-expansions of $\bar{\tau}$ and the flow components Φ_s and Φ_z.

Perform a Taylor expansion at $s = 0$ of the flow Φ with initial condition at $\tilde{\Sigma}_1^+$, we obtain

$$\begin{aligned} \Phi_s(\tau; s, \delta, z) &= s \cdot \xi_{s,0}^1(\tau; z) + \mathcal{O}_2(s) \\ \Phi_{\tilde{\beta}}(\tau; s, \delta, z) &= \tilde{\beta}_h(\tau) + s \cdot \xi_{s,0}^2(\tau; z) + \mathcal{O}_2(s) \\ \Phi_z(\tau; s, \delta, z) &= z + s \cdot \xi_{s,0}^3(\tau; z) + \mathcal{O}_2(s), \end{aligned} \tag{C.3}$$

with

$$\xi_{s,0}^1(\tau; z) = \partial_s \Phi_s(\tau; 0, \delta, z), \quad \xi_{s,0}^2(\tau; z) = \partial_s \Phi_{\tilde{\beta}}(\tau; 0, \delta, z), \quad \xi_{s,0}^3(\tau, z) = \partial_s \Phi_z(\tau; 0, \delta, z).$$

To obtain $\xi_{s,0}^1(\tau; z)$, we expand each side of the equation

$$\frac{d}{d\tau}\Phi_s(\tau; s, \delta, z) = F_s(\Phi_s(\tau; s, \delta, z), \Phi_{\tilde{\beta}}(\tau; s, \delta, z), \Phi_z(\tau; s, \delta, z)),$$

where F_s corresponds to the s-component of the vector field associated to motion (5.32), around the unperturbed solution $(0, \tilde{\beta}_h(\tau), z)$. We obtain

$$\begin{aligned} \frac{d}{d\tau}\Phi_s(\tau; s, \delta, z) &= s \cdot \xi_{s,0}^1{}'(\tau, z) + \mathcal{O}_2(s) \\ F_s(\Phi_s(\tau; s, \delta, z), \Phi_{\tilde{\beta}}(\tau; s, \delta, z), \Phi_z(\tau; s, \delta, z)) &= -\frac{m_0}{2} s \cdot \xi_{s,0}^1(\tau; z)\cos(\tilde{\beta}_h(\tau)) + \mathcal{O}_2(s). \end{aligned}$$

This gives the equation for $\xi_{s,0}^1$,

$$\begin{cases} \xi_{s,0}^1{}'(\tau; z) = -\frac{m_0}{2}\xi_{s,0}^1(\tau; z)\cos(\tilde{\beta}_h(\tau)) \\ \xi_{s,0}^1(0; z) = 1. \end{cases}$$

whose solution is

$$\xi_{s,0}^1(\tau; z) = \xi_{s,0}^1(\tau) = \cosh\left(\frac{m_0}{2}\tau\right) - \cos\delta\sinh\left(\frac{m_0}{2}\tau\right). \tag{C.4}$$

Following a similar procedure, we obtain

$$\xi^2_{s,0}(\tau; z) = 0, \quad \xi^3_{s,0}(\tau; z) = 0,$$

leading to the following result

$$\begin{aligned}
\Phi_s(\bar{\tau}(s; z); s, \delta, z) &= s \cdot \xi^1_{s,0}(\bar{\tau}(s; z); z) + \mathcal{O}_2(s) \\
\Phi_{\tilde{\beta}}(\tau; s, \delta, z) &= \tilde{\beta}_h(\tau) + \mathcal{O}_2(s) \\
\Phi_z(\bar{\tau}(s; z); s, \delta, z) &= z + \mathcal{O}_2(s).
\end{aligned}$$

Then, by the second equation the function $\bar{\tau}$ obtained by the Implicit Function Theorem is of the form

$$\bar{\tau}(s; z) = \tau_h + \mathcal{O}_2(s).$$

Hence, using the fact that $\tilde{\beta}_h(\tau_h) = \pi - \delta$, the expression of $\tilde{\beta}_h$ in (C.1) and the definition of $\xi^1_{s,0}(\tau; z)$ in (C.4), one can see that $\xi^1_{s,0}(\tau_h; z) = 1$ and therefore

$$\xi^1_{s,0}(\bar{\tau}(s; z); z) = 1 + \mathcal{O}_2(s).$$

This leads to

$$\Phi(\bar{\tau}(s; z); s, \delta, z) = (s + \mathcal{O}_2(s), \pi - \delta, z + \mathcal{O}_2(s)) \in \tilde{\Sigma}^-_2.$$

Taking the curve $\tilde{\gamma}_1(\nu) = (s_1(\delta; \nu), \delta, z_1(\delta; \nu))$ in (5.30) as the initial condition yields the curve

$$\tilde{\gamma}_2(\nu) = \mathcal{T}^+_{1,2}(s_1(\delta; \nu), \delta, z_1(\delta; \nu))$$

satisfying (5.34), where $\mathcal{T}^+_{1,2}$ is defined in (C.2).

The regularity of the map $\mathcal{T}^+_{1,2}$ with respect to s (consequence of (C.3)) and the curve $\tilde{\gamma}_1(\nu)$ (consequence of Lemma 5.5) allows us to compute the tangent vector $\tilde{\gamma}'_2(\nu)$ directly from (5.34), resulting in (5.35) and completing the proof.

D Proof of Lemma 6.3

This section is devoted to express the transition map provided by Theorem 5.3 in polar coordinates centered at P_1. To this end, we follow the notation of Sections 5 and 6, and we recall the following facts.

- The parameterizations $\chi_\pm$ of the invariant manifolds $W^s_\mu(S^-_{y_0})$ and $W^u_\mu(S^+_{x_0})$ provided in (5.15), (5.24) (see also (5.19)) satisfies the following properties

$$\begin{aligned}
\partial_z \chi_-(\delta, z) &= \partial_w \chi_+(\delta, w) = 1 + \mathcal{O}_4(\delta) \\
\partial_y \chi_-^{-1}(\delta, y) &= \partial_x \chi_+^{-1}(\delta, x) = 1 + \mathcal{O}_4(\delta),
\end{aligned} \tag{D.1}$$

  where $\chi_\pm^{-1}$ denote the inverse of the functions $z \mapsto \chi_\pm(\delta, z)$.

- By the definition and properties of ψ_- and ψ_+ in (5.12) and (5.22), the derivatives $\partial_y \psi_-(\delta, y)$ and $\partial_x \psi_+(\delta, x)$ satisfy

$$\partial_y \psi_-(\delta, y) = \partial_x \psi_+(\delta, x) = \mathcal{O}_2(\delta). \tag{D.2}$$

To prove Lemma 6.3, we first apply the changes of coordinates ψ and $\tilde{\psi}$ introduced in (2.12) and (2.14) respectively to translate the curves $\gamma_\infty^{u,<}$ and $\gamma_{S^-}^{s,<}$ defined in (6.11) into coordinates (θ, α).

This leads to curves parameterized as graphs as

$$\overline{\gamma}_\infty^{u,<} = \left\{ \left(\theta, \alpha_\infty^{u,<}(\theta)\right) : \theta \in B_\varepsilon(\theta_>) \right\}, \quad \overline{\gamma}_{S^-}^{s,<} = \left\{ \left(\theta, \alpha_{S^-}^{s,<}(\theta)\right) : \theta \in B_\varepsilon(\theta_>) \right\} \tag{D.3}$$

such that

$$\alpha_\infty^{u,<}(\theta) = -\arccos\left(\frac{\Theta_\infty^u(\theta) + \mu\delta^2\cos\theta - \delta^4}{\delta\sqrt{2(1-\mu)+\rho(\theta)}}\right), \quad \alpha_{S^-}^{s,<}(\theta) = -\arccos\left(\frac{\Theta_{S^-}^s(\theta) + \mu\delta^2\cos\theta - \delta^4}{\delta\sqrt{2(1-\mu)+\rho(\theta)}}\right).$$

To obtain approximations of $\alpha_\infty^{u,<}(\theta)$ and $\alpha_{S^-}^{s,<}(\theta)$ (and their derivatives), we recall the results from Proposition 2.6 and Proposition 2.7, which give us estimates for both angular momenta $\Theta_\infty^u(\theta)$ and $\Theta_{S^-}^s(\theta)$ respectively. In fact, $\rho(\theta) = \rho(\delta, \theta)$ (defined in (2.15)) admits the following approximation

$$\rho(\theta) = \mathcal{O}_1(\delta^6, \mu\delta^2), \quad \partial_\theta \rho(\theta) = \mathcal{O}_1(\mu\delta^4), \tag{D.4}$$

since $h = -\hat{\Theta}_0(\mu) = \mathcal{O}_1(\mu)$. Note that

- We take the $-$ sign for both $\alpha_\infty^{u,<}(\theta)$ and $\alpha_{S^-}^{s,<}(\theta)$ since we are close to the circle S^- (which is characterized by $\alpha = -\frac{\pi}{2}$ as shown in (2.20)).
- For $\theta = \theta_< \in B_\varepsilon(\theta_>)$ it is satisfied that $\alpha_\infty^{u,<}(\theta_<) = \alpha_{S^-}^{s,<}(\theta_<)$.

For further computations it will be necessary to approximate both derivatives $\alpha_\infty^{u,<\prime}(\theta_<)$ and $\alpha_{S^-}^{s,<\prime}(\theta_<)$, which are given by

$$\alpha_\infty^{u,<\prime}(\theta_<) = \frac{\Theta_\infty^u{}'(\theta_<)}{\sqrt{2(1-\mu)}\delta} + \mathcal{O}_1(\mu\delta), \quad \alpha_{S^-}^{s,<\prime}(\theta_<) = \frac{\Theta_{S^-}^s{}'(\theta_<)}{\sqrt{2(1-\mu)}\delta} + \mathcal{O}_1(\mu\delta) \tag{D.5}$$

where we have considered $0 < \mu \ll \delta$ small enough, and we have used the approximations in (D.4) and the fact that both $\Theta_\infty^u{}'(\theta_<)$ and $\Theta_{S^-}^s{}'(\theta_<)$ are of order $\mathcal{O}_1(\mu)$ (see Propositions 2.6 and 2.7).

Once in coordinates (θ, α), we apply the diffeomorphism Γ_- from Proposition 5.1 to express the curves $\overline{\gamma}_\infty^{u,<}$ and $\overline{\gamma}_{S^-}^{s,<}$ in (D.3) in coordinates $(\tilde{\beta}, z)$, yielding the following parameterizations in terms of $\theta \in B_\varepsilon(\theta_<)$

$$\tilde{\gamma}_\infty^{u,<}(\theta) = \left(\tilde{\beta}_\infty^{u,<}(\theta), z_\infty^{u,<}(\theta)\right), \quad \tilde{\gamma}_{S^-}^{s,<}(\theta) = \left(\tilde{\beta}_{S^-}^{s,<}(\theta), z_{S^-}^{s,<}(\theta)\right) \tag{D.6}$$

such that

$$\begin{aligned} \tilde{\beta}_\infty^{u,<}(\theta) &= \alpha_\infty^{u,<}(\theta) + \frac{\pi}{2} - \psi_-\left(\delta, y_\infty^{u,<}(\theta)\right), \quad z_\infty^{u,<}(\theta) = \chi_-^{-1}\left(\delta, y_\infty^{u,<}(\theta)\right), \\ \tilde{\beta}_{S^-}^{s,<}(\theta) &= \alpha_{S^-}^{s,<}(\theta) + \frac{\pi}{2} - \psi_-\left(\delta, y_{S^-}^{s,<}(\theta)\right), \quad z_{S^-}^{s,<}(\theta) = \chi_-^{-1}\left(\delta, y_{S^-}^{s,<}(\theta)\right), \end{aligned} \tag{D.7}$$

with

$$y_\infty^{u,<}(\theta) = \theta - 2\left(\alpha_\infty^{u,<}(\theta) + \frac{\pi}{2}\right), \quad y_{S^-}^{s,<}(\theta) = \theta - 2\left(\alpha_{S^-}^{s,<}(\theta) + \frac{\pi}{2}\right).$$

The goal is to express the curve $\tilde{\gamma}_\infty^{u,<}$ in (D.6) as a graph in terms of $\tilde{\beta}$ to apply Theorem 5.3.

To this end, we make use of the following fact: since the curve $\tilde{\gamma}_{S^-}^{s,<}(\theta) \subset W_\mu^s(S^-)$ for all $\theta \in B_\varepsilon(\theta_>)$, it satisfies that $\tilde{\beta}_{S^-}^{s,<}(\theta) = 0$ as stated in equation (5.4). Therefore, we can rewrite $\tilde{\beta}_\infty^{u,<}(\theta)$ in (D.7) as

$$\tilde{\beta}_\infty^{u,<}(\theta) = \tilde{\beta}_\infty^{u,<}(\theta) - \tilde{\beta}_{S^-}^{s,<}(\theta) = \alpha_\infty^{u,<}(\theta) - \alpha_{S^-}^{s,<}(\theta) - \left(\psi_-\left(\delta, y_\infty^{u,<}(\theta)\right) - \psi_-\left(\delta, y_{S^-}^{s,<}(\theta)\right)\right) \tag{D.8}$$

and therefore

$$\tilde{\beta}^{u,<\prime}_{\infty}(\theta_<) = \left(\alpha^{u,<\prime}_{\infty}(\theta_<) - \alpha^{s,<\prime}_{S^-}(\theta_<)\right)\left(1 - 2\partial_y\psi_-(\delta, y_*)\right),$$

where $y_* = y^{u,<}_{\infty}(\theta_<) = y^{s,<}_{S^-}(\theta_<)$.

Hence, using (D.2) and (D.5), $\tilde{\beta}^{u,<\prime}_{\infty}(\theta_<)$ admits the following approximation

$$\tilde{\beta}^{u,<\prime}_{\infty}(\theta_<) = \frac{d_-'(\theta_<)}{\sqrt{2(1-\mu)\delta}} + \mathcal{O}(\mu\delta), \tag{D.9}$$

where $d_-(\theta) = d_-\left(\theta, \hat{\Theta}_0(\mu)\right)$ is defined in (3.8). By (3.9), (3.10) and Lemma 3.2 it satisfies

$$C^{-1}\mu \leq |d_-'(\theta_<)| \leq C\mu, \tag{D.10}$$

for some $C > 0$ independent of μ and δ. This implies that, for $0 < \mu \ll \delta$ small enough, $\tilde{\beta}^{u,<\prime}_{\infty}(\theta_<) \neq 0$.

Hence, in a sufficiently small neighborhood of $\theta_<$ (for instance, we can consider $B_\varepsilon(\theta_>)$), the Inverse Function Theorem defines an inverse for $\tilde{\beta}^{u,<}_{\infty}(\theta)$, which we denote $\theta^{u,<}_{\infty}(\tilde{\beta})$, in an interval $\tilde{B}_u$ with $0 \in \tilde{B}_u$ (since $\tilde{\beta}^{u,<}_{\infty}(\theta_<) = \tilde{\beta}^{s,<}_{S^-}(\theta_<) = 0$) and

$$\theta^{u,<}_{\infty}(0) = \theta_<, \quad \theta^{u,<\prime}_{\infty}(0) = \frac{1}{\tilde{\beta}^{u,<\prime}_{\infty}(\theta_<)}. \tag{D.11}$$

Therefore, we can express the curve $\tilde{\gamma}^{u,<}_{\infty}$ in (D.6) as a graph in terms of $\tilde{\beta}$ of the form

$$\tilde{\gamma}^{u,<}_{\infty} = \left\{ \left(\tilde{\beta}, z^{u,<}_{\infty}(\tilde{\beta})\right) : \tilde{\beta} \in \tilde{B}_u \right\}, \tag{D.12}$$

with $z^{u,<}_{\infty}(\tilde{\beta}) = z^{u,<}_{\infty} \circ \theta^{u,<}_{\infty}(\tilde{\beta})$ satisfying

$$\begin{aligned}
z^{u,<\prime}_{\infty}(0) =& z^{u,<\prime}_{\infty}(\theta_<) \cdot \theta^{u,<\prime}_{\infty}(0) = \partial_y\chi_-^{-1}(\delta, y_*) \cdot y^{u,<\prime}_{\infty}(\theta_<) \cdot \theta^{u,<\prime}_{\infty}(0) \\
=& \left(1 + \mathcal{O}_4(\delta)\right)\left(1 - 2\alpha^{u,<\prime}_{\infty}(\theta_<)\right) \cdot \left(\tilde{\beta}^{u,<\prime}_{\infty}(\theta_<)\right)^{-1} \\
=& \left(1 + \mathcal{O}_4(\delta)\right)\left(\sqrt{2(1-\mu)}\frac{\delta}{d_-{}'(\theta_<)} - 2\frac{\Theta^u_{\infty}{}'(\theta_<)}{d_-{}'(\theta_<)} + \mathcal{O}_2(\delta)\right) \\
=& \left(1 + \mathcal{O}_4(\delta)\right)\sqrt{2(1-\mu)}\frac{\delta}{d_-{}'(\theta_<)},
\end{aligned} \tag{D.13}$$

with $d_-'(\theta_<)$ satisfying (D.10) and $y_* = y^{u,<}_{\infty}(\theta_<) = y^{s,<}_{S^-}(\theta_<)$. The result follows from (D.1), (D.5), (D.7), (D.9) and (D.11).

Denoting $\tilde{\beta} = \nu$, we are now ready to apply the transition map $\tilde{f}$ from Theorem 5.3 to the curve (D.12), which leads the following curve

$$\tilde{\gamma}^{u,>}_{\infty}(\nu) = (\tilde{\imath}^{u,>}_{\infty}(\nu), w^{u,>}_{\infty}(\nu)) = \left(-\nu + \mathcal{O}_1(\delta\nu), z^{u,<}_{\infty}(\nu) + \mathcal{O}_1(\delta^2\nu, \nu^2)\right),$$

for $\nu \in (0, \nu_0)$ with $\nu_0 > 0$ small enough, which satisfies

$$\tilde{\gamma}^{u,>\prime}_{\infty}(0) = (\tilde{\imath}^{u,>\prime}_{\infty}(0), w^{u,>\prime}_{\infty}(0)) = \left(-1 + \mathcal{O}_1(\delta), z^{u,<\prime}_{\infty}(0) + \mathcal{O}_1(\delta)\right). \tag{D.14}$$

This curve is expressed in coordinates $(\tilde{\imath}, w)$. Now, we apply the inverse of the diffeomorphism Γ_+ from Proposition 5.1 to express the curve $\tilde{\gamma}^{u,>}_{\infty}$ in coordinates (θ, α), which yields to the following parameterization

$$\overline{\gamma}^{u,>}_{\infty}(\nu) = \left(\theta^{u,>}_{\infty}(\nu), \alpha^{u,>}_{\infty}(\nu)\right) \tag{D.15}$$

such that

$$\theta_\infty^{u,>}(\nu) = \chi_+(\delta, w_\infty^{u,>}(\nu)) - 2\left(\tilde{\iota}_\infty^{u,>}(\nu) - \psi_+(\delta, \chi_+(\delta, w_\infty^{u,>}(\nu)))\right)$$
$$\alpha_\infty^{u,>}(\nu) = \tilde{\iota}_\infty^{u,>}(\nu) + \psi_+(\delta, \chi_+(\delta, w_\infty^{u,>}(\nu)) + \frac{\pi}{2}$$

where $\theta_\infty^{u,>}(\nu)$ satisfies $\theta_\infty^{u,>}(0) = \theta_> = \theta_< + \mathcal{O}_2(\delta)$ and

$$\begin{aligned}\theta_\infty^{u,>\prime}(0) &= (1 + 2\partial_x\psi_+(\delta, x_*)) \cdot \partial_w\chi_+(\delta, w_*) \cdot w_\infty^{u,>\prime}(0) - 2\tilde{\iota}_\infty^{u,>\prime}(0) \\ &= (1 + 2\partial_x\psi_+(\delta, x_*)) \cdot (1 + \mathcal{O}_4(\delta)) \cdot \left(z_\infty^{u,<\prime}(0) + \mathcal{O}_1(\delta)\right) + 2 + \mathcal{O}_1(\delta) \\ &= z_\infty^{u,<\prime}(0) + 2 + \mathcal{O}_1(\delta),\end{aligned}$$

where $x_* = \chi_+(\delta, w_*)$ and $w_* = w_\infty^{u,>}(0)$. This expression is obtained from (D.1), (D.2) and (D.14). It follows from (D.13) that, taking $0 < \mu \ll \delta$ small enough, $\theta_\infty^{u,>\prime}(0) \neq 0$.

Therefore, the Inverse Function Theorem ensures the existence of an inverse $\nu_\infty^{u,>}(\theta)$ for $\theta \in B_\varepsilon(\theta_>)$ satisfying

$$\nu_\infty^{u,>}(\theta_>) = 0, \quad \nu_\infty^{u,>\prime}(\theta_>) = \frac{1}{\theta_\infty^{u,>\prime}(0)}, \tag{D.16}$$

allowing us to express the curve $\overline{\gamma}_\infty^{u,>}(\nu)$ in (D.15) as a graph of the form

$$\overline{\gamma}_\infty^{u,>} = \left\{ \left(\theta, \alpha_\infty^{u,>}(\theta)\right) : \theta \in B_\varepsilon(\theta_>)\right\}, \tag{D.17}$$

such that

$$\alpha_\infty^{u,>}(\theta) = \alpha_\infty^{u,>} \circ \nu_\infty^{u,>}(\theta) = \tilde{\iota}_\infty^{u,>} \circ \nu_\infty^{u,>}(\theta) + \psi_+(\delta, \chi_+(\delta, w_\infty^{u,>} \circ \nu_\infty^{u,>}(\theta))) + \frac{\pi}{2}. \tag{D.18}$$

In order to get a proper approximation of both $\alpha_\infty^{u,>}(\theta_>)$ and $\alpha_\infty^{u,>\prime}(\theta_>)$, we use an analogous argument to (D.8), using in this case the curve $\gamma_{S^+}^{u,>}$ in (6.11) as a reference. To this end, following the changes (2.12), (2.14) and the diffeomorphism Γ_+ in Proposition 5.1, we obtain the curve $\gamma_{S^+}^{u,>}$ expressed in coordinates $(\tilde{\iota}, w)$, parameterized by $\theta \in B_\varepsilon(\theta_>)$ as follows

$$\tilde{\gamma}_{S^+}^{u,>} = \left\{ \left(\tilde{\iota}_{S^+}^{u,>}(\theta), w_{S^+}^{u,>}(\theta)\right) : \theta \in B_\varepsilon(\theta_>)\right\},$$

such that

$$\tilde{\iota}_{S^+}^{u,>}(\theta) = \alpha_{S^+}^{u,>}(\theta) - \frac{\pi}{2} - \psi_+\left(\delta, \chi_+(\delta, w_{S^+}^{u,>}(\theta))\right), \quad w_{S^+}^{u,>}(\theta) = \chi_+^{-1}(\delta, x_{S^+}^{u,>}(\theta)) \tag{D.19}$$

with

$$\alpha_{S^+}^{u,>}(\theta) = \arccos\left(\frac{\Theta_{S^+}^u(\theta) + \mu\delta^2\cos\theta - \delta^4}{\delta\sqrt{2(1-\mu) + \rho(\theta)}}\right), \quad x_{S^+}^{u,>}(\theta) = \theta - 2\left(\alpha_{S^+}^{u,>}(\theta) - \frac{\pi}{2}\right). \tag{D.20}$$

We recall that $w_* = w_\infty^{u,>}(0) = w_{S^+}^{u,>}(\theta_>)$ corresponds to the w-component of the intersection $p_>^*$ in (6.12).

Since $\tilde{\gamma}_{S^+}^{u,>} \subset W_\mu^u(S^+)$, by (5.7) the component $\tilde{\iota}_{S^+}^{u,>}(\theta) = 0$, yielding the following identity

$$\alpha_{S^+}^{u,>}(\theta) = \frac{\pi}{2} + \psi_+\left(\delta, \chi_+(\delta, w_{S^+}^{u,>}(\theta))\right).$$

Then, we can rewrite $\alpha_\infty^{u,>}(\theta)$ in (D.18) as

$$\alpha_\infty^{u,>}(\theta) = \alpha_{S^+}^{u,>}(\theta) + \tilde{\iota}_\infty^{u,>} \circ \nu_\infty^{u,>}(\theta) + \left(\psi_+(\delta, \chi_+(\delta, w_\infty^{u,>} \circ \nu_\infty^{u,>}(\theta))) - \psi_+\left(\delta, \chi_+(\delta, w_{S^+}^{u,>}(\theta))\right)\right)$$

Moreover, the derivative at $\theta_>$ is given by

$$\begin{aligned}\alpha_\infty^{u,>\prime}(\theta_>) =&\alpha_{S+}^{u,>\prime}(\theta_>) + \tilde{\iota}_\infty^{u,>\prime}(0)\cdot \nu_\infty^{u,>\prime}(\theta_>)\\ &+ \left(\partial_x\psi_+(\delta,x_*)\cdot\partial_w\chi_+(\delta,w_*)\right)\left(w_\infty^{u,>\prime}(0)\cdot\nu_\infty^{u,>\prime}(\theta_>) - w_{S+}^{u,>\prime}(\theta_>)\right).\end{aligned}$$

The expressions $\tilde{\iota}_\infty^{u,>\prime}(0)$ and $w_\infty^{u,>\prime}(0)$ from (D.14) and $\nu_\infty^{u,>\prime}(\theta_>)$ from (D.16) allow us to obtain

$$\begin{aligned}\tilde{\iota}_\infty^{u,>\prime}(0)\cdot\nu_\infty^{u,>\prime}(\theta_>) =&(-1+\mathcal{O}_1(\delta))\cdot\frac{1}{\theta_\infty^{u,>\prime}(0)} = (-1+\mathcal{O}_1(\delta))\cdot\frac{1}{z_\infty^{u,<\prime}(0)+2+\mathcal{O}_1(\delta)}\\ =&(-1+\mathcal{O}_1(\delta))\cdot\frac{1}{\frac{\delta\sqrt{2(1-\mu)}}{d_-'(\theta_<)} - 2\frac{\Theta_\infty^{u\prime}(\theta_<)}{d_-'(\theta_<)} + 2 + \mathcal{O}_1(\delta)}\\ =&(-1+\mathcal{O}_1(\delta))\cdot\frac{1}{\frac{\sqrt{2(1-\mu)}\delta}{d_-'(\theta_<)}\left(1 - 2\frac{\Theta_\infty^{u\prime}(\theta_<) - d_-'(\theta_<)}{\delta\sqrt{2(1-\mu)}} + \mathcal{O}_1(\mu)\right)}\\ =&(-1+\mathcal{O}_1(\delta))\left[\frac{d_-{}'(\theta_<)}{\delta\sqrt{2(1-\mu)}}\left(1+\frac{2\Theta_{S-}^{s}{}'(\theta_<)}{\delta\sqrt{2(1-\mu)}} + \mathcal{O}_1(\mu)\right)\right]\\ =&(-1+\mathcal{O}_1(\delta))\left[\frac{d_-{}'(\theta_>)}{\delta\sqrt{2(1-\mu)}}\left(1+\frac{2\Theta_{S-}^{s}{}'(\theta_>)}{\delta\sqrt{2(1-\mu)}} + \mathcal{O}_1(\mu)\right)\right],\\ w_\infty^{u,>\prime}(0)\cdot\nu_\infty^{u,>\prime}(\theta_>) =&\frac{z_\infty^{u,<\prime}(0)+\mathcal{O}_1(\delta)}{z_\infty^{u,<\prime}(0)+2+\mathcal{O}_1(\delta)} = 1 + \mathcal{O}_1\left(\frac{\mu}{\delta}\right).\end{aligned}$$

since $\theta_< = \theta_> + \mathcal{O}_2(\delta)$ so

$$\begin{aligned}d_-'(\theta_<) = d_-'(\theta_> + \mathcal{O}_2(\delta)) = d_-'(\theta_>) + \mathcal{O}_1(\mu\delta^2),\\ \Theta_{S-}^{s}{}'(\theta_<) = \Theta_{S-}^{s}{}'(\theta_> + \mathcal{O}_2(\delta)) = \Theta_{S-}^{s}{}'(\theta_>) + \mathcal{O}_1(\mu\delta^2).\end{aligned}$$

The computation of $w_{S+}^{u,>\prime}(\theta_>)$ is obtained from (D.19) and (D.20) as

$$w_{S+}^{u,>\prime}(\theta_>) = \partial_x\chi_+^{-1}(\delta,x_*)\cdot x_{S+}^{u,>\prime}(\theta_>) = (1+\mathcal{O}_4(\delta))\left(1-2\alpha_{S+}^{u,>\prime}(\theta_>)\right) = 1+\mathcal{O}_1\left(\frac{\mu}{\delta}\right)$$

given $0<\mu\ll\delta$ small enough, where x_* corresponds to the x-component of the intersection $p_>(\mu)$ in (6.12), $\partial_x\chi_+^{-1}(\delta,x_*)$ follows from the estimate (D.1) and $\alpha_{S+}^{u,>\prime}(\theta_>)$ is obtained from (D.20) (and whose approximation is equivalent to those in (D.5)). This result, along with the approximations of $\partial_w\chi_+(\delta,w)$ and $\partial_x\psi_+(\delta,x)$ in (D.1) and (D.2) respectively, give us the following result

$$\alpha_\infty^{u,>\prime}(\theta_>) = -\frac{\Theta_{S+}^{u}{}'(\theta_>) + d_-{}'(\theta_>)}{\sqrt{2(1-\mu)}\delta} + \mathcal{O}(\mu\delta). \tag{D.21}$$

Finally, applying the inverse of the changes (2.12) and (2.14) on the curve $\tilde{\gamma}_\infty^{u,>}$ in (D.17), we obtain the curve $\gamma_\infty^{u,>}$ in coordinates (θ,Θ) as a graph of the form

$$\gamma_\infty^{u,>} = \left\{(\theta,\Theta_\infty^{u,>}(\theta))\colon \theta\in B_\varepsilon(\theta_>)\right\},$$

where

$$\Theta_\infty^{u,>}(\theta) = \delta\sqrt{2(1-\mu)+\rho(\theta)}\cos\left(\alpha_\infty^{u,>}(\theta)\right) + \delta^4 - \mu\delta^2\cos\theta$$

with $\rho(\theta)$ as defined in (D.4). This expression and its derivative can be approximated by (D.21) respectively, yielding (6.13) and completing the proof.

References

[1] V.M. Alekseev. "Final motions in the three-body problem and symbolic dynamics". In: *Uspekhi Mat. Nauk* 36.4(220) (1981), pp. 161–176, 248. ISSN: 0042-1316.

[2] V.M. Alekseev. "Quasirandom dynamical systems". In: *Mat. Zametki* 6 (1969), pp. 489–498. ISSN: 0025-567X.

[3] V.M. Alekseev. "Quasirandom dynamical systems. II. One-dimensional nonlinear vibrations in a periodically perturbed field". In: *Mat. Sb. (N.S.)* 77(119) (1968), pp. 545–601. ISSN: 0368-8666.

[4] V.M. Alekseev. "Quasirandom dynamical systems. III. Quasirandom vibrations of one-dimensional oscillators". In: *Mat. Sb. (N.S.)* 78(120) (1969), pp. 3–50. ISSN: 0368-8666.

[5] V.I. Arnold, V.V. Kozlov, and A.I. Neishtadt. *Mathematical aspects of classical and celestial mechanics*. Third. Vol. 3. Encyclopaedia of Mathematical Sciences. [Dynamical systems. III], Translated from the Russian original by E. Khukhro. Springer-Verlag, Berlin, 2006, pp. xiv+518. ISBN: 978-3-540-28246-4; 3-540-28246-7.

[6] S.V. Bolotin. "Shadowing chains of collision orbits for the elliptic 3-body problem". In: *SPT 2004—Symmetry and perturbation theory*. World Sci. Publ., Hackensack, NJ, 2005, pp. 51–58. ISBN: 981-256-136-6. DOI: 10.1142/9789812702142_0007. URL: https://doi.org/10.1142/9789812702142_0007.

[7] S.V. Bolotin and R.S. MacKay. "Nonplanar second species periodic and chaotic trajectories for the circular restricted three-body problem". In: *Celestial Mech. Dynam. Astronom.* 94.4 (2006), pp. 433–449. ISSN: 0923-2958,1572-9478. DOI: 10.1007/s10569-006-9006-0. URL: https://doi.org/10.1007/s10569-006-9006-0.

[8] S.V. Bolotin and R.S. Mackay. "Periodic and chaotic trajectories of the second species for the n-centre problem". In: *Celestial Mech. Dynam. Astronom.* 77.1 (2000), pp. 49–75. ISSN: 0923-2958,1572-9478. DOI: 10.1023/A:1008393706818. URL: https://doi.org/10.1023/A:1008393706818.

[9] M.J. Capiński, S. Kepley, and J.D. Mireles James. "Computer assisted proofs for transverse collision and near collision orbits in the restricted three body problem". In: *J. Differential Equations* 366 (2023), pp. 132–191. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2023.03.053. URL: https://doi.org/10.1016/j.jde.2023.03.053.

[10] M.J. Capiński and A. Pasiut. *Oscillatory collision approach in the Earth-Moon restricted three body problem*. 2024. DOI: doi.org/10.48550/arXiv.2401.12386. arXiv: 2401.12386. URL: https://doi.org/10.48550/arXiv.2401.12386.

[11] A. Chenciner and J. Llibre. "A note on the existence of invariant punctured tori in the planar circular restricted three-body problem". In: *Ergodic Theory Dynam. Systems* 8* (1988), pp. 63–72. ISSN: 0143-3857,1469-4417. DOI: 10.1017/S0143385700009330. URL: https://doi.org/10.1017/S0143385700009330.

[12] A. Delshams et al. "Global instability in the restricted planar elliptic three body problem". In: *Comm. Math. Phys.* 366.3 (2019), pp. 1173–1228. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-018-3248-z. URL: https://doi.org/10.1007/s00220-018-3248-z.

[13] R.L. Devaney. "Triple collision in the planar isosceles three-body problem". In: *Invent. Math.* 60.3 (1980), pp. 249–267. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01390017. URL: https://doi.org/10.1007/BF01390017.

[14] M.S. ElBialy. "Triple collisions in the isosceles three body problem with small mass ratio". In: *Z. Angew. Math. Phys.* 40.5 (1989), pp. 645–664. ISSN: 0044-2275,1420-9039. DOI: 10.1007/BF00945869. URL: https://doi.org/10.1007/BF00945869.

[15] J. Féjoz. "Averaging the planar three-body problem in the neighborhood of double inner collisions". In: *J. Differential Equations* 175.1 (2001), pp. 175–187. ISSN: 0022-0396,1090-2732. DOI: 10.1006/jdeq.2000.3972. URL: https://doi.org/10.1006/jdeq.2000.3972.

[16] J. Féjoz. "Quasiperiodic motions in the planar three-body problem". In: *J. Differential Equations* 183.2 (2002), pp. 303–341. ISSN: 0022-0396,1090-2732. DOI: 10.1006/jdeq.2001.4117. URL: https://doi.org/10.1006/jdeq.2001.4117.
[17] N. Fenichel. "Asymptotic stability with rate conditions for dynamical systems". In: *Bull. Amer. Math. Soc.* 80 (1974), pp. 346–349. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1974-13498-1. URL: https://doi.org/10.1090/S0002-9904-1974-13498-1.
[18] N. Fenichel. "Asymptotic stability with rate conditions. II". In: *Indiana Univ. Math. J.* 26.1 (1977), pp. 81–93. ISSN: 0022-2518,1943-5258. DOI: 10.1512/iumj.1977.26.26006. URL: https://doi.org/10.1512/iumj.1977.26.26006.
[19] N. Fenichel. "Persistence and smoothness of invariant manifolds for flows". In: *Indiana Univ. Math. J.* 21 (1971/72), pp. 193–226. ISSN: 0022-2518,1943-5258. DOI: 10.1512/iumj.1971.21.21017. URL: https://doi.org/10.1512/iumj.1971.21.21017.
[20] S. Fleischer and A. Knauf. "Improbability of collisions in n-body systems". In: *Archive for Rational Mechanics and Analysis* 234 (2019), pp. 1007–1039.
[21] S. Fleischer and A. Knauf. "Improbability of wandering orbits passing through a sequence of Poincaré surfaces of decreasing size". In: *Archive for Rational Mechanics and Analysis* 231 (2019), pp. 1781–1800.
[22] J. Galante and V. Kaloshin. "Destruction of invariant curves in the restricted circular planar three body problem using comparison of action". In: *Duke Mathematical Journal* 159 (Aug. 2011). DOI: 10.1215/00127094-1415878.
[23] J. Galante and V. Kaloshin. "Destruction of invariant curves using the ordering condition". In: *Preprint* (2010).
[24] J. Galante and V. Kaloshin. "The method of spreading cumulative twist and its application to the restricted circular planar three body problem". In: *Preprint* (2010).
[25] A. Gorodetski and V. Kaloshin. "Hausdorff dimension of oscillatory motions for the restricted three body problems". In: *Preprint.* (2012).
[26] M. Guardia, V. Kaloshin, and J. Zhang. "Asymptotic density of collision orbits in the restricted circular planar 3 body problem". In: *Arch. Ration. Mech. Anal.* 233.2 (2019), pp. 799–836. ISSN: 0003-9527,1432-0673. DOI: 10.1007/s00205-019-01368-7. URL: https://doi.org/10.1007/s00205-019-01368-7.
[27] M. Guardia, P. Martín, and T. M-Seara. "Oscillatory motions for the restricted planar circular three body problem". In: *Invent. Math.* 203.2 (2016), pp. 417–492. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-015-0591-y. URL: https://doi.org/10.1007/s00222-015-0591-y.
[28] M. Guardia et al. "Hyperbolic dynamics and oscillatory motions in the 3 Body Problem". In: (July 2022). DOI: 10.48550/arXiv.2207.14351.
[29] M. Herman. "Some open problems in dynamical systems". In: *Proceedings of the International Congress of Mathematicians. Vol. II.* Sections 1–9, Held in Berlin, August 18–27, 1998, Doc. Math. **1998**, Extra Vol. II. EMS Press, Berlin, 1998, pp. 797–808.
[30] T. Kapela et al. "CAPD::DynSys: a flexible C++ toolbox for rigorous numerical analysis of dynamical systems". In: *Commun. Nonlinear Sci. Numer. Simul.* 101 (2021), Paper No. 105578, 26. ISSN: 1007-5704,1878-7274. DOI: 10.1016/j.cnsns.2020.105578. URL: https://doi.org/10.1016/j.cnsns.2020.105578.
[31] E.A. Lacomba and J. Llibre. "Transversal ejection-collision orbits for the restricted problem and the Hill's problem with applications". In: *J. Differential Equations* 74.1 (1988), pp. 69–85. ISSN: 0022-0396,1090-2732. DOI: 10.1016/0022-0396(88)90019-8. URL: https://doi.org/10.1016/0022-0396(88)90019-8.
[32] E.A. Lacomba and L. Losco. "Triple collisions in the isosceles 3-body problem". In: *Bull. Amer. Math. Soc. (N.S.)* 3.1 (1980), pp. 710–714. ISSN: 0273-0979,1088-9485. DOI: 10.1090/S0273-0979-1980-14802-8. URL: https://doi.org/10.1090/S0273-0979-1980-14802-8.

[33] J. Llibre. "On the restricted three-body problem when the mass parameter is small". In: *Celestial Mech.* 28.1-2 (1982), pp. 83–105. ISSN: 0008-8714. DOI: 10.1007/BF01230662. URL: https://doi.org/10.1007/BF01230662.

[34] J. Llibre and C. Simó. "Oscillatory solutions in the planar restricted three-body problem". In: *Math. Ann.* 248.2 (1980), pp. 153–184. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01421955. URL: https://doi.org/10.1007/BF01421955.

[35] T. M-Seara et al. "Generalized analytical results on n-ejection-collision orbits in the RTBP. Analysis of bifurcations". In: *J. Nonlinear Sci.* 33.1 (2023), Paper No. 17, 53. ISSN: 0938-8974,1432-1467. DOI: 10.1007/s00332-022-09873-y. URL: https://doi.org/10.1007/s00332-022-09873-y.

[36] J.P. Marco and L. Niederman. "Sur la construction des solutions de seconde espèce dans le problème plan restreint des trois corps". In: *Ann. Inst. H. Poincaré Phys. Théor.* 62.3 (1995), pp. 211–249. ISSN: 0246-0211. URL: http://www.numdam.org/item?id=AIHPA_1995__62_3_211_0.

[37] R. McGehee. "A stable manifold theorem for degenerate fixed points with applications to celestial mechanics". In: *J. Differential Equations* 14 (1973), pp. 70–88. ISSN: 0022-0396,1090-2732. DOI: 10.1016/0022-0396(73)90077-6. URL: https://doi.org/10.1016/0022-0396(73)90077-6.

[38] R. McGehee. "Parabolic orbits in the three-body problem". In: *Dynamical Systems*. Ed. by M.M. Peixoto. Academic Press, New York, 1973, pp. 249–254.

[39] R. McGehee. "Triple collision in the collinear three-body problem". In: *Invent. Math.* 27 (1974), pp. 191–227. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01390175. URL: https://doi.org/10.1007/BF01390175.

[40] R. Moeckel. "Chaotic dynamics near triple collision". In: *Arch. Rational Mech. Anal.* 107.1 (1989), pp. 37–69. ISSN: 0003-9527. DOI: 10.1007/BF00251426. URL: https://doi.org/10.1007/BF00251426.

[41] R. Moeckel. "Orbits of the three-body problem which pass infinitely close to triple collision". In: *Amer. J. Math.* 103.6 (1981), pp. 1323–1341. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2374233. URL: https://doi.org/10.2307/2374233.

[42] R. Moeckel. "Symbolic dynamics in the planar three-body problem". In: *Regul. Chaotic Dyn.* 12.5 (2007), pp. 449–475. ISSN: 1560-3547,1468-4845. DOI: 10.1134/S1560354707050012. URL: https://doi.org/10.1134/S1560354707050012.

[43] J. Moser. *Stable and random motions in dynamical systems: With special emphasis on celestial mechanics*. Vol. 1. Princeton university press, 2001.

[44] M. Ollé, O. Rodríguez, and J. Soler. "Analytical and numerical results on families of n-ejection-collision orbits in the RTBP". In: *Commun. Nonlinear Sci. Numer. Simul.* 90 (2020), pp. 105294, 27. ISSN: 1007-5704,1878-7274. DOI: 10.1016/j.cnsns.2020.105294. URL: https://doi.org/10.1016/j.cnsns.2020.105294.

[45] M. Ollé, Ò. Rodríguez, and J. Soler. "Ejection-collision orbits in the RTBP". In: *Commun. Nonlinear Sci. Numer. Simul.* 55 (2018), pp. 298–315. ISSN: 1007-5704,1878-7274. DOI: 10.1016/j.cnsns.2017.07.013. URL: https://doi.org/10.1016/j.cnsns.2017.07.013.

[46] P. Painlevé. *Leçons, sur la théorie analytique des équations différentielles: professées à Stockholm (septembre, octobre, novembre 1895) sur l'invitation de SM le roi de Suède et de Norwège*. Vol. 11. 9. A. Hermann, 1894.

[47] J. Paradela and S. Terracini. *Oscillatory Motions in the Restricted 3-body Problem: A functional analytic approach*. 2022. DOI: 10.48550/arXiv.2212.05684.

[48] C. Pinyol. "Ejection-collision orbits with the more massive primary in the planar elliptic restricted three body problem". In: *Celestial Mech. Dynam. Astronom.* 61.4 (1995), pp. 315–331. ISSN: 0923-2958,1572-9478. DOI: 10.1007/BF00049513. URL: https://doi.org/10.1007/BF00049513.

[49] D.G. Saari. "Improbability of collisions in Newtonian gravitational systems". In: *Trans. Amer. Math. Soc.* 162 (1971), 267–271, erratum, ibid. **168 (1972), 521**. ISSN: 0002-9947,1088-6850. DOI: 10.2307/1995752. URL: https://doi.org/10.2307/1995752.

[50] D.G. Saari. "Improbability of collisions in Newtonian gravitational systems. II". In: *Trans. Amer. Math. Soc.* 181 (1973), pp. 351–368. ISSN: 0002-9947,1088-6850. DOI: 10.2307/1996638. URL: https://doi.org/10.2307/1996638.

[51] C.L. Siegel. *Vorlesungen iiber Himmelsmechanik*. Springer, Berlin, 1956, pp. 18–178.

[52] C. Simó. "Analysis of triple collision in the isosceles problem". In: *Classical mechanics and dynamical systems (Medford, Mass., 1979)*. Vol. 70. Lect. Notes Pure Appl. Math. Dekker, New York, 1981, pp. 203–224. ISBN: 0-8247-1529-2.

[53] K. Sitnikov. "The existence of oscillatory motions in the three-body problems". In: *Soviet Physics Dokl., Soviet Physics. Doklady. A translation of the physics sections of Doklady Academii Nauk SSSR* 5 (1960), pp. 647–650.

[54] S. Wiggins. *Normally hyperbolic invariant manifolds in dynamical systems*. Vol. 105. Applied Mathematical Sciences. With the assistance of György Haller and Igor Mezić. Springer-Verlag, New York, 1994, pp. x+193. ISBN: 0-387-94205-X. DOI: 10.1007/978-1-4612-4312-0. URL: https://doi.org/10.1007/978-1-4612-4312-0.

[55] L. Zhao. "Quasi-periodic almost-collision orbits in the spatial three-body problem". In: *Comm. Pure Appl. Math.* 68.12 (2015), pp. 2144–2176. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.21539. URL: https://doi.org/10.1002/cpa.21539.