ON SMALL BREATHERS OF NONLINEAR KLEIN-GORDON EQUATIONS VIA EXPONENTIALLY SMALL HOMOCLINIC SPLITTING

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Abstract. Breathers are nontrivial time-periodic and spatially localized solutions of nonlinear dispersive partial differential equations (PDEs). Families of breathers have been found for certain integrable PDEs but are believed to be rare in non-integrable ones such as nonlinear Klein-Gordon equations. In this paper we show that small amplitude breathers of *any* temporal frequency do not exist for semilinear Klein-Gordon equations with generic analytic odd nonlinearities.

A breather with small amplitude exists only when its temporal frequency is close to be resonant with the linear Klein-Gordon dispersion relation. Our main result is that, for such frequencies, we rigorously identify the leading order term in the exponentially small (with respect to the small amplitude) obstruction to the existence of small breathers in terms of the so-called *Stokes constant*, which depends on the nonlinearity analytically, but is independent of the frequency. This gives a rigorous justification of a formal asymptotic argument by Kruskal and Segur [60] in the analysis of small breathers.

We rely on the spatial dynamics approach where breathers can be seen as homoclinic orbits. The birth of such small homoclinics is analyzed via a singular perturbation setting where a Bogdanov-Takens type bifurcation is coupled to infinitely many rapidly oscillatory directions. The leading order term of the exponentially small splitting between the stable/unstable invariant manifolds is obtained through a careful analysis of the analytic continuation of their parameterizations. This requires the study of another limit equation in the complexified evolution variable, the so-called *inner equation*.

Contents

1. Introduction

Breathers are nontrivial time-periodic and spatially localized solutions of nonlinear dispersive partial differential equations (PDEs). This kind of solutions play an important role in physical applications and the interest in their existence or breakdown gives rise to a fundamental problem in the mathematical study of the dynamics of such PDEs.

So far breathers have been constructed mostly for completely integrable PDEs. As far as the authors know, the *sine-Gordon equation*

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = 0, \tag{sG}$$

is one of the first PDEs found to admit a family of breathers (see e. g. [1]), which is given explicitly by

$$u^\omega(x,t) = 4\arctan\left(\frac{m}{\omega}\frac{\sin(\omega t)}{\cosh(mx)}\right), \quad m,\omega>0,\ m^2+\omega^2=1. \tag{1.1}$$

They are viewed as the locked states of a kink and an anti-kink in the integrable theory. Along with spatial and temporal translation, the breathers form a 3-dim surface in the infinite dimensional phase space of (sG).

1.1. Non-existence of small amplitude breathers.

The sine-Gordon equation (sG) is a particular case of the family of *nonlinear Klein-Gordon equations* in one space dimension. In this paper, we study the existence/non-existence of *small* breathers of a class of nonlinear Klein-Gordon equations

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \tag{1.2}$$

where the nonlinearity f satisfies

$$f(u) \text{ is a real-analytic odd function and } f(u) = \mathcal{O}(u^5) \text{ near } 0. \tag{1.3}$$

While their signs are natural restrictions, the coefficients 1 and $\frac{1}{3}$ in the above equation are not. In fact, given any nonlinear Klein-Gordon equation $(\partial_T^2 v - \partial_X^2)v + F(v) = 0$ with a smooth real valued odd function $F(v)$ with $F'(0) > 0$ and $F'''(0) < 0$, it is always possible to rescale $v(X,T) = Au(aX, aT)$ so that $u(x,t)$ satisfies (1.2).

Let $\omega > 0$ denote the temporal frequency of a possible breather $u(x,t)$ of (1.2). A solution $u(x,t)$ of (1.2) is a breather of temporal frequency ω if $u(x,t)$ is $\frac{2\pi}{\omega}$-periodic in t and

$$\lim_{x\to\pm\infty} u(x,\cdot) = 0$$

in some appropriate metric. Due to the Lagrangian structure of (1.2),

$$\mathbf{H} = \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} \left(\frac{1}{2}(\partial_x u)^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}u^2 + \frac{1}{12}u^4 + F(u)\right) dt, \quad \text{where } F(u) = \int_0^u f(s)ds = \mathcal{O}(|u|^6), \tag{1.4}$$

is a constant in x for any $\frac{2\pi}{\omega}$-periodic-in-t solutions of (1.2), which vanishes for any breather of temporal frequency ω.

Any real valued function $\frac{2\pi}{\omega}$-periodic in t can be expressed as a Fourier series

$$u(t) = \sum_{n=-\infty}^{+\infty} \left(-\frac{i}{2}\right) u_n e^{in\omega t}, \quad u_{-n} = -\overline{u_n}, \tag{1.5}$$

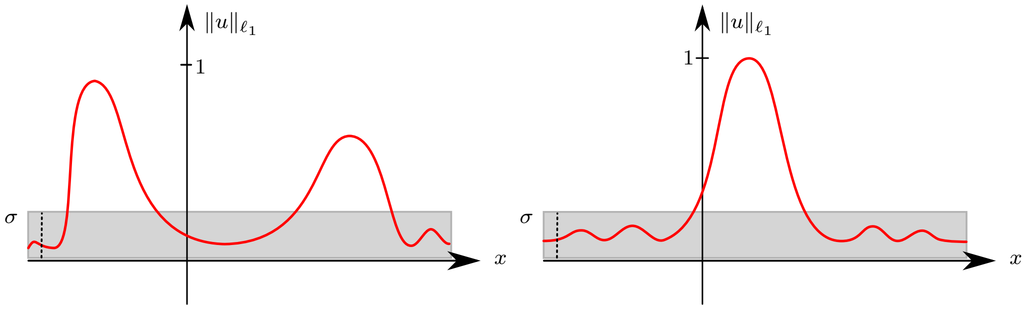

FIGURE 1. Multi-bump (left) and single-bump (right) functions according to Definition 1.1.

where the factor $-\frac{i}{2}$ is purely for the technical convenience when the problem is reduced to functions odd in t represented in Fourier sine series. We denote

$$\Pi_n[u] = u_n = \frac{i\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} u(t)e^{-in\omega t}dt, \qquad \|u\|_{\ell_1} = \sum_{n=-\infty}^{+\infty}|u_n| = \sum_{n=-\infty}^{+\infty}|\Pi_n[u]|. \tag{1.6}$$

Sometimes, with slight abuse of the notation, we also use $\Pi_n[u]$ to denote the mode $-\frac{i}{2}u_n e^{in\omega t}$. Just like $\|\cdot\|_{L_t^\infty}$, the above norm $\|\cdot\|_{\ell_1}$ is invariant under the rescaling in t, but controls the former. As we shall also take the advantage of the conservation (in the variable x) of $\mathbf{H}$, Sobolev norms like $\|\cdot\|_{H_t^k\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)}$ will be involved as well.

To state the first main result of the paper, we introduce the function space $\mathcal{F}_r$ of the nonlinearity f under consideration and the definition of single bump (in x) breathers (see Figure 1).

$$\begin{aligned}\mathcal{F}_r = \Big\{ & f : \{u\in\mathbb{C} : |u|<r\}\to\mathbb{C},\ f \text{ odd and real-analytic},\\ & f(u) = \sum_{k\geq 2} f_k u^{2k+1},\ f_k\in\mathbb{R},\ \|f\|_r := \sum_{k\geq 2}|f_k| r^{2k+1} < \infty\Big\},\end{aligned} \tag{1.7}$$

which is equivalent to the Banach space of real valued sequences $(f_k)_{k=2}^\infty$ with the above weighted ℓ_1 norm.

Definition 1.1. *Let $\sigma\in(0,1)$ and $\omega>0$. We say that a $\frac{2\pi}{\omega}$-periodic-in-t function $u(x,t)$ is σ-multi-bump in x in the ℓ_1 norm if there exist $x_1<x_2<x_3<x_4<x_5$ such that*

$$\max\{\|u(x_{j_1},\cdot)\|_{\ell_1} \mid j_1\in 1,3,5\} \leq \sigma\min\{\|u(x_{j_2},\cdot)\|_{\ell_1} \mid j_2\in 2,4\}.$$

A function $u(x,t)$ is said to be σ-single-bump if it is not σ-multi-bump.

Here x_2 and x_4 can be viewed as two bumps separated by a trough at x_3. The following theorem is the first main result of this paper. It will be a consequence of the more detailed Theorems 1.3 and 1.4 below.

Theorem 1.2. *Fix $r>0$. Then there exists an open and dense set $\mathcal{U}\subset\mathcal{F}_r$ such that for any $f\in\mathcal{U}$ the following holds. For any $\sigma\in(0,1)$, there exists $\rho^*>0$ such that there does not exist any solution $u(x,t)$ to* (1.2) *which:*

(1) *is $\frac{2\pi}{\omega}$-periodic in t for some $\omega>0$,*
(2) *is σ-single-bump in the ℓ_1 norm in the sense of Definition 1.1,*
(3) *satisfies that, as $|x|\to+\infty$,*

$$\|u(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x u(x,\cdot)\|_{L_t^2\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} \to 0, \tag{1.8}$$

(4) *satisfies*

$$\sup_{x\in\mathbb{R}}\|u(x,\cdot)\|_{\ell_1} < \min\{1,\rho^*\omega^{\frac{1}{2}}\}. \tag{1.9}$$

Some comments on Theorem 1.2 are in order.

(1) *About $\mathcal{U}$*: in the more detailed Theorems 1.3 and 1.4 below we give a more precise characterization of the set $\mathcal{U}\subset\mathcal{F}_r$ where Theorem 1.2 holds. Indeed, the complement of $\mathcal{U}$ in $\mathcal{F}_r$ is the preimage of zero of a certain analytic non-trivial function $C_{\text{in}}:\mathcal{F}_r\to\mathbb{C}$.

(2) *Regarding the smallness*: it is worth pointing out that, for a function $\frac{2\pi}{\omega}$-periodic in t, while the norm $\|\cdot\|_{\ell_1}$ is at the same level as $\|\cdot\|_{L^\infty_t}$ in scaling and controls the latter, the smallness in the theorem (see (1.9)) is measured uniformly in ω in terms of $\omega^{-\frac{1}{2}}\|u(x,\cdot)\|_{\ell_1}$. Even though this quantity looks to depend on $\omega \in \mathbb{R}^+$, in scaling it is comparable to $\|\cdot\|_{L^2_t(-\frac{\pi}{\omega},\frac{\pi}{\omega})}$ involved in the conserved (**H**). This norm is weaker than $\|\cdot\|_{H^1_t}$, clearly the above theorem also implies the nonexistence of single-bump breather solutions, small in the energy norm (as in (1.8)).

(3) Theorem 1.2 is only concerned with small *single-bump-in-x* breathers and it does not rule out possible small periodic-in-t solutions which decay as $|x| \to \infty$ but with multiple bumps. Clearly if $u(x,t)$ is σ-multi-bump, then it is also multi-bump for any $\sigma' \in (\sigma, 1)$. Hence the constant ρ_* of the smallness increases as σ increases.

(4) *Breathers with exponentially small tails*: while small single-bump breathers are not expected for (1.2) for most nonlinearities f, a more generic phenomenon is the existence of small breathers with exponentially small (with respect to the amplitude), but non-vanishing, tails for certain values of ω. Such solutions are usually called *generalized breathers* (see [43]). Proposition 1.5 below gives precise estimates on the tails of those generalized breathers.

Before stating the more detailed Theorems 1.3 and 1.4, let us put our result into some context.

Breather type solutions represent important structures in high energy physics *etc.* Moreover, they are of fundamental importance since they serve as building blocks organizing the infinite dimensional dynamics of the underlying evolutionary PDE. In [14], Chen, Liu, and Lu proved the soliton resolution of (sG) using the integrable theory. Namely, in certain weighted Sobolev norm, solutions to (sG) decay (at an algebraic rate in t) to a finite superposition of kinks, anti-kinks, and breathers, where breathers are the only spatially localized class. Therefore breather type structures could play a crucial role in the asymptotic dynamics of the nonlinear Klein-Gordon equations. In particular, unlike relative equilibria such as kinks, standing waves, *etc.*, breathers may be of arbitrarily small amplitudes and energy and thus give rise to obstacles to possible nonlinear dispersive decay or scattering of small energy solutions. (Small amplitude breathers become large in certain weighted norms adopted in some literatures, e.g. [34, 17, 14] *etc.*)

The (sG) breathers (1.1) are obtained based on the complete integrability of (sG). However, for non-integrable Klein-Gordon equations, the existence of (small amplitude) breathers is a completely different problem due to the lack of effective tools such as the inverse scattering method. It is a fundamental question to assert whether the existence of breathers is a special phenomenon due to the integrability or it occurs more generally. In fact, the existence of breathers for non-integrable nonlinear wave equations is expected to be rare[1] (see [20, 62, 12]).

In the seminal work [60] from 1987, Kruskal and Segur used an ingenious formal asymptotic expansion to show the nonexistence of small $\mathcal{O}(\varepsilon)$ amplitude breathers in a class of nonlinear Klein-Gordon equations for ω which is ε^2-close to the resonant frequency $\omega = 1$. The obstacle to solving for breathers is exponentially small in $\varepsilon \ll 1$. For the past more than thirty years, as far as the authors know, no rigorous justification of their leading order exponentially small asymptotics had been given for such nonlinear PDEs. A fundamental part of the proof of Theorem 1.2 is to provide a rigorous proof of Kruskal and Segur's formal argument (for odd analytic nonlinearities) as well as rule out the existence of breathers for other frequencies (either close to other resonances or away from resonances). This is stated more precisely in Theorem 1.3 below.

Among other works on small breathers, in [39], the authors proved that small breathers odd in x do not exist for (1.2) by establishing certain asymptotic stability in the phase space of odd-in-x functions. The oddness is, however, contrary to the well-known examples – the (sG) breathers (1.1) are even in x. Breathers have also been proven not to exist for some generalized KdV equations and the Benjamin-Ono equation, see [51, 52]. In [13], small breathers of a 1-dim nonlinear wave equation in periodic (in x) media were obtained. They play an important role in theoretical scenarios where photonic crystals are used as optical storage. In this model, the periodic media causes the spectra of the linearized problem to be rather different, and this makes the existence of small breathers possible.

For breathers of $\mathcal{O}(1)$ amplitude, in the classical works [20, 21, 12], Denzler and Birnir-McKean-Weinstein studied the rigidity of breathers, namely, the persistence of (infinite subfamilies of) breathers (1.1) when (sG) is perturbed as

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = \varepsilon\Delta(u) + \mathcal{O}(\varepsilon^2),$$

[1] On the contrary breathers are more likely to exist in Hamiltonian systems on lattices, see for instance [46, 45, 72, 56, 57].

where Δ is an analytic function in a small neighborhood of $u = 0$. They proved that breathers corresponding to infinitely many amplitudes $m = \sqrt{1-\omega^2}$ persist only if $\Delta(u)$ results from a trivial rescaling of (sG). In [37], (sG) was also singled out as the only 1-dim nonlinear Klein-Gordon equation admitting breathers in certain form (see also [47]). These rigidity results are consistent with the generic non-existence of breathers. Even though for small amplitude, (1.2) might also be viewed as close to (sG) in the C^∞ class, it does not help much in the analysis of the exponential small obstruction to the existence of breathers (see Theorem 1.3 below) since the nonlinearity of the former is not a perturbation to that of the latter in the analytic function class.

The nonlinear Klein-Gordon equation (1.2) is also quite different from the one studied in [63] (see also [11]). The spatial variable in [63, 11] is taken in $\mathbb{R}^3$ (which gives stronger dispersion than in $\mathbb{R}$) and an extra potential term $V(x)u$ is added. This term creates an isolated oscillatory eigenvalue of the linear problem whose interaction with the continuous spectra leads to slow radiation. In contrast, (1.2) does not contain a potential term and its small breathers have temporal frequencies slightly less than 1, which is the end point of the continuous spectrum of (1.2) linearized at 0. In [58], temporally periodic and spatially decaying solutions were found for the nonlinear Klein-Gordon equation with cubic nonlinearity, i.e. $f(u) = 0$, for $x \in \mathbb{R}^3$. These solutions are close to some steady solutions (not necessarily small) with $\mathcal{O}(1/|x|)$ spatial decay. Such decay, which is too slow for the solutions to be in the energy space, is due to the 3-dim Helmholtz equation, whose solutions would only be in L^∞ and oscillate if $x \in \mathbb{R}^1$. Hence these solutions are more analogous to the breathers with tails constructed in [61] in 1-dim.

1.2. **Main quantitative results: leading order of the exponentially small obstruction.** Theorem 1.2 is a consequence of the more detailed Theorems 1.3 and 1.4 below. In seeking small breathers, which are conceptually born from the end point of the continuous spectra of the linear Klein-Gordon equation, it is essential that the temporal frequency ω is close to resonant. Hence, to state the more detailed theorem, we divide $\omega \in \mathbb{R}^+$ into two primary classes

$$I_k(\varepsilon_0) = \left[\sqrt{\frac{1}{k(k+\varepsilon_0^2)}}, \frac{1}{k}\right), \; k \in \mathbb{N}, \qquad \text{and} \qquad J_k(\varepsilon_0) = \left[\frac{1}{k+1}, \sqrt{\frac{1}{k(k+\varepsilon_0^2)}}\right), \; k \in \mathbb{N}\cup\{0\}, \tag{1.10}$$

where $0 < \varepsilon_0 \le 1/2$ is a parameter to be determined later. Note $J_0(\varepsilon_0) = [1,\infty)$ and $(0,\infty) = (\cup_{k\in\mathbb{N}} I_k) \bigcup (\cup_{k\ge 0} J_k)$. We shall comment more on these sets in the context of spatial dynamics in Section 1.4.

Theorem 1.3. *Fix $r > 0$ and consider $f \in \mathcal{F}_r$ (see (1.7)), then the following statements hold.*

(1) *There exists $\rho_1^* > 0$ such that for any $\varepsilon_0 \in (0, 1/2]$, $\omega \in J_k(\varepsilon_0)$, $k \in \mathbb{N}\cup\{0\}$, if $u(x,t)$ is a $\frac{2\pi}{\omega}$-periodic-in-t solution to (1.2) satisfying*

$$\|u(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x u(x,\cdot)\|_{L_t^2\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} \to 0, \tag{1.11}$$

as $x \to +\infty$ or $-\infty$, then

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell_1} \ge \rho_1^* \min\{1,\, \varepsilon_0 \omega^{\frac{1}{2}}\}. \tag{1.12}$$

(2) *There exists $C_{\rm in} \in \mathbb{C}$ and $\rho_2^* > 0$ depending on f such that, for any $y_0 > 0$, there exist $\varepsilon_0, M > 0$ such that for any*

$$\omega = \sqrt{\frac{1}{k(k+\varepsilon^2)}} \in I_k(\varepsilon_0), \;\; \forall\, k \ge 1, \tag{1.13}$$

there exist unique $\frac{2\pi}{k\omega}$-periodic and odd in t solutions $u_{\rm wk}^\star(x,t)$, $\star = s, u$, to (1.2), only containing Fourier modes $n \in k\mathbb{Z}$ with odd $\frac{n}{k}$ in (1.5), such that

(a) *For $x \ge -\frac{y_0}{\varepsilon\sqrt{k}\omega}$ for $\star = s$ and $x \le \frac{y_0}{\varepsilon\sqrt{k}\omega}$ for $\star = u$, they can be approximated as*

$$\left\| \left(1 - \frac{1}{(k\omega)^2}\partial_t^2\right) \left(\begin{pmatrix} u_{\rm wk}^\star(x,t) \\ \frac{\partial_x u_{\rm wk}^\star(x,t)}{\varepsilon\sqrt{k}\omega} \end{pmatrix} - \varepsilon\sqrt{k}\omega \begin{pmatrix} v^h(\varepsilon\sqrt{k}\omega x) \\ (v^h)'(\varepsilon\sqrt{k}\omega x) \end{pmatrix} \sin k\omega t \right) \right\|_{\ell_1} \le M k^{-\frac{3}{2}} \varepsilon^3 v^h(\varepsilon\sqrt{k}\omega x), \tag{1.14}$$

where $v^h(y) = \frac{2\sqrt{2}}{\cosh y}$;

(b) *They also satisfy* $\Pi_k\left[\partial_x u^\star_{\rm wk}(0,\cdot)\right]=0$, $\star=s,u$, *and*

$$\left\|\left(|-\partial_t^2-1|^{\frac12}(u^u_{\rm wk}-u^s_{\rm wk})+i\partial_x(u^u_{\rm wk}-u^s_{\rm wk})\right)(0,t)-4\sqrt{2}C_{\rm in}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}}\sin 3k\omega t\right\|_{\ell_1}\le\frac{Me^{-\frac{\sqrt{2k}\pi}{\varepsilon}}}{\frac12\log k-\log\varepsilon}. \tag{1.15}$$

(c) *A* $\frac{2\pi}{\omega}$*-periodic-in-*t *solution* $u(x,t)$ *to* (1.2) *satisfies*

$$\sup_{x\in\mathbb{R}}\|u(x,\cdot)\|_{\ell_1}\le\rho_2^*\omega^{\frac12} \tag{1.16}$$

and (1.11) *as* $x\to-\infty$ *(or as* $x\to+\infty$*) iff* $u^u_{\rm wk}$ *satisfies* (1.16) *and* $u(x,t)=u^u_{\rm wk}(x+x_0,t+t_0)$ *(or* $u(x,t)=u^s_{\rm wk}(x+x_0,t+t_0)$*) for some* $x_0,t_0\in\mathbb{R}$*. Consequently, there exists a solution* $u(x,t)$ *to* (1.2) *satisfying* (1.16) *and* (1.11) *as* $|x|\to\infty$ *iff there exists* $r\in\mathbb{R}$ *such that* $u^u_{\rm wk}(x+r,t)=u^s_{\rm wk}(x,t)$ *for all* x *and* t *and satisfies* (1.16).

In this theorem, while the non-existence of small breathers is confirmed in the case of temporal frequency $\omega\in J_k(\varepsilon_0)$, the only possible (up to translations) candidates $u^\star_{\rm wk}(x,t)$, $\star=s,u$, of small breathers with $\omega\in I_k(\varepsilon_0)$ are identified along with optimal estimates. In particular, it gives a necessary and sufficient condition (in statement (2c)) on the existence of small breathers that $u^\star_{\rm wk}$, $\star=s,u$, must remain small for all $x\in\mathbb{R}$ and coincide after a translation. *The most important result of the theorem is statement (2b)* which rigorously identifies the exponentially small leading order term and its coefficient $C_{\rm in}$ in the splitting of $u^\star_{\rm wk}$, $\star=s,u$, when they get close in an infinite dimensional space (of periodic functions of the variable t) in their first opportunity in x.

From Theorem 1.3(2ab), one may verify that $x=0$ is the only critical point of $\Pi_k[u^\star_{\rm wk}(x,\cdot)]$ for $\pm x\le\frac{1}{\varepsilon\sqrt{k}\omega}$, $\star=u,s$. Therefore, if $C_{\rm in}\neq0$, then there does not exist $|r|\le\frac{1}{\varepsilon\sqrt{k}\omega}$ such that $u^s_{\rm wk}(x,t)\equiv u^u_{\rm wk}(x+r,t)$. Hence $C_{\rm in}\neq0$ excludes the existence of small single bump breathers due to (1.14) and (1.15), which are the simplest and the most natural class of small breathers including those given in (1.1) for (sG) (see Figure 1).

Note then that Theorem 1.3 is conditional, since it proves the nonexistence of single-bump small amplitude breathers *provided* the constant $C_{\rm in}=C_{\rm in}(f)\in\mathbb{C}$ satisfies $C_{\rm in}\neq0$. In particular, *it proves Theorem 1.2* as long as $C_{\rm in}(f)\neq0$ for an open and dense set of $f\in\mathcal{F}_r$. Next theorem, proven in Section 11, shows that this is indeed the case. In fact, we also give an explicit family of nonlinearities $f(\mu,u)$, involving a parameter μ, such that $C_{\rm in}(f)\neq0$ for all but a discrete set of $\mu\in\mathbb{R}$.

Theorem 1.4. *Fix* $r>0$*. The map* $C_{\rm in}:\mathcal{F}_r\to\mathbb{C}$ *introduced in Theorem 1.3(2) is analytic and non-constant. Moreover, the set* $\mathcal{U}=\mathcal{F}_r\setminus C_{\rm in}^{-1}(0)$ *is open and dense.*

As already mentioned, even if small single-bump breathers are not expected to exist for (1.2) with most f, the more generic phenomenon is the existence of small breathers with non-vanishing tails which are exponentially small with respect to the amplitude. This is stated in the next proposition.

Proposition 1.5. *Fix* $r>0$ *and consider* $f\in\mathcal{F}_r$*, then the following holds. There exist* $\varepsilon_0,M>0$ *such that for any* $k\in\mathbb{N}$ *and* $\omega=\sqrt{\frac{1}{k(k+\varepsilon^2)}}\in I_k(\varepsilon_0)$*:*

1. *There always exist* $\frac{2\pi}{\omega}$*-periodic-in-*t *solutions* $u(x,t)$ *such that*
$$\left\||-\partial_t^2-1|^{\frac12}(u-u^\star_{\rm wk})(x,\cdot)\right\|_{L^2_t(-\frac{\pi}{\omega},\frac{\pi}{\omega})}+\left\|\partial_x(u-u^\star_{\rm wk})(x,\cdot)\right\|_{L^2_t(-\frac{\pi}{\omega},\frac{\pi}{\omega})}\le Mk^{\frac12}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}},$$
for both all $x\ge0$ *with* $\star=s$ *and* $x\le0$ *with* $\star=u$*.*
2. *Suppose that the constant* $C_{\rm in}$ *introduced in Theorem 1.3 satisfies* $C_{\rm in}\neq0$*. Then, the breather with tails* $u(x,t)$ *given by item (1) also satisfies*
$$\left\||-\partial_t^2-1|^{\frac12}(u-u^\star_{\rm wk})(x,\cdot)\right\|_{L^2_t(-\frac{\pi}{\omega},\frac{\pi}{\omega})}+\left\|\partial_x(u-u^\star_{\rm wk})(x,\cdot)\right\|_{L^2_t(-\frac{\pi}{\omega},\frac{\pi}{\omega})}\ge\frac{|C_{\rm in}|}{M}k^{\frac12}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}},$$
for both all $x\ge0$ *with* $\star=s$ *and* $x\le0$ *with* $\star=u$*.*

We give several remarks on Theorems 1.3, 1.4 and Proposition 1.5.

1. The constant $C_{\rm in}$ introduced in Theorem 1.3 is often referred to as the *Stokes constant* in the literature, which is the coefficient of the leading order term in the exponentially small obstruction in (1.15). We emphasize that the non-existence of small single bump amplitude breathers holds for *all* frequencies under the *single* condition $C_{\rm in}\neq0$. This constant depends on the full jet of the real

analytic nonlinearity f, but is independent of k or ω. No simple closed formula has been identified for $C_{\rm in}$ in the literature. We expect that one should be able to develop a computer assisted proof to check the nonvanishing of $C_{\rm in}$ for given nonlinearities (following the ideas developed in [4] for a 3-dimensional Hopf-zero bifurcation). In Section 11.2 below, we conjecture a formula of the Stokes constant in terms of a series.

(2) The relative scale between u and $\partial_x u$ in the estimates in Theorem 1.3(2) is consistent with the quadratic part of the Hamiltonian $\mathbf{H}$, where $|-\partial_t^2-1|$ is somewhat degenerate of the order $\mathcal{O}(k^{-1}\varepsilon^2)$ when applied to the k-th Fourier mode $e^{ik\omega t}$.

(3) The generalized small amplitude breathers given by Proposition 1.5 have frequency ω slightly smaller than each $\frac{1}{k}$. This $\frac{1}{k}$ is consistent with the fact that small (sG) breathers (1.1) have periods slightly greater than $2k\pi$. Each of these breather-like solutions to (1.2) with exponentially small tails is the superposition of a small exponentially localized-in-x wave of order $\mathcal{O}(\varepsilon k^{-\frac12} e^{-\varepsilon k^{-\frac12}|x|})$ with an L^∞_{xt} correction up to order $\mathcal{O}(k^{\frac12}\varepsilon^{-1}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}})$. In the generic non-degenerate case of $C_{\rm in}\neq 0$, the infimum of the tails of such generalized breathers is also bounded below by this exponential order.

(4) In contrast to the fact that the breathers of (sG) form a 3-dim manifold in the infinite dimensional space of solutions, the breathers with exponentially small tails actually form a family of finite codimension (see Proposition 2.2 below).

(5) As $u^\star_{\rm wk}$, $\star = u, s$, are special solutions of high regularity, the norm in Theorem 1.3(2ab) actually could be refined to be H^n_t for any $n\geq 0$. In contrast, since the set of breathers with exponentially small tails is of finite codimension in the energy space of the spatial dynamics, the norms in Proposition 1.5 arising from the quadratic part of the Hamiltonian $\mathbf{H}$ are not expected to be improved.

(6) In the generic case of $C_{\rm in}\neq 0$ provided by Theorem 1.4 which implies that (1.2) does not have small breathers, the asymptotic behavior of small solutions in the energy space $H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$ is a natural but intriguing question. Even though the breathers with exponentially small tails $u(x,t)$ obtained in Proposition 1.5 are not in the energy space $H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$, they still shed some light on the dynamics of (1.2). Take $k=1$ for simplicity. Let $\gamma\in C_0^\infty(\mathbb{R},\mathbb{R})$ be a cut-off function satisfying $\gamma(s)=1$ for $|s|\leq 1$ and $u(x,t)$ be a sufficiently smooth breather like solution with exponentially small tails (see Proposition 1.5). Consider the solution $\tilde u(x,t)$ to (1.2) with initial value $\gamma(\frac{1}{\varepsilon^3}e^{-\frac{\sqrt2\pi}{\varepsilon}}x)\big(u(x,0),\partial_t u(x,0)\big)$. Its $H^1_x\times L^2_x$ norm is of the order $\mathcal{O}(\sqrt{\varepsilon})$. The propagation speed of (1.2) being equal to 1 implies that $\tilde u$ is periodic in t for $|x|,|t|\leq \mathcal{O}(\varepsilon^{-3}e^{\frac{\sqrt2\pi}{\varepsilon}})$. Hence, this exponential time scale has to be relevant in studying the asymptotic dynamics of small energy solutions of (1.2).

Breathers with small tails as well as some other similar types of solutions had already been obtained, but often with only exponentially small *upper bound estimates* on the tails, instead of their precise orders or lower bounds (and without the explicit exponent $-\frac{\sqrt{2k}\pi}{\varepsilon}$). In [43], Lu derived breathers with tails bounded by $\mathcal{O}(e^{-\frac{c}{\varepsilon}})$ for some unspecified $c>0$. In a sequence of papers, Groves and Schneider considered small amplitude modulating pulse solutions to a class of semilinear [30] and quasilinear [31, 32] reversible wave equations. These are solutions consisting of pulse-like spatially localized envelopes advancing in the laboratory frame and modulating an underlying wave-train of a fixed wave number $\xi_0>0$, which are time-periodic in a moving frame of reference. They would become breathers if $\xi_0=0$. For quasilinear reversible wave equations, Groves and Schneider constructed solutions $u(x,t)$ of this type with tails bounded by $\mathcal{O}(e^{-\frac{c}{\sqrt\varepsilon}})$ but only defined for $|x|\leq\mathcal{O}(e^{\frac{c}{\sqrt\varepsilon}})$. The finite length of the domain in x was mainly due to difficulties arising in quasilinear PDEs. In the semilinear case, such solutions could be derived globally in $x\in\mathbb{R}$ with the same $\mathcal{O}(e^{-\frac{c}{\sqrt\varepsilon}})$ estimates on the tails. The upper bounds of the tails in these papers were obtained by making the error terms small through consecutive applications of partial normal forms, e. g. as in [53, 36].

The proof of Theorems 1.2, 1.3 and Proposition 1.5 rely on the spatial dynamics method (see, e.g. [38, 71]). This method is often an effective approach in constructing certain coherent structures for nonlinear PDEs where a spatial variable x plays a distinct role. In such a framework, the desired solutions are sought as special solutions in an evolutionary system where this x is treated as the dynamic variable.

We fix a temporal frequency $\omega>0$ and consider in the rescaled variable $\tau=\omega t$,

$$\omega^2\partial_\tau^2 u-\partial_x^2 u+u-\frac13 u^3-f(u)=0. \tag{1.17}$$

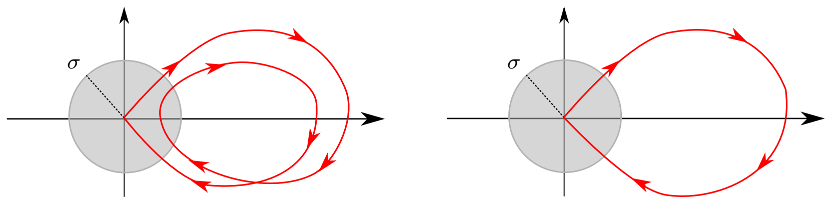

FIGURE 2. Multi-bump (left) and single-bump (right) solutions in the spatial dynamics framework.

Considering x as the evolutionary variable, it is a globally well-posed infinite dimensional Hamiltonian system in appropriate spaces of 2π-periodic-in-τ functions where its Hamiltonian can be derived from $\mathbf{H}$.

Breathers of (1.2) correspond to 2π-periodic-in-τ solutions of (1.17) which decay to 0 as $|x| \to \infty$, namely, orbits homoclinic to the equilibrium 0 due to the intersection of its stable and unstable manifolds. Note that, on the one hand, the notions of single-bump and multi-bump breathers (see Definition 1.1) get translated to homoclinics as in Figure 2. On the other hand, small amplitude breathers correspond to small homoclinic loops. From this point of view, the proof of Theorem 1.3 will rely on analyzing the (finite dimensional) stable and unstable invariant manifolds of $u = 0$ and on whether their intersections lead to *small* homoclinic loops. Proposition 1.5 will rely on analyzing the center-stable and center-unstable invariant manifolds and constructing small homoclinic loops to the center manifold.

In the next section we first present an abstract setting for analyzing (the breakdown of) small homoclinic loops through a dynamical system approach. Then, we show how the Klein Gordon equation (1.17) fits into this framework regarded from the spatial dynamics point of view.

1.3. **Birth of small homoclinics via "eigenvalue collision": exponentially small splitting of separatrices.** Let us consider an N-dimensional system, $N \leq \infty$, (P_α) involving a parameter $\alpha \in I \subset \mathbb{R}$, which has a steady state at 0. We want to analyze whether this steady state has small homoclinic loops.

Assume the following happens in (P_α).

(a) *"Eigenvalue collision"* at $\alpha = 0$. Namely, in a neighborhood of $0 \in \mathbb{C}$, there are exactly two eigenvalues $\pm\lambda(\alpha) \sim \pm\sqrt{\alpha}$ (modulo symmetries, but counting the algebraic multiplicity) of the linearization of (P_α) at 0. As α increases, they move towards 0 from the imaginary axis $i\mathbb{R}$ and then move into the real axis $\mathbb{R}$ after coinciding at 0 when $\alpha = 0$.

(b) For $\alpha > 0$, the *normal form* of the local nonlinear system (P_α) near 0 projected to the 2-dimensional eigenspace $\mathcal{M}$ associated to $\pm\lambda(\alpha)$ is equivalent to

$$\ddot{u} - \lambda(\alpha)^2 u + u^m = 0, \quad m \geq 2, \tag{1.18}$$

where the "+" sign matters only when m is odd. Apparently this normal form system has one or two small homoclinic orbits of amplitude $\mathcal{O}\big(\lambda(\alpha)^{\frac{2}{m-1}}\big)$ for $0 < \alpha \ll 1$.

(c) The system (P_α) has a first integral which is locally positive definite in the center manifold around the steady state.

Note that, under these assumptions, small homoclinic loops cannot exist either of $\alpha < 0$ or if $\alpha > 0$ is "not close to 0". One first observes that they cannot exist in the center manifold $W^c(0)$ since the steady state is isolated at its level of energy inside $W^c(0)$. Hence homoclinic orbits exist only at the intersection of the stable and unstable invariant manifolds $W^{s,u}(0)$. However, if α is not small, then the dynamics inside the stable and unstable manifolds $W^{s,u}(0)$ are conjugate to the linear dynamics in a large neighborhood of 0. Then any orbit in them (and in particular any possible homoclinic orbits) must first go away at a uniform distance from 0. In conclusion, small homoclinic loops can only exist for small $\alpha > 0$.

If the whole system (P_α) is 2-dimensional, $\alpha = 0$ corresponds to one type of the Bogdanov-Takens bifurcations. In this case the existence of a conserved quantity leads to the existence of small homoclinic orbit for all small $\alpha > 0$.

When (P_α) is of higher dimensions, then the dynamics in the directions transverse to $\mathcal{M}$ is at a fast scale and thus (P_α) is a typical singular perturbation system.

If the fast dynamic is hyperbolic, then by the standard *normally hyperbolic* invariant manifold theory, a 2-dimensional slow manifold $\mathcal{M}_\alpha$ persists for $0 < \alpha \ll 1$ and the existence of small homoclinic orbit can be reduced to the above 2-dim case on $\mathcal{M}_\alpha$. This mechanism indeed happens in the construction of some special solutions in some nonlinear PDEs including [13] (or more of the elliptic type PDE, see, e.g. [50]).

However, if there are fast elliptic/oscillatory directions (as happens for the Klein-Gordon equation (1.17)), then there does not necessarily exist a slow manifold and one cannot reduce (P_α) to 2 dimensions. Without such reduction, one is forced to find small homoclinic orbits as the intersection of the low dimensional stable and unstable manifolds $W^{u,s}(0)$ of 0 close to $\mathcal{M}$ in the N-dimensional phase space of (P_α), but this is highly unlikely simply by counting the dimensions. Homoclinic orbits are generated via such eigenvalue collision mechanism only in some very lucky/rare systems, such as the completely integrable (sG) where the family of breathers actually can be also extended to large amplitudes.

In a generic system of the above *normally elliptic case* with rapid oscillations, while the stable and unstable invariant manifolds, denoted by $W^{u,s}(0)$, do not intersect, the existence of a conserved quantity whose Hessian is positive definite in the center direction of the linearized (P_α) at 0 often ensures the transverse intersection of the center-stable and center-unstable manifolds $W^{cs,cu}(0)$. This intersection yields a finite co-dimensional tube homoclinic to the center manifold, which corresponds to generalized breathers for the Klein-Gordon equation (1.17)). Moreover, the distance between $W^{u,s}(0)$ determines how close this homoclinic tube is to 0 which, in the present paper, corresponds to how small the tails of the generalized breathers of the nonlinear Klein-Gordon equations can be.

Regarding the distance between $W^{u,s}(0)$, the strong averaging effect of the fast oscillations makes $W^{u,s}(0)$ very close to each other – usually $\mathcal{O}(\alpha^n)$ if (P_α) has finite smoothness and $\mathcal{O}(e^{-\frac{1}{\alpha^\delta}})$ if (P_α) analytic. A leading order approximation such as the one obtained in Theorem 1.3(2ab) provides accurate information of this distance, usually called *splitting distance*[2].

To summarize, the mechanism of eigenvalue collision leading to a Bogdanov-Takens type bifurcation embedded in a normally elliptic singular perturbation problem is primarily responsible for the birth of small homoclinic orbits/breathers with small tails for the nonlinear Klein-Gordon equation (1.17). It yields exact breathers in some very special cases such as the completely integrable (sG).

In fact, this general mechanism leads to what is usually called *exponentially small splitting of separatrixs*, a phenomenon that usually arises in analytic systems with two time scales with i.) fast oscillations and ii.) slow hyperbolic dynamics with a homoclinic loop (also called separatrix), as in the setting explained above. Other settings where this phenomenon occurs are the resonances of nearly integrable Hamiltonian systems and close to the identity area preserving maps. Analysis of such phenomena is fundamental in the construction of unstable behaviors in these models such as Arnold diffusion or chaotic dynamics.

The study of the exponentially small splitting of separatrixs goes back to the seminal paper by Lazutkin [40], which dealt with the standard map. His strategy can be described as follows:

(1) The singular limit (1.18) has a homoclinic orbit whose time parameterization is analytic in a strip containing the real line and has singularities in the complex plane.
(2) One looks for parameterizations of the perturbed invariant manifolds which are close to the unperturbed homoclinic. They can be extended to complex values of the parameter which are close to the singularities of the unperturbed homoclinic with smallest imaginary part.
(3) One analyzes the difference between the perturbed stable and unstable manifolds close to these singularities. To this end, one has to look for the leading order of the perturbed invariant manifolds in these complex domains. Then, one is encountered with two different situations:
    (i) In some problems, the perturbed invariant manifolds are also well approximated by the unperturbed homoclinic solution near its singularities. In this case, one can show that the classical Melnikov method gives the first order of the difference between these manifolds.
    (ii) In most of the problems, like the problem at hand, the unperturbed homoclinic is not a good approximation of the perturbed invariant manifolds in these complex domains. Therefore, one must look for new first order approximations. These first orders are solutions of the so-called *inner equation*, which is a singular limit equation independent of the perturbative parameter. The analysis of this equation gives the asymptotic formula for the difference between the invariant manifolds. In particular, it provides the Stokes constant C_{in} appearing in Theorem 1.3.
(4) The last step is to translate the analysis in the complex domain to the real parameterizations of the invariant manifolds.

[2]It also sheds light for the future study of scattering maps [18] induced by the homoclinic tube and multi-bump homoclinics.

In the present paper we apply this strategy to the nonlinear Klein-Gordon equation (1.17), or equivalently (1.2). It is explained heuristically in more detail in Sections 2.1 and 2.2 below.

In the last decades this strategy or similar ones relying on analytic continuation of the parameterization of the invariant manifolds has been applied to various problems mostly in *finite dimensions*. The first category (case 3(i) above), where the Melnikov function provides the leading order of the splitting distance, includes fast periodic forcing of integrable Hamiltonian systems [35, 59, 19, 27] and non-generic unfoldings of the Hopf-zero singularity [9]. This category can also be handled by other methods such as direct series expansions [24, 69, 70]. Problems falling into the second category (case 3(ii) above), where one needs an inner equation, are more common. Among them are near identity maps [28, 26, 48, 49], resonances of nearly integrable Hamiltonian systems [29, 54, 3, 8], and generic local bifurcations [41, 42, 10, 5, 6, 7, 25, 23]. This category can also be handled by a different method, the so-called continuous averaging by Treschev [68].

Exponentially small splitting phenomena also arise in the construction of solitary and traveling waves in PDEs and lattices [2, 22, 65, 41, 64, 42, 66, 67, 55]. However, in all the above papers involving leading order analysis of the exponentially smallness the fast oscillatory dimentions are finite (often two). As far as the authors know, the present paper is *the first one dealing with an infinite number of oscillatory directions*.

1.4. **The spatial dynamics approach for the Klein-Gordon equation.** We devote this section to implement the spatial dynamics approach for equation 1.17 and to write it as a system having the features of the class of models P_α introduced in Section 1.3. To this end, we denote

$$g(u) = \frac{1}{3}u^3 + f(u). \tag{1.19}$$

In terms of the Fourier series expansion (1.5) (see also (1.6)), the equation (1.17) reads

$$(\partial_x^2 + n^2\omega^2 - 1)u_n = -\Pi_n\left[g(u)\right], \qquad n \in \mathbb{Z}. \tag{1.20}$$

The eigenvalues of the linearization equation at 0, that is

$$\partial_x^2 u - \omega^2 \partial_\tau^2 u - u = 0,$$

are $\pm\nu_n$, where

$$\nu_n = \sqrt{1 - n^2\omega^2},$$

and their eigenfunctions can be calculated using the Fourier modes.

Consider $\omega \in [\frac{1}{k+1}, \frac{1}{k})$ for some $k \geq 0$. For $0 \leq |n| \leq k$, the eigenvalues $\pm\nu_n \in \mathbb{R} \setminus \{0\}$ are hyperbolic, while the center eigenvalues $\pm\nu_n = \pm i\vartheta_n$, $\vartheta_n = \sqrt{n^2\omega^2 - 1}$, correspond to $|n| \geq k+1$. Recall the two primary classes of intervals $I_k(\varepsilon_0)$, $k \in \mathbb{N}$, and $J_k(\varepsilon_0)$, $k \in \mathbb{N} \cup \{0\}$, of the frequency ω defined in (1.10) for some $\varepsilon_0 \in (0, \frac{1}{2})$. Clearly the dimension of the hyperbolic eigenspace of 0 increases by 1 as the frequency ω decreases through $\frac{1}{k}$ moving from $J_{k-1}(\varepsilon_0)$ into $I_k(\varepsilon_0)$.

In the strongly hyperbolic case of $\omega \in J_k(\varepsilon_0)$, $k \geq 0$, the smallest hyperbolic eigenvalue satisfies

$$\nu_k > \tfrac{\varepsilon_0}{\sqrt{k+\varepsilon_0^2}} > \min\{1, \tfrac{\varepsilon_0}{2\sqrt{k}}\}.$$

Based on the general local invariant manifold theory (see, e.g. Theorem 4.4 in [16]) and this spectral gap along with the cubic nonlinearity of wave type equation (1.17), one expects that the local stable/unstable manifolds of 0 are close to the stable/unstable subspaces in a neighborhood of 0 of radius of the order $\mathcal{O}\big(\min\{1, \frac{\varepsilon_0}{2\sqrt{k}}\}\big)$ and all orbits on both these manifolds leave such neighborhood eventually. This argument is carried out uniformly in k and ω in Section 9 and statement (1) of Theorem 1.3 follows consequently. Therefore they cannot intersect in such a neighborhood to produce small homoclinic orbits.

In contrast to the above case, when ω decreases through $\frac{1}{k}$ and enters $I_k(\varepsilon_0)$, $k \geq 1$, ν_k can be arbitrarily small. The linearized (1.17) is only weakly hyperbolic in the k-th modes – the newly generated hyperbolic directions – and small homoclinics might be generated through a Bogdanov-Takens bifurcation as described in Section 1.3. The fact $\omega \in I_k(\varepsilon_0)$ is consistent with that the periods of small (sG) breathers (1.1) are close to $2k\pi$. The different scales in x in these weakly hyperbolic directions and the other much faster directions make the local dynamics of (1.17) near 0 a singular perturbation problem. More precisely, let

$$\varepsilon = \sqrt{\frac{1}{k}\left(\frac{1}{\omega^2} - k^2\right)} \in (0, \varepsilon_0) \tag{1.21}$$

and consider the following rescaling of the amplitude and x,

$$u = \varepsilon\sqrt{k}\omega v \text{ and } y = \varepsilon\sqrt{k}\omega x. \tag{1.22}$$

Thus, $u(x,\tau)$ satisfies (1.17) if, and only if, $v(y,\tau)$ satisfies

$$\partial_y^2 v - \frac{1}{\varepsilon^2 k}\partial_\tau^2 v - \frac{1}{\varepsilon^2 k\omega^2} v + \frac{1}{3}v^3 + \frac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3} f\left(\varepsilon\sqrt{k}\omega v\right) = 0, \tag{1.23}$$

which is a Hamiltonian PDE in the dynamical variable y with the Hamiltonian

$$\mathcal{H}(v,\partial_y v) = \int_{\mathbb{T}} \left(\frac{(\partial_y v)^2}{2} + \frac{(\partial_\tau v)^2}{2\varepsilon^2 k} - \frac{v^2}{2\varepsilon^2 k\omega^2} + \frac{v^4}{12} + \frac{F(\varepsilon\sqrt{k}\omega v)}{\varepsilon^4 k^2\omega^4} \right) d\tau. \tag{1.24}$$

Using the projection Π_n defined in (1.6) and denoting $\cdot = d/dy$, we obtain (see (1.20)),

$$\ddot{v}_n = -\frac{(n^2\omega^2 - 1)}{\varepsilon^2 k\omega^2} v_n - \frac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3}\Pi_n\left[g(\varepsilon\sqrt{k}\omega v)\right],\ n \in \mathbb{Z}. \tag{1.25}$$

By (1.21),

$$\lambda_n = \sqrt{\frac{1}{k}\left|n^2 - \frac{1}{\omega^2}\right|} \geq \frac{1}{2}, \quad \text{for each} \ \ |n| \neq k.$$

Using this notation, (1.25) becomes

$$\begin{cases} \ddot{v}_n = \dfrac{\lambda_n^2}{\varepsilon^2} v_n - \dfrac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3}\Pi_n\left[g(\varepsilon\sqrt{k}\omega v)\right], & |n| < k, \\ \ddot{v}_{\pm k} = v_{\pm k} - \dfrac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3}\Pi_{\pm k}\left[g(\varepsilon\sqrt{k}\omega v)\right], & \\ \ddot{v}_n = -\dfrac{\lambda_n^2}{\varepsilon^2} v_n - \dfrac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3}\Pi_n\left[g(\varepsilon\sqrt{k}\omega v)\right], & |n| > k. \end{cases} \tag{1.26}$$

Notice $v_{-n} = -\overline{v_n}$ and $(\varepsilon k^{\frac{1}{2}}\omega)^{-3} g(\varepsilon\sqrt{k}\omega v) = \mathcal{O}(|v|^3)$ is smooth with bounds uniform in $\varepsilon\sqrt{k}\omega$. The stable and unstable invariant manifolds $W^s(0)$ and $W^u(0)$ of $2k+1$ real dimensions correspond to solutions v^s and v^u of (1.26) satisfying the asymptotic conditions[3]

$$\lim_{y\to+\infty} v_n^s(y) = \lim_{y\to+\infty} \dot{v}_n^s(y) = \lim_{y\to-\infty} v_n^u(y) = \lim_{y\to-\infty} \dot{v}_n^u(y) = 0, \qquad \text{for all} \ \ n \in \mathbb{Z}. \tag{1.27}$$

The singular perturbation problem (1.26) can be written as

$$\begin{cases} \varepsilon\dot{v}_n = \lambda_n w_n \\ \varepsilon\dot{w}_n = \lambda_n v_n - \lambda_n^{-1}\varepsilon^{-1}k^{-\frac{3}{2}}\omega^{-3}\Pi_n\left[g(\varepsilon\sqrt{k}v)\right], & |n| < k, \\ \varepsilon\dot{w}_n = -\lambda_n v_n - \lambda_n^{-1}\varepsilon^{-1}k^{-\frac{3}{2}}\omega^{-3}\Pi_n\left[g(\varepsilon\sqrt{k}\omega v)\right], & |n| > k, \\ \ddot{v}_{\pm k} = v_{\pm k} - (\varepsilon\sqrt{k}\omega)^{-3}\Pi_{\pm k}\left[g(\varepsilon\sqrt{k}\omega v)\right]. & \end{cases} \tag{1.28}$$

The formal singular limit of this system as $\varepsilon \to 0$ defines a critical manifold

$$\mathcal{M} = \{(v,w) \mid v_n = w_n = 0 \quad \text{for all} \ \ n \neq \pm k\}$$

of real dimension 4 due to $v_{-n} = -\overline{v_n}$. The limiting dynamics on $\mathcal{M}$ is given by the Duffing equation

$$\ddot{v}_k = v_k - \frac{1}{3}\Pi_k\left[\left(\mathrm{Im}(v_k e^{ik\tau})\right)^3\right] = v_k - \frac{1}{4}|v_k|^2 v_k, \tag{1.29}$$

which is integrable with the phase symmetry. It is known that in (1.29) the 2-dimensional stable and unstable manifolds of 0 coincide. In particular, it has a unique real homoclinic orbit to 0 satisfying $v_k > 0$ and $\dot{v}_k(0) = 0$, which is given by (see Figure 3)

$$v_k = v^h(y) = \frac{2\sqrt{2}}{\cosh(y)}. \tag{1.30}$$

---

[3]The Hamiltonian restricted to the center manifold is positive definitive locally around $u = 0$ for all $\omega > 0$ and, therefore, all orbits backward/forward asymptotic to $u = 0$ must belong to the unstable/stable manifold (see Corollary 9.4 below).

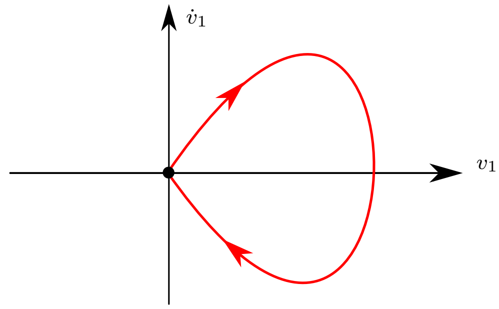

FIGURE 3. Real positive homoclinic (1.30) to 0 of the Duffing equation (1.29) in the critical manifold $\mathcal{M}$.

For $0 < \varepsilon \ll 1$, the special solutions $u_{\rm wk}^\star$, $\star = u, s$, given in Theorem 1.3(2) and principally supported in the k-modes, are the ones on the $(2k+1)$-dimensional invariant manifolds $W^\star(0)$ with the weakest decay, which are natural deformations of $v^h(y)\sin k\tau$ with proper rescaling. In Section 10 we prove that $u_{\rm wk}^\star$ is the only possible intersection of $W^\star(0)$, $\star = u, s$, in an $\mathcal{O}(k^{-\frac{1}{2}})$ neighborhood of 0 (much greater than the amplitude $\mathcal{O}(\varepsilon k^{-\frac{1}{2}})$ of $u_{\rm wk}^\star$).

Most of the analysis in the paper is devoted to identifying the exponentially small leading order term of the splitting $(u_{\rm wk}^u - u_{\rm wk}^s)|_{x=0}$, where $x = 0$ corresponds to the first opportunity when they get close, and deriving the leading order coefficient $C_{\rm in}$ (see Theorem 1.3(2b)).

Structure of the paper. The core of the proof of most of the results in Theorem 1.3(2ab) is for the case $k = 1$, $\omega \in I_1(\varepsilon_0)$, and under the oddness assumption in t. The main results for this particular setting are stated in Section 2: Theorem 2.1 deals with the break up of single bump breathers and Proposition 2.2 deals with the existence of generalized breathers. The proof of Theorem 2.1 is given in Section 3 (Section 4 - 7 contain the proofs of some of the statements in Section 3). Then, Proposition 2.2 is proven in Section 8. Section 9 is devoted to prove the nonexistence of breathers for frequencies which are far from resonant, that is item (1) of Theorem 1.3. Section 10 explains the reduction of the general case of close to resonant frequencies to that considered in Section 2: oddness in t assumption and $\omega \in I_1(\varepsilon_0)$. This completes the prove of item (2) of Theorem 1.3. Finally, in Section 11, we prove Theorem 1.4.

2. ANALYSIS OF THE FIRST BIFURCATION $(k = 1)$ WITH ODDNESS ASSUMPTION IN t

We devote this section to analyze the stable and unstable manifolds of $v = 0$ and their splitting for equation (1.23) with $k = 1$ and $\omega \in I_1(\varepsilon_0)$ (see (1.10)). We also analyze the center-stable and center-unstable manifolds to construct the generalized breathers provided by Proposition 1.5.

To make the function space setting precise, recall the norm $\|\cdot\|_{\ell_1}$ defined in (1.6) which is simply the ℓ_1 norm of the Fourier coefficients in τ. Since

$$\|f_1 f_2\|_{\ell_1} \le \|f_1\|_{\ell_1}\|f_2\|_{\ell_1},$$

treating y as the evolution variable, the local-in-y well-posedness of (1.23) with $(v, \partial_y v)(y, \cdot) \in \mathbf{X}$, where

$$\mathbf{X} := \{(v, w) \mid v, w \text{ are } 2\pi\text{-periodic in } \tau \text{ and } \|(v, w)\|_{\mathbf{X}} := \|v\|_{\ell_1} + \|(1 + |\partial_\tau|)^{-1} w\|_{\ell_1} < \infty\}, \tag{2.1}$$

follows from a standard procedure. Here the operator $|\partial_\tau|$ is simply the multiplication of $|n|$ to the n-th Fourier modes for each n. For some results where the conservation of energy is used, we also consider the energy space $H_\tau^1 \times L_\tau^2$ which is a dense subspace of $\mathbf{X}$ where (1.23) is also well-posed.

Due to the oddness assumptions on f, the subspace

$$\mathbf{X}_o = \{(v, w) \in \mathbf{X} \mid v, w \text{ are odd in } \tau\} \subset \mathbf{X} \tag{2.2}$$

of 2π-periodic odd functions of τ is invariant under the flow of (1.23), so we first restrict the analysis to this subspace. For such odd functions (of real values) of τ, the Fourier series (1.5) turns out to be

$$v(t) = \sum_{n=-\infty}^{+\infty} \left(-\frac{i}{2}\right) \mathrm{sgn}(n) v_{|n|} e^{in\tau} = \sum_{n \ge 1} v_n \sin(n\tau), \quad \tau = \omega t, \;\; \Pi_n[v] = v_n \in \mathbb{R}, \; n \in \mathbb{N}. \tag{2.3}$$

With a slight abuse of notation, sometimes we may also use $\Pi_n[v]$ to denote the n-th mode $v_n \sin n\tau$. Later in Section 10, we extend the analysis to the general setting.

As explained in Section 1.4, we refer to the analysis in the setting of $k=1$ and $\omega \in I_1(\varepsilon_0)$ as the *first bifurcation.* Indeed, for $\omega \in I_1(\varepsilon_0)$, the linearization around $v=0$ possesses (in the odd-in-t functions space $\mathbf{X}_o$) a pair of weak hyperbolic eigenvalues and all the other eigenvalues are elliptic. In particular the stable and unstable manifolds, $W^s(0)$ and $W^u(0)$, are one dimensional.

Next theorem gives an asymptotic formula for the splitting between $W^u(0)$ and $W^s(0)$ in the cross section

$$\Sigma = \{(v, \partial_y v) \in \mathbf{X}_o : \Pi_1[\partial_y v] = 0\}, \tag{2.4}$$

(see Figure 4).

Theorem 2.1. *Fix $r>0$. Consider the equation* (1.23) *with $f \in \mathcal{F}_r$ (equivalently* (1.26) *or* (1.28)*) for $k=1$ and $\omega \in I_1(\varepsilon_0)$ defined as in* (1.13)*. Then, there exist a constant $C_{\rm in} \in \mathbb{C}$ such that for any fixed $y_0>0$ there exists $\varepsilon_0, M>0$ such that, for every $0<\varepsilon\le\varepsilon_0$, the following statements hold.*

(1) *The invariant manifolds $W^u(0)$ and $W^s(0)$ of* (1.23) *in $\mathbf{X}_o$ correspond to unique solutions $v^u(y,\tau)$ and $v^s(y,\tau)$ of* (1.26) *satisfying* (1.27)*, which are real-analytic in y, 2π-periodic in τ, and satisfy $\Pi_1[\partial_y v^{u,s}](0)=0$, respectively. Moreover, $\Pi_{2l}[v^{u,s}]\equiv 0$, for every $l\in\mathbb{N}$ and*

$$\left\|\partial_\tau^2\big(v^u(y,\tau)-v^h(y)\sin\tau\big)\right\|_{\ell_1} + \left\|\partial_\tau^2\partial_y\big(v^u(y,\tau)-v^h(y)\sin\tau\big)\right\|_{\ell_1} \le M\varepsilon^2 v^h(y) \quad \text{for } y\le y_0,$$
$$\left\|\partial_\tau^2\big(v^s(y,\tau)-v^h(y)\sin\tau\big)\right\|_{\ell_1} + \left\|\partial_\tau^2\partial_y\big(v^s(y,\tau)-v^h(y)\sin\tau\big)\right\|_{\ell_1} \le M\varepsilon^2 v^h(y) \quad \text{for } y\ge -y_0,$$

where v^h is the homoclinic orbit given in (1.30).

(2) *At $y=0$, their difference satisfies*

$$\left\|\big(|-\partial_\tau^2-\omega^{-2}|^{\frac12}(v^u-v^s)+i\varepsilon\partial_y(v^u-v^s)\big)(0,\tau) - \frac{4\sqrt2}{\varepsilon}e^{-\frac{\pi\sqrt2}{\varepsilon}}C_{\rm in}\sin(3\tau)\right\|_{\ell_1} \le \frac{Me^{-\frac{\pi\sqrt2}{\varepsilon}}}{\varepsilon\log(\varepsilon^{-1})}. \tag{2.5}$$

We highlight that Theorem 2.1 is concerned with the distance between the stable and unstable invariant manifolds at the first crossing with the transversal section Σ. This does not exclude intersections at further crossings and thus existence of multi-bump breathers. See Figures 2 and 4.

Theorem 2.1 proves statements in Theorem 1.3(2ab) for $k=1$ and $\omega\in I_1(\varepsilon_0)$ (restricted to the odd in t setting) which deal with the one-dimensional stable and unstable manifolds .

The next proposition analyzes the intersection between the center-stable and center-unstable invariant manifolds of $v=0$.

Proposition 2.2. *Fix $r>0$. Consider the equation* (1.23) *with $f\in\mathcal{F}_r$ for $k=1$ and $\omega\in I_1(\varepsilon_0)$ defined as in* (1.13)*. For any fixed $y_0>0$, there exists $\varepsilon_0, M>0$ such that, for every $0<\varepsilon\le\varepsilon_0$, the following statements hold.*

Let $W\subset\Sigma$ be the intersection near $(v^h(0)\sin\tau, 0)$ of the center-stable manifold $W^{cs}(0)$ and center-unstable manifold $W^{cu}(0)$ of (1.23) *in $\mathbf{X}_o$ when they intersect the hyperplane Σ for the first time in y. Then,*

(1) *Let*

$$\begin{aligned}\mathcal{N} = \{(v,\partial_y v) \mid\ & \varepsilon^{-1}\left\||-\partial_\tau^2-\omega^{-2}|^{\frac12}\big(v-v^\star(0)\big)\right\|_{L^2} + \|\partial_y v-\partial_y v^\star(0)\|_{L^2} \\ & \le M\big(\varepsilon^{-1}\left\||-\partial_\tau^2-\omega^{-2}|^{\frac12}\big(v^u(0)-v^s(0)\big)\right\|_{L^2} + \|\partial_y v^u(0)-\partial_y v^s(0)\|_{L^2}\big),\ \star=u,s\}\end{aligned}.$$

Then $W\cap\mathcal{N}\neq\emptyset$ and the Hamiltonian $\mathcal{H}$ evaluated at solutions in $W\cap\mathcal{N}$ satisfies

$$\tfrac{1}{M}\inf_{W\cap\mathcal{N}}\mathcal{H} \le \varepsilon^{-2}\left\||-\partial_\tau^2-\omega^{-2}|^{\frac12}\big(v^u(0)-v^s(0)\big)\right\|_{L^2}^2 + \|\partial_y v^u(0)-\partial_y v^s(0)\|_{L^2}^2 \le M\inf_{W\cap\mathcal{N}}\mathcal{H}.$$

(2) *Each $(v,\partial_y v)\in W$ corresponds to a single bump homoclinic orbit $\big(v(y,\tau),\partial_y v(y,\tau)\big)$ to $W^c(0)$, i.e $(v,\partial_y v)$ is asymptotic to two orbits $\big(v_c^\pm(y),\partial_y v_c^\pm(y)\big)$ in the center manifold $W^c(0)$ as $y\to\pm\infty$. Moreover, $(v,\partial_y v)$ satisfies*

$$\tfrac{1}{M}\mathcal{H}(v,\partial_y v) \le \varepsilon^{-2}\left\||-\partial_\tau^2-\omega^{-2}|^{\frac12}\big(v(y)-v^\star(y)\big)\right\|_{L^2}^2 + \|\partial_y v(y)-\partial_y v^\star(y)\|_{L^2}^2 \le M\mathcal{H}(v,\partial_y v), \tag{2.6}$$

for $y\ge -y_0$ with $\star=s$ and $y\le y_0$ with $\star=u$.

(3) *If $v^u=v^s$, where v^u, v^s are the solutions obtained in Theorem 2.1, then it is a homoclinic orbit to 0, otherwise the intersection $W^{cs}(0)\cap W^{cu}(0)$, which is codimension 2 in $\mathbf{X}_o$, is transverse in $\mathcal{N}$.*

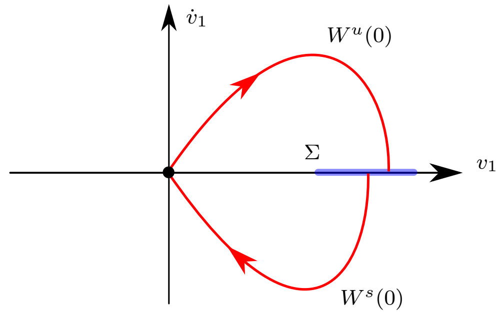

FIGURE 4. The (infinite dimensional) transverse section Σ (see (2.4)) where we measure the distance between the perturbed manifolds $W^u(0)$ and $W^s(0)$.

Remark 2.3. *In the case $v^u \neq v^s$, the transversality of the intersection of the codimension-1 $W^{cs}(0)$ and $W^{cu}(0)$ actually implies that a dense subset of W consists of functions smooth in τ. See Remark 8.1 below.*

This proposition implies Proposition 1.5 for $k = 1$ and $\omega \in I_1(\varepsilon_0)$, which deal with generalized breathers with exponentially small tails. Indeed, Proposition 2.2 implies the existence of a family of orbits homoclinic to the center manifold with exponentially small energy. They correspond to breather like solutions $u(x, t)$ of (1.2) which are $\frac{2\pi}{\omega}$-periodic in t and decaying in x like $\mathcal{O}(\varepsilon e^{-\varepsilon|x|})$ subject to perturbations whose $L^\infty_x(H^1_t \times L^2_t)$ norm is bounded by $\mathcal{O}(\frac{1}{\varepsilon} e^{-\frac{\sqrt{2}\pi}{\varepsilon}})$.

Theorem 2.1 and Proposition 2.2 are proven in Sections 3 and 8 respectively. We devote the rest of this section to give some heuristics on the proof of Theorem 2.1, in particular, on why the distance between the invariant manifolds is exponentially small and on how to obtain its asymptotic formula.

2.1. **Heuristics of the proof of Theorem 2.1; exponentially small bounds.** Looking at formula (2.5) one can see that the distance between the one dimensional invariant manifolds $W^u(0)$ and $W^s(0)$ of (1.23) is exponentially small in ε. In this section we give some intuition why and we explain which are the steps needed to obtain upper bounds on this distance. Later, in Section 2.2, we show how to obtain the asymptotic formula (2.5) for it.

Since the invariant manifolds are one-dimensional, one can parameterize them as solutions of the second order equation (1.26) for $k = 1$, which satisfies

$$\begin{aligned} \ddot{v}_1 &= v_1 - \frac{v_1^3}{4} + \mathcal{O}_{\ell_1}(\widetilde{\Pi} v) + \mathcal{O}(\varepsilon^2) \\ \varepsilon^2 \ddot{v}_n &= -\lambda_n^2 v_n + \mathcal{O}_{\ell_1}(\varepsilon^2 v^3), \quad n \geq 2, \end{aligned}$$

for small $\widetilde{\Pi} v$, where we have introduced the following notation, which is also used in the forthcoming sections

$$\widetilde{\Pi}(v) = v - \Pi_1(v) \sin \tau = \sum_{n \geq 2} v_n \sin(n\tau). \tag{2.7}$$

Imposing decay at infinity (as $y \to +\infty$ for $W^s(0)$, as $y \to -\infty$ for $W^u(0)$) and $\partial_y v_1^{u,s}(0) = 0$, item (1) of Theorem 2.1 looks natural: the distance between the perturbed and unperturbed manifolds $(v_1, \widetilde{\Pi} v) = (v^h, 0)$ is of the same order as the perturbation (notice the singular character of the model and the different size of each component of the vector field). These estimates can be proven through a fixed point argument by using the standard Perron method.

Even if the perturbed invariant manifolds are $\mathcal{O}(\varepsilon^2)$ close to the unperturbed ones, the singular character of the model makes their difference beyond all orders in ε, in fact exponentially small. Let us give some heuristic ideas of why this phenomenon happens. We have chosen parameterizations such that $\partial_y v_1^{u,s}(0) = 0$. Moreover, as the system conserves the Hamiltonian, both manifolds belong to the energy level of the saddle-center critical point $v = 0$. Therefore, the difference $v_1^u - v_1^s$ at $y = 0$ can be recovered from the differences projected to the rest of directions, namely $\widetilde{\Pi}(v^u - v^s)$ and $\widetilde{\Pi}(\partial_y v^u - \partial_y v^s)$. Thus, we focus on measuring these differences. Let us write the equations for these components as a first order equation for $n \geq 3$ (recall that $\Pi_{2l} v^{u,s} = 0$ for $l \geq 0$),

$$\begin{aligned} \dot{v}_n &= w_n \\ \dot{w}_n &= -\frac{\lambda_n^2}{\varepsilon^2} v_n + \frac{1}{\varepsilon^3\omega^3}\Pi_n\left[g(\varepsilon\omega v)\right]. \end{aligned}$$

As the parameterizations of both invariant manifolds satisfy the same equation, their difference

$$(\Delta_n, \Xi_n) \triangleq (v_n^u - v_n^s, \partial_y v_n^u - \partial_y v_n^s)$$

satisfies a linear equation for $n \geq 3$,

$$\begin{aligned} \dot{\Delta}_n &= \Xi_n \\ \dot{\Xi}_n &= -\frac{\lambda_n^2}{\varepsilon^2}\Delta_n + \Pi_n\left[M(v^u, v^s)\Delta\right]. \end{aligned}$$

Since the last term is much smaller than the oscillating one, to give a heuristic idea of the phenomenon taking place, let us assume that $M = 0$. Then, the system becomes a linear system of constant coefficients which we can diagonalize by taking

$$\begin{aligned} \Gamma_n &= \lambda_n\Delta_n + i\varepsilon\Xi_n \\ \Theta_n &= \lambda_n\Delta_n - i\varepsilon\Xi_n \end{aligned} \tag{2.8}$$

to obtain

$$\begin{aligned} \dot{\Gamma}_n &= -i\frac{\lambda_n}{\varepsilon}\Gamma_n \\ \dot{\Theta}_n &= i\frac{\lambda_n}{\varepsilon}\Theta_n. \end{aligned}$$

The solutions of this system can be easily computed as

$$\begin{aligned} \Gamma_n(y) &= e^{-i\frac{\lambda_n}{\varepsilon}(y-y^+)}\Gamma_n(y^+) \\ \Theta_n(y) &= e^{i\frac{\lambda_n}{\varepsilon}(y-y^-)}\Theta_n(y^-) \end{aligned}$$

for any points $y^\pm$.

By the definition of (Γ_n, Θ_n) in (2.8), one has

$$\begin{aligned} \Gamma_n(y^+) &= \lambda_n(v_n^u(y^+) - v_n^s(y^+)) + i\varepsilon(\partial_y v_n^u(y^+) - \partial_y v_n^s(y^+)) \\ \Theta_n(y^-) &= \lambda_n(v_n^u(y^-) - v_n^s(y^-)) - i\varepsilon(\partial_y v_n^u(y^-) - \partial_y v_n^s(y^-)). \end{aligned}$$

The main observation here is that, if we are able to extend both the stable and unstable manifolds $v_n^{u,s}(y)$ to some complex values $y^\pm = \pm i\sigma$, $\sigma > 0$, one obtains the following estimates for $y \in \mathbb{R}$ near $y = 0$,

$$\begin{aligned} |\Gamma_n(y)| &\leq e^{-\frac{\lambda_n\sigma}{\varepsilon}}|\Gamma_n(i\sigma)| \\ |\Theta_n(y)| &\leq e^{-\frac{\lambda_n\sigma}{\varepsilon}}|\Theta_n(-i\sigma)|, \end{aligned}$$

which are exponentially small in ε and strongly depend on the size of the unstable/stable solutions at the complex points $y^\pm = \pm i\sigma$.

For the nonlinear system, we will find the solutions

$$v_n^s(y) \text{ for } \operatorname{Re} y \geq 0, \qquad v_n^u(y) \text{ for } \operatorname{Re} y \leq 0$$

as perturbations of the singular limit solution $v_1 = v^h(y)$, $v_n = 0$, $n \geq 2$, where $v^h(y)$ is the unperturbed homoclinic solution (1.30). As this function has poles of order one at the points $y^\pm = \pm i\pi/2$, it is natural to expect that the optimal value for σ is in a neighborhood on the lower side of $\pi/2$. In Theorem 3.1 to be proved in Section 4 we show that, for y close to $\pm i\pi/2$,

$$v^{s,u}(y,\tau) = v^h(y)\sin\tau + \mathcal{O}_{\ell_1}\left(\frac{\varepsilon^2}{|y^2 + \frac{\pi^2}{4}|^3}\right) = \frac{2\sqrt{2}}{\cosh y}\sin\tau + \mathcal{O}_{\ell_1}\left(\frac{\varepsilon^2}{|y^2 + \frac{\pi^2}{4}|^3}\right). \tag{2.9}$$

Therefore, we see that

$$|v^{u,s}(y^\pm)| \lesssim \frac{1}{\varepsilon}, \quad \text{at } \; y_\pm = \pm i\sigma, \quad \text{with } \sigma = \frac{\pi}{2} - \kappa\varepsilon \text{ and some } \; \kappa > 0.$$

Consequently, $|\Gamma_n(y^+)| \lesssim \frac{1}{\varepsilon}$, $|\Theta_n(y^-)| \lesssim \frac{1}{\varepsilon}$. Therefore, one expects that for $y \in \mathbb{R}$ close to $y = 0$,

$$|\Gamma_n(y)| \lesssim \frac{1}{\varepsilon} e^{-\frac{\lambda_n \pi}{2\varepsilon}}, \qquad |\Theta_n(y)| \lesssim \frac{1}{\varepsilon} e^{-\frac{\lambda_n \pi}{2\varepsilon}}.$$

As $\lambda_n \geq \lambda_3 = 2\sqrt{2} + \mathcal{O}(\varepsilon)$ for $n \geq 3$, one obtains an upper bound for the difference

$$|\Gamma_n(y)| \lesssim \frac{1}{\varepsilon} e^{-\frac{\sqrt{2}\pi}{\varepsilon}}, \qquad |\Theta_n(y)| \lesssim \frac{1}{\varepsilon} e^{-\frac{\sqrt{2}\pi}{\varepsilon}}$$

and similar bounds are satisfied by $\Delta_n(y) = v_n^u(y) - v_n^s(y)$.

Observe that these bounds fit with the estimates (2.5) given in Theorem 2.1. However, Theorem 2.1 gives certainly more information since it provides an asymptotic formula for Δ.

2.2. **Strategy of the proof of Theorem 2.1; exponentially small asymptotics.** As we have seen in Section 2.1, to obtain an asymptotic formula of $v^s - v^u$, one needs a deeper study of these functions near $y^\pm = \pm i(\pi/2 - \kappa\varepsilon)$ for some $\kappa > 0$.

According to (2.9), for real values of y, the invariant manifolds are ε^2-perturbations of the unperturbed homoclinic orbit, but, when $y \mp \frac{i\pi}{2} = \mathcal{O}(\varepsilon)$ we have that both the homoclinic and error term become of the same size $\mathcal{O}(\frac{1}{\varepsilon})$ and therefore $v^{s,u}(y,\tau)$ are not well approximated by the homoclinic solution $v^h(y)\sin\tau$ anymore. Thus, we look for suitable leading orders of $v^{s,u}(y,\tau)$ for y such that $y \mp \frac{i\pi}{2} = \mathcal{O}(\varepsilon)$.

We focus on the singularity $y = i\pi/2$ (the same analysis can be performed near the singularity $y = -i\pi/2$ analogously). We proceed as follows. We perform the singular change to the *inner variable*

$$z = \varepsilon^{-1}\left(y - i\frac{\pi}{2}\right)$$

and the scaling

$$\phi(z,\tau) = \varepsilon v\left(i\frac{\pi}{2} + \varepsilon z, \tau\right).$$

From (1.23), one can deduce the equation satisfied by $\phi(z,\tau)$,

$$\partial_z^2\phi - \partial_\tau^2\phi - \frac{1}{\omega^2}\phi + \frac{1}{3}\phi^3 + \frac{1}{\omega^3}f(\omega\phi) = 0, \qquad \omega = \frac{1}{\sqrt{1+\varepsilon^2}}.$$

The first order of this equation corresponds to the regular limit $\varepsilon = 0$, which gives the so-called *inner equation*

$$\partial_z^2\phi^0 - \partial_\tau^2\phi^0 - \phi^0 + \frac{1}{3}(\phi^0)^3 + f(\phi^0) = 0.$$

The estimates (2.9) show, that, after these changes, the stable/unstable manifolds behave as

$$\phi^{s,u}(z,\tau) = -\frac{2\sqrt{2}i}{z}\sin\tau + \mathcal{O}_{\ell_1}\left(\frac{1}{z^3}\right).$$

Therefore, it is natural to look for solutions of the inner equation which "match" these asymptotics. This is done in Theorem 3.3 below where we obtain and analyze two solutions $\phi^{0,u}$, $\phi^{0,s}$ of the inner equation which are the first order of the unstable/stable manifolds "close to the singularity" $y = i\pi/2$. They are of the form

$$\begin{aligned} \phi^{0,s}(z,\tau) &= -\frac{2\sqrt{2}i}{z}\sin\tau + \psi^s(z,\tau), \text{ for } \operatorname{Re} z > 0 \\ \phi^{0,u}(z,\tau) &= -\frac{2\sqrt{2}i}{z}\sin\tau + \psi^u(z,\tau), \text{ for } \operatorname{Re} z < 0, \end{aligned} \tag{2.10}$$

with $\psi^{s,u} = \mathcal{O}(\frac{1}{z^3})$ in suitable complex domains satisfying $|z| \geq \kappa$ and containing the negative imaginary axis $\Im z \leq -\kappa$ (recall that $z = \varepsilon^{-1}(y - \frac{i\pi}{2})$ and therefore $y = 0$ lies on this negative imaginary axis of z). Again these solutions contain only odd modes in τ.

Moreover, in Theorem 3.3 we provide a formula for the difference of these two solutions which reads

$$\phi^{0,u}(-ir,\tau) - \phi^{0,s}(-ir,\tau) = e^{-2\sqrt{2}r}\left(C_{\text{in}}\sin(3\tau) + \mathcal{O}_{\ell_1}\left(\frac{1}{r}\right)\right) \qquad \text{as} \qquad r \to +\infty. \tag{2.11}$$

This asymptotic formula can be obtained relying on different techniques. In the literature, one can find proofs relying either on fixed point arguments for the difference [3, 10] or on Borel resummation techniques and Écalle Resurgence Theory [54] (applied to finite dimensional inner equations). In the present paper, we obtain this formula through a different method relying on invariant manifolds and foliations for a an ill-posed associated PDE, which is more in line with the techniques used throughout the paper. Let us give here a (very) heuristic idea of the origin of this result from this new point of view.

Writing a solution of the inner equation as $\phi^0 = \sum \phi_n^0 \sin(n\tau)$ we obtain

$$\begin{aligned} \frac{d^2}{dz^2}\phi_1^0 - \frac{1}{4}(\phi_1^0)^3 &= F_1\left(\phi^0\right) \\ \frac{d^2}{dz^2}\phi_n^0 + \mu_n^2\phi_n^0 &= F_n\left(\phi^0\right),\ n \geq 3, \end{aligned} \tag{2.12}$$

where $' = d/dz$, $\mu_n = \sqrt{n^2-1}$, and F_n contain higher order terms.

Let us assume that $F_n = 0$ to give an heuristic idea of the process. First, let us make the change $z = -ir$ and write system (2.12) as a first order system through the change

$$\Psi_c(r) = \left(\phi_1^0(-ir), -i\frac{d}{dz}\phi_1^0(-ir)\right), \quad \Psi_{n,\pm} = -i\frac{d}{dz}\phi_n^0(-ir) \pm \sqrt{n^2-1}\phi_n^0(-ir), \quad \Psi^{\pm} = (\Psi_{2l-1,\pm})_{l=2}^{\infty},$$

which gives

$$\begin{aligned} \frac{d}{dr}\Psi_c^1 &= \Psi_c^2 & \qquad \frac{d}{dr}\Psi_{n,-} &= -\sqrt{n^2-1}\Psi_{n,-} \\ \frac{d}{dr}\Psi_c^2 &= \frac{1}{4}\left(\Psi_c^1\right)^3 & \qquad \frac{d}{dr}\Psi_{n,+} &= \sqrt{n^2-1}\Psi_{n,+}. \end{aligned}$$

Observe that $\Psi = 0$ is a critical point with a center manifold W^c given by $\Psi^+ = \Psi^- = 0$, and a center-stable manifold W^{cs} given by $\Psi^+ = 0$. Moreover, W^{cs} possesses the classical stable foliation. Indeed, given a point $\Psi = (\Psi^c, \Psi^-, 0) \in W^{cs}$ there exists a point $\Psi_b = (\Psi^c, 0, 0) \in W^c$ such that $|\Phi_r(\Psi) - \Phi_r(\Psi_b)| \leq \mathcal{O}(e^{-2r})$, as $2 \in (0, \sqrt{3^2-1})$, where Φ_r is the flow on W^{cs} which is well defined for $r \geq 0$. The points whose trajectories are asymptotic to a given $\Psi_b \in W^c$ form a leaf of the foliation.

This foliation allows us to give an asymptotic formula for $\phi^{0,s}(z) - \phi^{0,u}(z)$:

- The first observation is that our solutions $\phi^{0,s}(z)$, $\phi^{0,u}(z)$, when restricted to the negative imaginary axis away from 0 and written in these coordinates, correspond to $\Psi^{u,s}(r) = (\Psi_c^{u,s}, \Psi_-^{u,s}, \Psi_+^{u,s})(r)$ satisfying
$$\lim_{r\to+\infty} \Psi^{s,u}(r) = 0.$$
Therefore, they belong to W^{cs} and, in this simplified model, should have the "unstable coordinate" $\Psi_+^{u,s}(r) \equiv 0$.
- The second observation is that we know, by (2.10), that
$$|\Psi^u(r) - \Psi^s(r)| \leq \mathcal{O}_{\ell_1}\left(\frac{1}{r^3}\right),\ \text{as } r \to +\infty,$$
which implies that they should have the same "central coordinate" ($\Psi_c^u(r) = \Psi_c^s(r)$ in this simplified model) and therefore they belong to the same leaf in the stable foliation. One can see this fact using the linearized fundamental solutions in the central coordinates which give: $\Psi_c^u(r) - \Psi_c^s(r) \sim c_1 r^{-2} + c_2 r^3$ and the decay of this difference immediately gives $c_1 = c_2 = 0$.
- Now that we know that $\Psi^{u,s}(r) = (\Psi_c(r), \Psi_-^{u,s}(r), 0)$, we only need to compute the difference in the stable coordinate $\beta_-(r) = \Psi_-^u(r) - \Psi_-^s(r)$ which satisfies:
$$\frac{d}{dr}\beta_- = A\beta_-, \quad A = \text{diag}(-\sqrt{n^2-1})$$
and this immediately implies that
$$\beta_-(r) = e^{(r-r_0)A}\beta_-(r_0) = e^{-2\sqrt{2}(r-r_0)}\beta_{3,-}(r_0)\sin(3\tau) + \mathcal{O}_{\ell_1}\left(e^{-3r}\right).$$
Calling $C = e^{2\sqrt{2}r_0}\beta_{3,-}(r_0)$ we have
$$\lim_{r\to+\infty} e^{2\sqrt{2}r}\beta_-(r) - C\sin 3\tau = 0.$$

Using these ideas, in Theorem 3.3 below, we incorporate the dismissed higher order terms (see (2.12)) and give a complete proof of the asymptotic formula for the difference between the solutions of the inner equation. Note that the constant C above corresponds to the constant C_{in} in (2.11).

Once we obtain the difference between the inner solutions $\phi^{0,u}-\phi^{0,s}$, we must show that this difference gives indeed a first order of the difference between the perturbed invariant manifolds. That is, we must estimate the function $(\phi^u-\phi^s)-(\phi^{0,u}-\phi^{0,s})$ in some appropriate complex domain. To this end, it starts with showing that the solutions of the inner equation $\phi^{0,u}(z)$, $\phi^{0,s}(z)$, when written in the original variables $y=i\frac{\pi}{2}+\varepsilon z$, are good approximations of the stable and unstable solutions v^u, v^s for y satisfying $y\mp\frac{i\pi}{2}=\mathcal{O}(\varepsilon)$. Such analysis is done in Theorem 3.6.

From such estimates, applying the ideas in Section 2.1, we obtain smaller exponentially small errors at $y=0$. This shows that the difference of $\phi^{0,u}-\phi^{0,s}$ provides the main term of the exponentially small distance between v^u and v^s and that the first order of this distance is given by the Stokes constant C_{in}.

3. Description of the Proof of Theorem 2.1

We describe the main steps of the proof of Theorem 2.1 where $k=1$ and the odd symmetry of functions in τ is assumed.

3.1. **Estimates of the invariant manifolds in complex domains.** In order to estimate the distance between the perturbed invariant manifolds $W^s(0)$ and $W^u(0)$ in Σ, we consider suitable parameterizations for them. Since the invariant manifolds $W^u(0)$ and $W^s(0)$ are one dimensional, they are the images of solutions v^u and v^s of (1.26) with the asymptotic conditions

$$\lim_{y\to+\infty} v^s(y,\tau)=\lim_{y\to-\infty} v^u(y,\tau)=0, \qquad \text{for all } \tau\in\mathbb{T}.$$

We write equation (1.26) as

$$\begin{cases} \ddot{v}_1 = v_1 - \dfrac{v_1^3}{4} + \left(-\dfrac{1}{\varepsilon^3\omega^3}\Pi_1\left[g(\varepsilon\omega v)\right] + \dfrac{v_1^3}{4}\right), \\ \ddot{v}_n = -\dfrac{\lambda_n^2}{\varepsilon^2}v_n - \dfrac{1}{\varepsilon^3\omega^3}\Pi_n\left[g(\varepsilon\omega v)\right],\ n\geq 2, \end{cases}$$

where Π_n is the Fourier projection given by (2.3) and g is given by (1.19).

We study the solutions v^u, v^s as perturbations of the homoclinic orbit $v^h(y)\sin\tau$ given by (1.30), which satisfies $\ddot{v}^h=v^h-(v^h)^3/4$. Thus, we set

$$\xi(y,\tau)=v(y,\tau)-v^h(y)\sin\tau=\sum_{n\geq 1}\xi_n(y)\sin(n\tau),$$

whose Fourier coefficients satisfy

$$\begin{cases} \ddot{\xi}_1 = \xi_1 - \dfrac{3(v^h)^2\xi_1}{4} - \dfrac{3v^h\xi_1^2}{4} - \dfrac{\xi_1^3}{4} + \left(-\dfrac{1}{\varepsilon^3\omega^3}\Pi_1(g(\varepsilon\omega(\xi+v^h\sin\tau))) + \dfrac{(\xi_1+v^h)^3}{4}\right), \\ \ddot{\xi}_n = -\dfrac{\lambda_n^2}{\varepsilon^2}\xi_n - \dfrac{1}{\varepsilon^3\omega^3}\Pi_n(g(\varepsilon\omega(\xi+v^h\sin\tau))), \qquad n\geq 2. \end{cases}$$

Define the operators

$$\mathcal{L}(\xi)=\left(\ddot{\xi}_1-\xi_1+\frac{3(v^h)^2\xi_1}{4}\right)\sin\tau+\sum_{n\geq 2}\left(\ddot{\xi}_n+\frac{\lambda_n^2}{\varepsilon^2}\xi_n\right)\sin(n\tau), \tag{3.1}$$

$$\mathcal{F}(\xi)=-\frac{1}{\varepsilon^3\omega^3}g(\varepsilon\omega(\xi+v^h\sin\tau))+\left(\frac{(\xi_1+v^h)^3}{4}-\frac{3v^h\xi_1^2}{4}-\frac{\xi_1^3}{4}\right)\sin\tau. \tag{3.2}$$

To obtain solutions $v^\star$, $\star=u,s$, of (1.23) satisfying (1.27) is equivalent to find solutions $\xi^\star$ of the functional equation

$$\mathcal{L}(\xi)=\mathcal{F}(\xi), \tag{3.3}$$

satisfying

$$\lim_{y\to-\infty}\xi^u(y,\tau)=\lim_{y\to\infty}\xi^s(y,\tau)=0, \qquad \text{for all} \quad \tau\in\mathbb{T}. \tag{3.4}$$

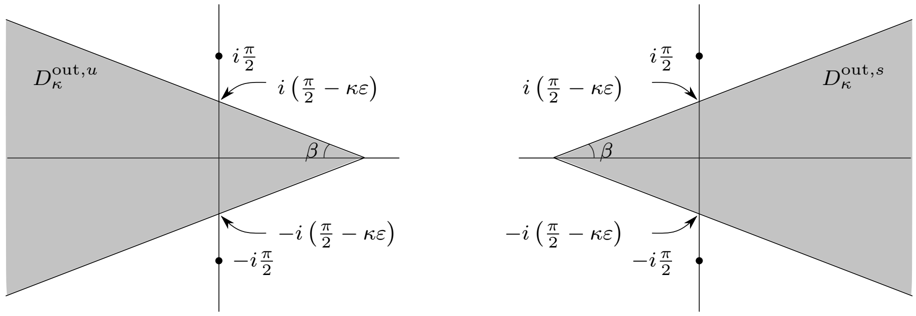

FIGURE 5. Outer domains $D_\kappa^{\mathrm{out},u}$ and $D_\kappa^{\mathrm{out},s}$.

We analyze these parameterizations in the following complex sectorial domains, usually called *outer domains*,

$$
\begin{aligned}
D_\kappa^{\mathrm{out},u} &= \left\{y\in\mathbb{C}; |\mathrm{Im}(y)|\le -\tan\beta\,\mathrm{Re}(y)+\frac{\pi}{2}-\kappa\varepsilon\right\}\\
D_\kappa^{\mathrm{out},s} &= \left\{y\in\mathbb{C}; -y\in D_\kappa^{\mathrm{out},u}\right\},
\end{aligned}
\tag{3.5}
$$

where $0<\beta<\pi/4$ is a fixed angle independent of ε and $\kappa\ge 1$ (see Figure 5). Observe that $D_\kappa^{\mathrm{out},*}$, $*=u,s$, reach domains at a $\kappa\varepsilon$–distance of the singularities $y=\pm i\pi/2$ of v^h (see Section 2.1).

The next theorem proves the existence and estimates of the functions ξ^u,ξ^s. It is proven in Section 4.

Theorem 3.1 (Outer). *Consider the equation* (1.23) *with $k=1$. There exist $\kappa_0\ge 1$ big enough and $\varepsilon_0>0$ small enough, such that, for each $0<\varepsilon\le\varepsilon_0$ and $\kappa\ge\kappa_0$, the invariant manifolds $W^\star(0)\subset\mathbf{X}_o$ of* (1.23)*, $\star=u,s$, are parameterized as unique solutions of equation* (1.23) *by*

$$v^\star(y,\tau)=v^h(y)\sin\tau+\xi^\star(y,\tau),\ y\in D_\kappa^{\mathrm{out},\star},\ \tau\in\mathbb{T},$$

where v^h is given by (1.30) *and $\xi^\star: D_\kappa^{\mathrm{out},\star}\times\mathbb{T}\to\mathbb{C}$ are functions real-analytic in the variable y such that*

(1) *They satisfy the asymptotic condition* (3.4)*, $\partial_y\Pi_1[\xi^\star](0)=0$ and $\Pi_{2l}[\xi^\star](y)\equiv 0$ for $l\in\mathbb{N}$.*
(2) *There exists a constant $M_1>0$ independent of ε and κ, such that*

$$
\begin{aligned}
\|\xi^\star\|_{\ell_1}(y) &\le \frac{M_1\varepsilon^2}{|\cosh(y)|} \qquad \text{for} \quad y\in D_\kappa^{\mathrm{out},\star}\cap\{|\,\mathrm{Re}(y)|>1\}\\
\|\xi^\star\|_{\ell_1}(y) &\le \frac{M_1\varepsilon^2}{|y^2+\pi^2/4|^3} \qquad \text{for} \quad y\in D_\kappa^{\mathrm{out},\star}\cap\{|\,\mathrm{Re}(y)|\le 1\}.
\end{aligned}
$$

Moreover, the derivatives of $\xi^\star$ can be bounded as

(1) *For $y\in D_\kappa^{\mathrm{out},\star}\cap\{|\,\mathrm{Re}(y)|>1\}$,*

$$\left\|\partial_\tau^2\xi^\star\right\|_{\ell_1}(y),\ \|\partial_\tau^2\partial_y\xi^\star\|(y)\le\frac{M_1\varepsilon^2}{|\cosh(y)|}.$$

(2) *For $y\in D_\kappa^{\mathrm{out},\star}\cap\{|\,\mathrm{Re}(y)|\le 1\}$,*

$$\left\|\partial_\tau^2\xi^\star\right\|_{\ell_1}(y)\le\frac{M_1\varepsilon^2}{|y^2+\pi^2/4|^3}\quad\text{and}\quad\left\|\partial_\tau^2\partial_y\xi^\star\right\|_{\ell_1}(y)\le\frac{M_1\varepsilon^2}{|y^2+\pi^2/4|^4}.$$

Remark 3.2. *While the 1-dim stable and unstable manifolds of the equilibrium 0 are determined by their exponential asymptotic behavior as $y\to\pm\infty$ where the freedom of translation in y is fixed by $\partial_y\Pi_1[\xi^{u,s}]=0$, it is important that the precise order of the error $\xi^{u,s}=\mathcal{O}(\frac{\varepsilon^2}{|y^2+\frac{\pi^2}{4}|^3})$ is obtained near the singularity $y=\pm\frac{\pi}{2}i$. This does not only allows one to identify the correct scaling leading to the limit of the inner equation in the next subsection, but also uniquely fix the solutions of the inner equation optimally approximating $v^{u,s}$.*

3.2. **Analysis close to the singularities.** Notice that the parameterizations $v^\star(y,\tau)$ of $W^\star(0)$, $\star = u, s$ given by Theorem 3.1, are ε^2-close to the homoclinic orbit $v^h(y)\sin(\tau)$ for $y \in \mathbb{R} \cap D_\kappa^{\mathrm{out},\star}$. Nevertheless, at distance $\mathcal{O}(\varepsilon)$ of the poles $y = \pm i\pi/2$ of v^h, $v^h \sim \varepsilon^{-1}$ has comparable size to the error $\xi^\star \sim \varepsilon^{-1}$.

To obtain a first order approximation of the invariant manifolds at distance $\mathcal{O}(\varepsilon)$ of the poles $y = \pm i\pi/2$ we proceed as follows. We focus on the singularity $y = i\pi/2$ since similar results can be derived near the singularity $y = -i\pi/2$ by the conjugacy. Consider the *inner variable*

$$z = \varepsilon^{-1}\left(y - i\frac{\pi}{2}\right) \tag{3.6}$$

and the scaling

$$\phi(z,\tau) = \varepsilon v\left(i\frac{\pi}{2} + \varepsilon z, \tau\right). \tag{3.7}$$

Writing equation (1.23) for $\phi(z,\tau)$ and recalling $\omega = (1+\varepsilon^2)^{-\frac{1}{2}}$, we obtain

$$\partial_z^2\phi - \partial_\tau^2\phi - \frac{1}{\omega^2}\phi + \frac{1}{3}\phi^3 + \frac{1}{\omega^3}f(\omega\phi) = 0. \tag{3.8}$$

This equation coincides with the original Klein-Gordon equation (1.17)[4]. However, notice that now the evolution variable is $z = x - i\dfrac{\pi}{2\varepsilon}$.

The first order of (3.8) corresponds to the regular limit $\varepsilon = 0$, which gives the so-called *inner equation*

$$\partial_z^2\phi^0 - \partial_\tau^2\phi^0 - \phi^0 + \frac{1}{3}(\phi^0)^3 + f(\phi^0) = 0. \tag{3.9}$$

We are interested in identifying certain solutions of (3.9) with the same first order of the outer solutions $v^{u,s}(y,\tau)$ given in Theorem 3.1 near the pole $y = i\pi/2$. Therefore, we look for solutions $\phi^{0,*}(z,\tau)$, $* = u, s$, of (3.9) which have the same leading order expansion as $\phi^{u,s}(z,\tau) = \varepsilon v^{u,s}\left(i\left(\frac{\pi}{2} + \varepsilon z\right), \tau\right)$. Near the pole $y = i\pi/2$, by Theorem 3.1 we have

$$v^{u,s}(y,\tau) = v^h(y)\sin\tau + \mathcal{O}\left(\frac{\varepsilon^2}{(y - i\pi/2)^3}\right) = \frac{-2\sqrt{2}i}{y - i\pi/2}\sin\tau + \mathcal{O}(y - i\pi/2) + \mathcal{O}\left(\frac{\varepsilon^2}{(y - i\pi/2)^3}\right)$$

which, in the inner variables (3.6) and (3.7), corresponds to

$$\phi^{u,s}(z,\tau) = \frac{-2\sqrt{2}i}{z}\sin\tau + \mathcal{O}(\varepsilon^2 z) + \mathcal{O}\left(\frac{1}{z^3}\right).$$

Taking into account the change of variables (3.6) and the shape of the outer domains (3.5), this asymptotic condition must hold for $\operatorname{Im} z < 0$ and $\operatorname{Re} z < 0$ for ϕ^u and for $\operatorname{Im} z < 0$ and $\operatorname{Re} z > 0$ for ϕ^s.

Therefore we consider the *inner domains*

$$\begin{aligned} D_{\theta,\kappa}^{u,\mathrm{in}} &= \{z \in \mathbb{C}; |\operatorname{Im}(z)| > \tan\theta \operatorname{Re}(z) + \kappa\}, \\ D_{\theta,\kappa}^{s,\mathrm{in}} &= \{z \in \mathbb{C}; -z \in D_{\theta,\kappa}^{u,\mathrm{in}}\}, \end{aligned} \tag{3.10}$$

for $0 < \theta < \pi/6$ and $\kappa > 0$ (see Figure 6), and we look for solutions of the inner equation of the form

$$\phi^{0,\star}(z,\tau) = \frac{-2\sqrt{2}i}{z}\sin\tau + \psi^\star(z,\tau), \quad \text{with} \quad \psi^\star = \mathcal{O}\left(\frac{1}{z^3}\right), \quad \text{for} \quad (z,\tau) \in D_{\theta,\kappa}^{\star,\mathrm{in}} \times \mathbb{T},\ \star = u, s.$$

We present the results concerning the existence of these solutions $\phi^{0,\star}$ of (3.8), $\star = u, s$. Moreover we give an asymptotic expression for the difference $\phi^{0,u}(z,\tau) - \phi^{0,s}(z,\tau)$ as $\operatorname{Im}(z) \to -\infty$, which will be crucial to compute the first order of the difference $v^u - v^s$. The following theorem will be proved in Section 5.

Theorem 3.3 (Inner). *Let $\theta \in (0, \frac{\pi}{6})$ and $r > 0$ be fixed and consider $f \in \mathcal{F}_r$. There exists $\kappa_0 \geq 1$ big enough such that, for each $\kappa \geq \kappa_0$,*

[4]Warning: It is the original one for $\psi = \omega\phi$, but will be analyzed near a singular complex function.

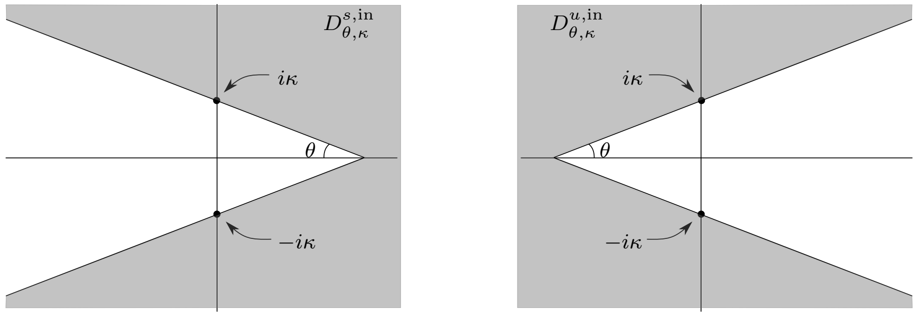

FIGURE 6. Inner domains $D^{s,\mathrm{in}}_{\theta,\kappa}$ and $D^{u,\mathrm{in}}_{\theta,\kappa}$.

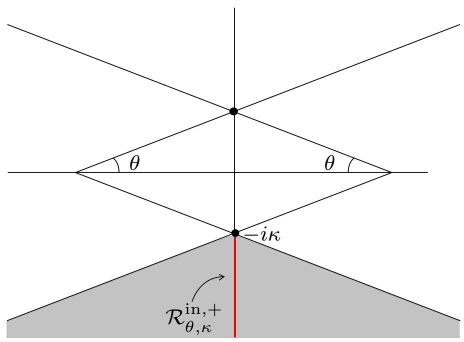

FIGURE 7. Domain $\mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}$.

(1) *Equation (3.9) has two solutions $\phi^{0,\star} : D^{\star,\mathrm{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C}$, $\star = u, s$, given by*

$$\phi^{0,\star}(z,\tau) = -\frac{2\sqrt{2}i}{z}\sin\tau + \psi^\star(z,\tau), \tag{3.11}$$

which are analytic in the variable z. Moreover, $\Pi_{2l}\left[\phi^{0,\star}\right] \equiv 0$ for every $l \in \mathbb{N}$, and there exists a constant $M_2 > 0$ independent of κ such that, for every $z \in D^{\star,\mathrm{in}}_{\theta,\kappa}$ and $z' \in D^{\star,\mathrm{in}}_{2\theta,4\kappa}$

$$\left\|\partial_\tau^2\psi^\star\right\|_{\ell_1}(z) \le \frac{M_2}{|z|^3}, \quad \left\|\partial_\tau^2\partial_z\psi^\star\right\|_{\ell_1}(z') \le \frac{M_2}{|z'|^4}. \tag{3.12}$$

(2) *The difference $\Delta\phi^0(z,\tau) = \phi^{0,u}(z,\tau) - \phi^{0,s}(z,\tau)$ is given by (see Figure 7),*

$$\Delta\phi^0(z,\tau) = e^{-i\mu_3 z}\left(C_{\mathrm{in}}\sin(3\tau) + \chi(z,\tau)\right), \quad z \in \mathcal{R}^{\mathrm{in},+}_{\theta,\kappa} = D^{u,\mathrm{in}}_{\theta,\kappa} \cap D^{s,\mathrm{in}}_{\theta,\kappa} \cap \{z;\ \mathrm{Re}(z) = 0, \mathrm{Im}(z) < 0\} \tag{3.13}$$

where $\mu_3 = 2\sqrt{2}$, $C_{\mathrm{in}} \in \mathbb{C}$ is a constant, and χ is analytic in z and satisfies that, for $z \in \mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}$,

$$\|\partial_\tau\chi\|_{\ell_1}(z) \le \frac{M_2}{|z|} \quad \text{and} \quad \|\partial_z\chi\|_{\ell_1}(z) \le \frac{M_2}{|z|^2}.$$

(3) *The constant $C_{\mathrm{in}} = C_{\mathrm{in}}(f) \in \mathbb{C}$ depends analytically on $f \in \mathcal{F}_r$.*

Remark 3.4. *It is interesting to see that the stable and unstable solutions $\phi^{0,\star}$, $\star = u, s$, are identified by the $\mathcal{O}(|z|^{-1})$ decay as $\mathrm{Re}\, z \to \pm\infty$, where the same Lyapunov-Perron approach works. The freedom of translation in z, which causes a variation of the order $\mathcal{O}(|z|^{-2})$ is fixed by the $\mathcal{O}(|z|^{-3})$ restriction of the error terms. The splitting $\Delta\phi^0$ between $\phi^{0,u}$ and $\phi^{0,s}$ would turn out to be the principal part of the splitting between v^u and v^s. The leading order form of $\Delta\phi^0$ can be understood in two different perspectives. On the one hand, it is related to the Borel summation of divergent power series and the readers are referred to Section 11.2 for related discussions and our conjecture on how to compute C_{in}. On the other hand, along the real direction of z, the inner equation (3.9) is hyperbolic in the PDE sense and oscillatory. However, when we view it along the imaginary axis, it becomes strongly hyperbolic in the dynamical systems sense and elliptic in the PDE*

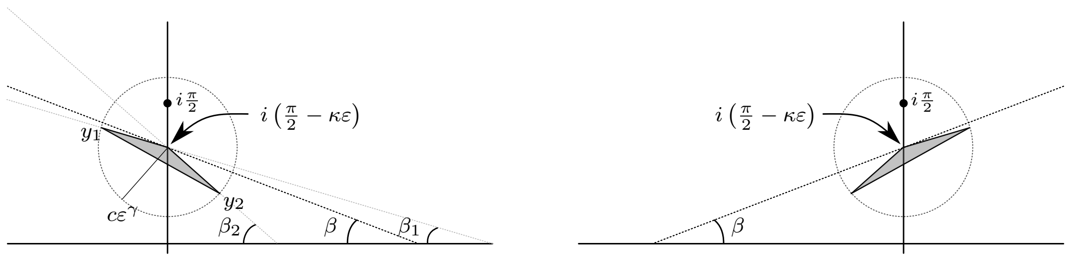

FIGURE 8. Matching domains $D^{\mathrm{mch},u}_{+,\kappa}$ (on the left) and $D^{\mathrm{mch},s}_{+,\kappa}$ (on the right).

sense (and dynamically ill-posed). All the originally oscillatory directions become hyperbolic in the dynamical systems sense and thus in particular the stable manifolds become infinite dimensional containing $\phi^{0,\star}$. The splitting $\Delta\phi^0$ is dominated by the weakest exponential decay rate and the Stokes constant C_{in} basically comes from the difference between the weakest stable coordinates of $\phi^{0,u}$ and $\phi^{0,s}$.

Remark 3.5. *The constant $C_{\mathrm{in}}(f)$ in Theorem 3.3 is the constant appearing in Theorem 2.1. Theorem 3.3(3) provides its analyticity with respect to f as stated in Theorem 1.4. In Section 11 we prove that, typically, it does not vanish.*

Our next step is to prove that the solutions of the inner equation obtained in Theorem 3.3 are good approximations of the parameterizations $v^*(y,\tau)$, $*=u,s$, obtained in Theorem 3.1 near the pole $y=i\pi/2$. To prove this fact we introduce the following *matching domains*.

Take $0<\gamma<1$, $0<\beta_1<\beta<\beta_2<\pi/4$ constants independent of ε and κ. Then, we consider the points $y_j\in\mathbb{C}$, $j=1,2$ satisfying

(1) $\mathrm{Im}(y_j)=-\tan\beta_j\,\mathrm{Re}(y_j)+\pi/2-\kappa\varepsilon$;
(2) $|y_j-i(\pi/2-\kappa\varepsilon)|=\varepsilon^\gamma$;
(3) $\mathrm{Re}(y_1)<0$ and $\mathrm{Re}(y_2)>0$.
(4) $e^{5(\pi-\beta_1)}-e^{-5\beta_2}\neq 0$.

Note that $\mathrm{Im}(y_2)<\frac{\pi}{2}-\kappa\varepsilon<\mathrm{Im}(y_1)$. Then, consider the following *matching domains* (see Figure 8),

$$
\begin{aligned}
D^{\mathrm{mch},u}_{+,\kappa} &= \Big\{y\in\mathbb{C};\ \mathrm{Im}(y)\le -\tan\beta_1\,\mathrm{Re}(y)+\pi/2-\kappa\varepsilon,\ \mathrm{Im}(y)\le -\tan\beta_2\,\mathrm{Re}(y)+\pi/2-\kappa\varepsilon,\\
&\qquad \mathrm{Im}(y)\ge \mathrm{Im}(y_1)-\tan\left(\frac{\beta_1+\beta_2}{2}\right)(\mathrm{Re}(y)-\mathrm{Re}(y_1))\Big\},\\
D^{\mathrm{mch},s}_{+,\kappa} &= \left\{y\in\mathbb{C};-\overline{y}\in D^{\mathrm{mch},u}_{+,\kappa}\right\}.
\end{aligned}
\tag{3.14}
$$

Notice that there exist constants $M_1,M_2>0$ independent of ε and κ such that

$$
\begin{aligned}
M_1\varepsilon^\gamma \le |y_j-i\pi/2|\le M_2\varepsilon^\gamma, &\qquad j=1,2,\\
M_1\kappa\varepsilon \le |y-i\pi/2|\le M_2\varepsilon^\gamma, &\qquad \text{for}\quad y\in D^{\mathrm{mch},u}_{+,\kappa}.
\end{aligned}
$$

In terms of the inner variable z (see (3.6)), the matching domains are given by

$$
\mathcal{D}^{\mathrm{mch},\star}_{+,\kappa}=\{z\in\mathbb{C};\varepsilon z+i\pi/2\in D^{\mathrm{mch},\star}_{+,\kappa}\},\ \star=u,s.
$$

Notice that,

$$
\begin{aligned}
M_1\varepsilon^{\gamma-1}\le |z_j|\le M_2\varepsilon^{\gamma-1}, &\qquad j=1,2,\\
M_1\kappa\le |z|\le M_2\varepsilon^{\gamma-1} &\qquad \text{for},\quad z\in\mathcal{D}^{\mathrm{mch},u}_{+,\kappa}.
\end{aligned}
$$

where z_1 and z_2 are the vertices of the inner domain y_1 and y_2, respectively, expressed in the inner variable.

Next theorem estimates the difference in the matching domains (3.14) between the functions ϕ^*, $*=u,s$ in (3.7) and the functions $\phi^{0,*}$, $*=u,s$, given by Theorem 3.3. The theorem is proven in Section 6.

Theorem 3.6 (Matching)**.** *Fix $\gamma\in(1/3,1)$. Let $\phi^\star(z,\tau)=\varepsilon v^\star(i\pi/2+\varepsilon z,\tau)$, $\star=u,s$, where $v^\star$ is the parameterization obtained in Theorem 3.1. Then, there exist $\varepsilon_0,\delta_0>0$ sufficiently small such that, for each $0<\varepsilon\le\varepsilon_0$ and κ satisfying $\kappa\varepsilon^{1-\gamma}+\frac{|\log\varepsilon|}{\kappa^2}\le\delta_0$, and $z\in\mathcal{D}^{\mathrm{mch},\star}_{+,\kappa}$,*

$$
\phi^\star(z,\tau)=\phi^{0,\star}(z,\tau)+\varphi^\star(z,\tau),
$$

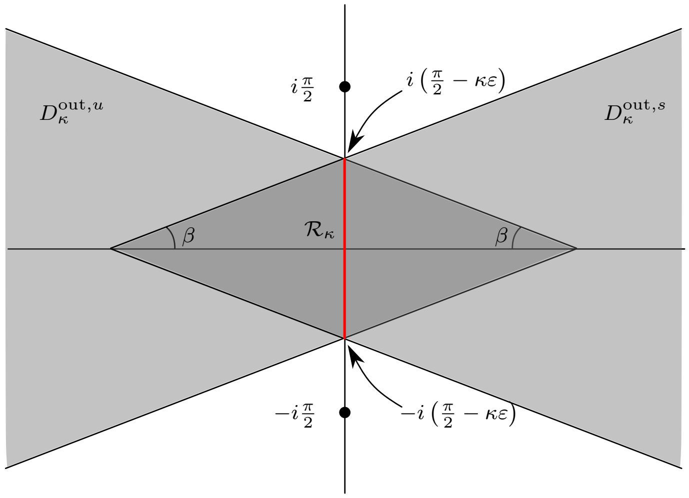

FIGURE 9. Domain $\mathcal{R}_\kappa$.

where $\phi^{0,\star}$ is the solution of the inner equation (3.9) *obtained in Theorem 3.3, and $\varphi^\star$ satisfies that for $(z,\tau)\in\mathcal{D}^{\mathrm{mch},\star}_{+,\kappa}$*

$$\|\partial_\tau^2\varphi^\star\|_{\ell_1}(z)\leq\frac{M_3(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1})|\log\varepsilon|}{|z|^2}\quad \textit{and}\quad \|\partial_\tau^2\partial_z\varphi^\star\|_{\ell_1}(z)\leq\frac{M_3(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1})|\log\varepsilon|}{\kappa|z|^2},$$

where $M_3>0$ is a constant independent of ε and κ.

Remark 3.7. *Notice that $\gamma=1/2$ minimizes the size of $\|\varphi^\star\|_{\ell_1,2}$, $\star=u,s$, in Theorem 3.6. In this case,*

$$\|\partial_\tau^2\varphi^\star\|_{\ell_1,2}\leq M|\log\varepsilon|\varepsilon^{1/2}|z|^{-2}.$$

Remark 3.8. *The idea to obtain the above matching estimate is that y_1 and y_2 are connected by a segment with nontrivial slope in the complex plane, where the linear part of the problem becomes somewhat elliptic in the 1-dim variable z (in the PDE sense) except in the direction of the mode $\sin\tau$. Therefore $\varphi^\star$, $\star=u,s$, is nicely determined by the values at y_1 and y_2 which simply come from the asymptotic form $\phi^{0,\star}$ and $\varphi^\star$. The order $\mathcal{O}(|z|^{-2})$ is largely determined by the mode $\sin\tau$.*

3.3. **The distance between the invariant manifolds.** Our next step is to give an asymptotic formula for the difference

$$\Delta(y,\tau)=v^u(y,\tau)-v^s(y,\tau)=\xi^u(y,\tau)-\xi^s(y,\tau),\tag{3.15}$$

where $\xi^{u,s}$ are the functions obtained in Theorem 3.1 (recall that $\Pi_{2l}[\Delta v]=0$ for every $l\geq0$), in the domain (see Figure 9).

$$\mathcal{R}_\kappa=D_\kappa^{\mathrm{out},u}\cap D_\kappa^{\mathrm{out},s}\cap i\mathbb{R},$$

Next lemma shows that the difference Δ satisfies a linear equation.

Lemma 3.9. *The function Δ introduced in* (3.15) *satisfies the linear equation*

$$\mathcal{L}(\Delta)=\Pi_1\left[\eta_1(y,\tau)\Pi_1[\Delta]\sin\tau+\eta_2(y,\tau)\widetilde{\Pi}[\Delta]\right]\sin\tau+\widetilde{\Pi}[\eta_3(y,\tau)\Delta],$$

where $\mathcal{L}$ is the operator given in (3.1) *and $\eta_j:\mathcal{R}_\kappa\times\mathbb{T}\to\mathbb{C}$, $j=1,2,3$, are functions analytic in y. Moreover, there exists a constant $M>0$ independent of κ and ε such that*

$$\|\eta_1\|_{\ell_1}(y)\leq\frac{M\varepsilon^2}{|y^2+\pi^2/4|^4},\quad \textit{and}\quad \|\eta_2\|_{\ell_1}(y),\|\eta_3\|_{\ell_1}(y)\leq\frac{M}{|y^2+\pi^2/4|^2}.$$

Proof. From (3.3) and Theorem 3.1, we have that

$$\mathcal{L}(\Delta)=\mathcal{F}(\xi^u)-\mathcal{F}(\xi^s),$$

where $\mathcal{F}$ is the operator given in (3.2). Using the expression of $\mathcal{F}$ (see also (4.5)), we obtain that

$$\begin{aligned}\mathcal{F}(\xi^u)-\mathcal{F}(\xi^s)=&-\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi^u+v^h\sin\tau))-g(\varepsilon\omega(\xi^s+v^h\sin\tau))\right]\\&-\Pi_1\left[(\xi_1^u+v^h)^2\sin^2\tau\widetilde{\Pi}(\xi^u)-(\xi_1^s+v^h)^2\sin^2\tau\widetilde{\Pi}(\xi^s)\right]\sin\tau\\&-\Pi_1\left[(\xi_1^u+v^h)\sin\tau(\widetilde{\Pi}[\xi^u])^2-(\xi_1^s+v^h)\sin\tau(\widetilde{\Pi}[\xi^s])^2\right.\\&\left.+\frac{1}{3}\left((\widetilde{\Pi}[\xi^u])^3-(\widetilde{\Pi}[\xi^s])^3\right)\right]\sin\tau\\&+\left(-\frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(\xi^u+v^h\sin\tau))-f(\varepsilon\omega(\xi^s+v^h\sin\tau))\right]\right.\\&\left.-\frac{3v^h\left((\xi_1^u)^2-(\xi_1^s)^2\right)}{4}-\frac{(\xi_1^u)^3-(\xi_1^s)^3}{4}\right)\sin\tau.\end{aligned}$$

The proof follows from calculations based in the power series expansion of g and f and the estimates

$$\left|v^h(y)\right|\leq\frac{M}{|y^2+\pi^2/4|},\qquad\|\xi^{u,s}\|_{\ell_1}(y)\leq\frac{M\varepsilon^2}{|y^2+\pi^2/4|^3}$$

obtained in Theorem 3.1. □

The idea to obtain the exponentially small splitting estimate is that $y^\pm=\pm i(\frac{\pi}{2}-\kappa\varepsilon)$ (see Figure 9) are connected by a vertical segment where the linear operator $\mathcal{L}$ becomes elliptic (in the PDE sense) in the 1-dim variable y except in the direction of the mode $\sin\tau$. This has two implications: a.) the solution is determined by the values at the two boundary points $y^\pm$ and b.) the Green's function principally in the form of exponential functions leads to the desired splitting estimate at $y=0$. The mode $\sin\tau$ seems to be an exception. Recalling $\partial_y\Pi_1[\Delta]|_{y=0}=0$, the splitting in the direction $\Pi_1[\Delta]|_{y=0}$ will be handled by the conservation of energy due to the Hamiltonian structure.

As explained in Section 2.1, to prove that the distance between the stable and unstable manifold is exponentially small is crucial the fact that the model considered has a conserved quantity. Indeed, if the system would not have a first integral, the distance between the invariant manifolds would be "tipically" of order of some power of ε. Therefore, in this section we must rely on the conservation of energy to analyze Δ.

Let us rewrite equation (1.23) as

$$\begin{cases}\partial_y v=w,\\ \partial_y w=\dfrac{1}{\varepsilon^2}\partial_\tau^2 v+\dfrac{1}{\varepsilon^2\omega^2}v-\dfrac{1}{3}v^3-\dfrac{1}{\varepsilon^3\omega^3}f\left(\varepsilon\omega v\right),\end{cases}$$

which is Hamiltonian with respect to

$$\mathcal{H}(v,w)=\frac{1}{\pi}\int_{\mathbb{T}}\left(\frac{w^2}{2}+\frac{(\partial_\tau v)^2}{2\varepsilon^2}-\frac{v^2}{2\varepsilon^2\omega^2}+\frac{v^4}{12}+\frac{F(\varepsilon\omega v)}{\varepsilon^4\omega^4}\right)d\tau,$$

where F is an analytic function such that $F'(z)=f(z)$ and $F(z)=\mathcal{O}(z^6)$.

Notice that the solutions $v^\star(y,\tau)$ of (1.23), $\star=u,s$, obtained in Theorem 3.1 are contained in the energy level $\{\mathcal{H}=0\}$. We use the Hamiltonian $\mathcal{H}$ to obtain the variable $\Pi_1[\Delta]$ in terms of the variables $\widetilde{\Pi}[\Delta]$, $\Pi_1[\Xi]$ and $\widetilde{\Pi}[\Xi]$ where $\Xi=\partial_y\Delta=w^u-w^s=\partial_y v^u-\partial_y v^s$.

Lemma 3.10. *The functions Δ, Ξ satisfy*

$$\Pi_1[\Delta](y)=\frac{\dot{v}^h(y)}{\ddot{v}^h(y)}\Pi_1[\Xi](y)+A(\Xi)(y)+B(\widetilde{\Pi}[\Delta])(y),\tag{3.16}$$

where A and B are linear operators such that, for $y\in\mathcal{R}_\kappa$,

(1) $|A(\Xi)(y)|\leq\dfrac{M\varepsilon^2}{|y^2+\pi^2/4|}\|\Xi\|_{\ell_1}(y)$
(2) $|B(\widetilde{\Pi}[\Delta])(y)|\leq M\|\widetilde{\Pi}[\Delta]\|_{\ell_1}(y)$.

Proof. As the projections Π_1 and $\widetilde{\Pi}$ are orthogonal (see (2.3) and (2.7)), $\mathcal{H}$ is given by

$$\mathcal{H}(v,w)=\frac{(\Pi_1[w])^2}{2}-\frac{(\Pi_1[v])^2}{2}+\frac{1}{\pi}\int_{\mathbb{T}}\left(\frac{(\widetilde{\Pi}[w])^2}{2}+\frac{(\partial_\tau\widetilde{\Pi}[v])^2}{2\varepsilon^2}-\frac{(\widetilde{\Pi}[v])^2}{2\varepsilon^2\omega^2}+\frac{v^4}{12}+\frac{F(\varepsilon\omega v)}{\varepsilon^4\omega^4}\right)d\tau.$$

Using that $\mathcal{H}(v^\star,w^\star)=0$, $\star=u,s$, integrating by parts the ∂_τ term and the Mean Value Theorem, we have that

$$\begin{aligned}0=&\mathcal{H}(v^u,w^u)-\mathcal{H}(v^s,w^s)\\ =&\frac{\Pi_1[w^u]+\Pi_1[w^s]}{2}\Pi_1[\Xi]-\frac{\Pi_1[v^u]+\Pi_1[v^s]}{2}\Pi_1[\Delta]\\ &+\frac{1}{\pi}\int_{\mathbb{T}}\left[\frac{\widetilde{\Pi}[w^u]+\widetilde{\Pi}[w^s]}{2}\widetilde{\Pi}[\Xi]-\frac{1}{\varepsilon^2}\frac{\partial_\tau^2\widetilde{\Pi}[v^u]+\partial_\tau^2\widetilde{\Pi}[v^s]}{2}\widetilde{\Pi}[\Delta]-\frac{\widetilde{\Pi}[v^u]+\widetilde{\Pi}[v^s]}{2\varepsilon^2\omega^2}\widetilde{\Pi}[\Delta]\right]d\tau\\ &+\frac{1}{\pi}\int_{\mathbb{T}}\left[\frac{(v^u)^3+(v^u)^2(v^s)+(v^u)(v^s)^2+(v^s)^3}{12}\Delta+\left(\frac{1}{\varepsilon^3\omega^3}\int_0^1 f(\varepsilon\omega(\sigma v^u+(1-\sigma)\sigma v^s))d\sigma\right)\Delta\right]d\tau.\end{aligned}$$

Using

$$v^\star=v^h\sin(\tau)+\xi^\star(y,\tau),\quad \ddot{v}^h=v^h-(v^h)^3/4=\sqrt{2}(\cosh(2y)-3)\operatorname{sech}^3(y),$$

and observing that $\ddot{v}^h(y)$ is strictly negative, for every $y=i\widetilde{y}$ with $\widetilde{y}\in(-\pi/2,\pi/2)$, one has

$$0=-\ddot{v}^h(1+a(y))\Pi_1[\Delta]+\dot{v}^h\Pi_1[\Xi]+\widetilde{A}(\Xi)+\widetilde{B}(\widetilde{\Pi}[\Delta])$$

By the estimates in Theorem 3.1 and using that $\ddot{v}^h(y)$ has a third order pole at $y=\pm i\pi/2$, we have

$$|a(y)|\leq\frac{M\varepsilon^2}{|y^2+\pi^2/4|^2}\leq\frac{M}{\kappa^2},\qquad\text{for}\quad y\in\mathcal{R}_\kappa \tag{3.17}$$

and, also for $y\in\mathcal{R}_\kappa$,

$$\left|\widetilde{A}(\Xi)\right|(y)\leq\frac{M\varepsilon^2}{|y^2+\pi^2/4|^4}\|\Xi\|_{\ell_1}(y)\qquad\text{and}\qquad\left|\widetilde{B}(\widetilde{\Pi}[\Delta])\right|(y)\leq\frac{M}{|y^2+\pi^2/4|^3}\|\widetilde{\Pi}[\Delta]\|_{\ell_1}(y).$$

Moreover, using the estimate (3.17) and taking κ big enough, we have

$$|D(y)^{-1}|\leq M|y^2+\pi^2/4|^3,\ y\in\mathcal{R}_\kappa,\ \text{where } D(y)=\ddot{v}^h(y)\,(1+a(y))\,.$$

Hence, it follows that

$$\Pi_1[\Delta]=\frac{\dot{v}^h\Pi_1[\Xi]+\widetilde{A}(\Xi)+\widetilde{B}(\widetilde{\Pi}[\Delta])}{\ddot{v}^h(1+a)}=\frac{\dot{v}^h}{\ddot{v}^h}\Pi_1[\Xi]+A(\Xi)+B(\widetilde{\Pi}[\Delta]),$$

where A and B are the linear operators

$$\begin{aligned}A(\Xi)(y)&=\frac{\widetilde{A}(\Xi)(y)}{\ddot{v}^h(y)(1+a(y))}-\frac{\dot{v}^h(y)}{\ddot{v}^h(y)(1+a(y))}a(y)\Pi_1[\Xi](y)\\ B(\widetilde{\Pi}[\Delta])&=\frac{\widetilde{B}(\widetilde{\Pi}[\Delta])}{\ddot{v}^h(y)(1+a(y))}.\end{aligned}$$

The proof of the proposition follows directly from the estimates of $\widetilde{A}(\Xi)$, $\widetilde{B}(\widetilde{\Pi}[\Delta])$, a and the fact that $\ddot{v}^h$ and $\dot{v}^h$ have a third and second order pole at the points $y=\pm i\pi/2$, respectively. □

Lemma 3.10 allows to study the difference between the invariant manifolds without keeping track of the component Δ_1. In other words, we use coordinates $(\Pi_1 w,\widetilde{\Pi}v,\widetilde{\Pi}w)$ to analyze the level of energy $\mathcal{H}=0$ and therefore we measure the difference between the functions v^u and v^s through the components $(\Xi_1,\widetilde{\Pi}\Delta,\widetilde{\Pi}\Xi)$. The inconvience of the energy reduction is that the equation loses the second order structure since it also depends on $\Xi=\partial_y\Delta$.

To capture the exponentially small behavior of $(\widetilde{\Pi}\Delta, \widetilde{\Pi}\Xi)$ it is convenient to write the second order equation as a first order system in diagonal form. Thus, we define

$$\begin{aligned}\Gamma &= \sum_{k\geq 1}\Gamma_{2k+1}(y)\sin((2k+1)\tau), \quad \Gamma_{2k+1} = \lambda_{2k+1}\Delta_{2k+1} + i\varepsilon\Xi_{2k+1}\\ \Theta &= \sum_{k\geq 1}\Theta_{2k+1}(y)\sin((2k+1)\tau), \quad \Theta_{2k+1} = \lambda_{2k+1}\Delta_{2k+1} - i\varepsilon\Xi_{2k+1}.\end{aligned} \tag{3.18}$$

From now on we measure the difference between the invariant manifolds (within the energy level $\mathcal{H} = 0$) by the difference "vector"

$$\widetilde{\Delta} = (\Xi_1, \Gamma, \Theta). \tag{3.19}$$

Notice that the estimates of Theorem 3.1 imply that Δ satisfies

$$\sum_{k\geq 1}\lambda_{2k+1}^2|\Delta_{2k+1}(y)| \leq M\|\partial_\tau^2\xi^u\|_{\ell_1}(y) + M\|\partial_\tau^2\xi^s\|_{\ell_1}(y) \leq \frac{M\varepsilon^2}{|y^2+\pi^4/4|^3},$$

along with a similar estimate on Ξ, therefore the functions Γ and Θ are well defined for $y \in \mathcal{R}_\kappa$ and satisfy

$$\sum_{k\geq 1}\lambda_{2k+1}|\Gamma_{2k+1}(y)| \leq \frac{M\varepsilon^2}{|y^2+\pi^4/4|^3} \quad \textit{and} \quad \sum_{k\geq 1}\lambda_{2k+1}|\Theta_{2k+1}(y)| \leq \frac{M\varepsilon^2}{|y^2+\pi^4/4|^3}.$$

Proposition 3.11. *The function $\widetilde{\Delta} = (\Xi_1, \Gamma, \Theta)$ satisfies the equation*

$$\widetilde{\mathcal{L}}(\widetilde{\Delta}) = \mathcal{M}(\widetilde{\Delta}),$$

where $\widetilde{\mathcal{L}}$ is the differential operator

$$\begin{aligned}\widetilde{\mathcal{L}}(\Xi_1,\Gamma,\Theta) = &\Bigg(\dot{\Xi}_1 - \frac{\dddot{v}^h}{\ddot{v}^h}\Xi_1, \sum_{k\geq 1}\left(\dot{\Gamma}_{2k+1} + i\frac{\lambda_{2k+1}}{\varepsilon}\Gamma_{2k+1}\right)\sin((2k+1)\tau),\\ &\sum_{k\geq 1}\left(\dot{\Theta}_{2k+1} - i\frac{\lambda_{2k+1}}{\varepsilon}\Theta_{2k+1}\right)\sin((2k+1)\tau)\Bigg)\end{aligned} \tag{3.20}$$

and $\mathcal{M}$ is a linear operator which can be written as

$$\mathcal{M}(\Xi_1,\Gamma,\Theta) = \begin{pmatrix} m_W(y)\Xi_1 + \mathcal{M}_W(\Gamma,\Theta)\\ m_{\mathrm{osc}}(y,\tau)\Xi_1 + \mathcal{M}_{\mathrm{osc}}(\Gamma,\Theta)\\ -m_{\mathrm{osc}}(y,\tau)\Xi_1 - \mathcal{M}_{\mathrm{osc}}(\Gamma,\Theta)\end{pmatrix}, \tag{3.21}$$

where $m_W : \mathcal{R}_\kappa \to \mathbb{C}$, $m_{\mathrm{osc}} : \mathcal{R}_\kappa \times \mathbb{T} \to \mathbb{C}$ are functions analytic in y satisfying

$$|m_W(y)| \leq \frac{M\varepsilon^2}{|y^2+\pi^2/4|^3} \qquad \textit{and} \qquad \|m_{\mathrm{osc}}\|_{\ell_1}(y) \leq \frac{M\varepsilon}{|y^2+\pi^2/4|}$$

and $\mathcal{M}_W$, $\mathcal{M}_{\mathrm{osc}}$ are linear operators such that, for $y \in \mathcal{R}_\kappa$,

$$\begin{aligned}|\mathcal{M}_W(\Gamma,\Theta)(y)| &\leq \frac{M}{|y^2+\pi^2/4|^2}\left(\|\Gamma\|_{\ell_1}(y) + \|\Theta\|_{\ell_1}(y)\right)\\ \left\|\mathcal{M}_{\mathrm{osc}}(\Gamma,\Theta)\right\|_{\ell_1}(y) &\leq \frac{M\varepsilon}{|y^2+\pi^2/4|^2}\left(\|\Gamma\|_{\ell_1}(y) + \|\Theta\|_{\ell_1}(y)\right),\end{aligned}$$

where $M > 0$ is a constant independent of ε and κ .

Proof. From (3.18) and Proposition 3.9, we have that, for each $k \geq 1$,

$$\begin{aligned}\dot{\Gamma}_{2k+1} &= \lambda_{2k+1}\Xi_{2k+1} + i\varepsilon\ddot{\Delta}_{2k+1}\\ &= \lambda_{2k+1}\Xi_{2k+1} + i\varepsilon\left(-\frac{\lambda_{2k+1}^2}{\varepsilon^2}\Delta_{2k+1} + \Pi_{2k+1}\left[\eta_3(y,\tau)\Delta\right]\right)\\ &= -i\frac{\lambda_{2k+1}}{\varepsilon}\Gamma_{2k+1} + i\varepsilon\Pi_{2k+1}\left[\eta_3(y,\tau)\Delta\right].\end{aligned} \tag{3.22}$$

Analogously, for each $k \geq 1$,

$$\dot{\Theta}_{2k+1} = i\frac{\lambda_{2k+1}}{\varepsilon}\Theta_{2k+1} - i\varepsilon\Pi_{2k+1}\left[\eta_3(y,\tau)\Delta\right]. \tag{3.23}$$

Moreover, for the variable Ξ_1, by (3.1) and Proposition 3.9, we have that

$$\dot{\Xi}_1 = \left(1 - \frac{3(v^h)^2}{4}\right)\Delta_1 + \Pi_1\left[\eta_1(y,\tau)\Delta_1\sin(\tau) + \eta_2(y,\tau)\widetilde{\Pi}[\Delta]\right].$$

Using (3.16) for Δ_1 and $\left(1 - \frac{3(v^h)^2}{4}\right)\dot{v}^h = \dddot{v}^h$, we obtain

$$\begin{aligned}
\dot{\Xi}_1 &= \frac{\dddot{v}^h}{\ddot{v}^h}\Xi_1 + \left(1 - \frac{3(v^h)^2}{4}\right)\left(A(\Xi) + B(\widetilde{\Pi}[\Delta])\right)\\
&\quad + \Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi) + B(\widetilde{\Pi}[\Delta])\right)\sin(\tau) + \eta_2(y,\tau)\widetilde{\Pi}[\Delta]\right]\\
&= \frac{\dddot{v}^h}{\ddot{v}^h}\Xi_1 + \left(1 - \frac{3(v^h)^2}{4}\right)A(\Xi_1\sin(\tau)) + \Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi_1\sin(\tau))\right)\sin\tau\right]\\
&\quad + \left(1 - \frac{3(v^h)^2}{4}\right)\left(A(\widetilde{\Pi}[\Xi]) + B(\widetilde{\Pi}[\Delta])\right)\\
&\quad + \Pi_1\left[\eta_1(y,\tau)\left(A(\widetilde{\Pi}[\Xi]) + B(\widetilde{\Pi}[\Delta])\right)\sin\tau + \eta_2(y,\tau)\widetilde{\Pi}[\Delta]\right].
\end{aligned}$$

Finally, using (3.18),

$$\begin{aligned}
\dot{\Xi}_1 = &\frac{\dddot{v}^h}{\ddot{v}^h}\Xi_1 + \left(1 - \frac{3(v^h)^2}{4}\right)A(\Xi_1\sin\tau) + \Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi_1\sin\tau)\right)\sin\tau\right]\\
&+ \left(1 - \frac{3(v^h)^2}{4}\right)\left(\frac{1}{2i\varepsilon}A(\Gamma - \Theta) + B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right)\\
&+ \Pi_1\left[\eta_1(y,\tau)\left(\frac{1}{2i\varepsilon}A(\Gamma - \Theta)\right)\sin(\tau) + \eta_1(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin\tau\right.\\
&\left. + \eta_2(y,\tau)\left(\sum_{n\geq 2}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right].
\end{aligned}$$

For the other components, as

$$\begin{aligned}
i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\Delta\right] &= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi) + B(\widetilde{\Pi}[\Delta])\right)\sin(\tau) + \widetilde{\Pi}[\Delta]\right)\right]\\
&= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi_1\sin(\tau))\right)\sin(\tau)\right]\\
&\quad + i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(A(\widetilde{\Pi}[\Xi])\sin(\tau) + B(\widetilde{\Pi}[\Delta])\sin(\tau) + \widetilde{\Pi}[\Delta]\right)\right]\\
&= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi_1\sin(\tau))\right)\sin(\tau)\right]\\
&\quad + i\varepsilon\widetilde{\Pi}\left[\frac{\eta_3(y,\tau)}{2i\varepsilon}A(\Gamma - \Theta)\sin(\tau) + \eta_3(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin(\tau)\right.\\
&\quad\left. + \eta_3(y,\tau)\sum_{k\geq 1}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right]
\end{aligned}$$

the proof is concluded by using (3.22) and (3.23) and taking

$$
\begin{aligned}
m_W(y)\Xi_1 &= \left(1-\frac{3(v^h)^2}{4}\right)A(\Xi_1\sin\tau)+\Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot v^h}{\ddot v^h}\Xi_1+A(\Xi_1\sin\tau)\right)\sin\tau\right]\\
\mathcal{M}_W(\Gamma,\Theta) &= \left(1-\frac{3(v^h)^2}{4}\right)\left(\frac{1}{2i\varepsilon}A(\Gamma-\Theta)+B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right)\\
&\quad+\Pi_1\left[\frac{\eta_1(y,\tau)}{2i\varepsilon}A(\Gamma-\Theta)\sin(\tau)+\eta_1(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin\tau\right.\\
&\quad\left.+\eta_2(y,\tau)\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right]\\
m_{\mathrm{osc}}(y,\tau)\Xi_1 &= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot v^h}{\ddot v^h}\Xi_1+A(\Xi_1\sin\tau)\right)\sin\tau\right]\\
\mathcal{M}_{\mathrm{osc}}(\Gamma,\Theta) &= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\frac{1}{2i\varepsilon}A(\Gamma-\Theta)\sin\tau+B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin\tau\right.\right.\\
&\quad\left.\left.+\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right],
\end{aligned}
$$

and using the bounds for the functions η_j, $j=1,2,3$ and the operators A and B provided in Propositions 3.9 and 3.10. ☐

We characterize the function $\widetilde{\Delta}$ as the *unique solution* of a certain integral equation. To this end, we introduce some notation. Given a sequence $a=(a_{2k+1})_{k\geq 1}$, we define the functions

$$
\begin{aligned}
\mathcal{I}_\Gamma(a)(y,\tau) &= \sum_{k\geq 1}a_{2k+1}e^{-i\frac{\lambda_{2k+1}}{\varepsilon}y}\sin((2k+1)\tau)\\
\mathcal{I}_\Theta(a)(y,\tau) &= \sum_{k\geq 1}a_{2k+1}e^{i\frac{\lambda_{2k+1}}{\varepsilon}y}\sin((2k+1)\tau).
\end{aligned}
\tag{3.24}
$$

We also define the following linear operator, which is a right inverse of the operator $\widetilde{\mathcal{L}}$ in (3.20),

$$
\mathcal{P}(f,g,h)=\left(\mathcal{P}^W(f),\mathcal{P}^\Gamma(g),\mathcal{P}^\Theta(h)\right),
\tag{3.25}
$$

where

$$
\begin{aligned}
\mathcal{P}^W(f) &= \ddot v^h(y)\int_0^y\frac{f(s)}{\ddot v^h(s)}ds\\
\mathcal{P}^\Gamma(g) &= \sum_{k\geq 1}\mathcal{P}^\Gamma_{2k+1}(g)\sin((2k+1)\tau), \qquad \mathcal{P}^\Gamma_{2k+1}(g)(y)=\int_{y^+}^y e^{i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)}\Pi_{2k+1}[g](s)ds\\
\mathcal{P}^\Theta(h) &= \sum_{k\geq 1}\mathcal{P}^\Theta_{2k+1}(h)\sin((2k+1)\tau), \qquad \mathcal{P}^\Theta_{2k+1}(h)(y)=\int_{y^-}^y e^{-i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)}\Pi_{2k+1}[h](s)ds
\end{aligned}
$$

and

$$
y^\pm=\pm i\left(\frac{\pi}{2}-\kappa\varepsilon\right).
$$

Using the just introduced functions and operators and recalling that by, Theorem 3.1, $\Xi_1(0)=\partial_y\xi_1^u(0)-\partial_y\xi_1^s(0)=0$, it can be easily checked that the function $\widetilde{\Delta}$ must satisfy the integral equation

$$
\widetilde{\Delta}=(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d))+\widetilde{\mathcal{M}}(\widetilde{\Delta}), \qquad \text{with} \qquad \widetilde{\mathcal{M}}(\widetilde{\Delta})=\mathcal{P}\circ\mathcal{M}(\widetilde{\Delta}),
\tag{3.26}
$$

where $\mathcal{M}$ is given by (3.21) and $\mathcal{I}_\Gamma(c)$, $\mathcal{I}_\Theta(d)$ are given in (3.24) with

$$
c_{2k+1}=\Gamma_{2k+1}(y^+)e^{i\frac{\lambda_{2k+1}}{\varepsilon}y^+} \qquad \text{and} \qquad d_{2k+1}=\Theta_{2k+1}(y^-)e^{-i\frac{\lambda_{2k+1}}{\varepsilon}y^-},
\tag{3.27}
$$

(note that $\Gamma(y^+,\tau) = \mathcal{I}_\Gamma(c)(y^+,\tau)$ and $\Theta(y^-,\tau) = \mathcal{I}_\Theta(d)(y^-,\tau)$).

Now we are ready to define the leading order of the function $\widetilde{\Delta}$. We first give some heuristic explanation. In Section 7, we shall first show that $\widetilde{M}$ is small and thus we expect that the main term of $\widetilde{\Delta}$ for the (Γ,Θ) is given by $(\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d))$. Let us analyze how these functions behave. We do the reasoning for Γ since the one for Θ is analogous.

$$\mathcal{I}_\Gamma(c)(y,\tau) = \sum_{k\geq 1} \Gamma_{2k+1}(y^+) e^{-i\frac{\lambda_{2k+1}}{\varepsilon}(y-y^+)} \sin((2k+1)\tau)$$

Recalling $\lambda_3 = \sqrt{8-\varepsilon^2}$, $\mu_3 = 2\sqrt{2}$, (3.18), that $\Gamma_{2k+1}(y^+) = \lambda_{2k+1}\Delta_{2k+1}(y^+) + i\varepsilon\Xi_{2k+1}(y^+)$ and using Theorem 3.6 to approximate the functions $v^{u,s}$ at the point $y = y^+$ by the corresponding solutions of the inner equation (see Theorem 3.3) and the asymptotic formula for the difference between $\phi^{0,u}$ and $\phi^{0,s}$ at $z^+ = (y^+ - i\pi/2)/\varepsilon$, also in Theorem 3.3, one has

$$\begin{aligned}\Gamma_3(y^+) &= 2\frac{\lambda_3}{\varepsilon} e^{-i\mu_3 \frac{y^+ - i\frac{\pi}{2}}{\varepsilon}} \left(C_{\text{in}} + \mathcal{O}\left(\frac{1}{\kappa}\right)\right) + \text{h.o.t}\\ \Gamma_{2k+1}(y^+) &= \frac{1}{\varepsilon} e^{-i\mu_3 \frac{y^+ - i\frac{\pi}{2}}{\varepsilon}} \mathcal{O}\left(\frac{1}{\kappa}\right) + \text{h.o.t.}\end{aligned}$$

Therefore,

$$\mathcal{I}_\Gamma(c)(y) = \frac{2\lambda_3}{\varepsilon} e^{-i2\sqrt{2}\frac{y-i\frac{\pi}{2}}{\varepsilon}} \left(C_{\text{in}}\sin 3\tau + \mathcal{O}\left(\frac{1}{\kappa}\right)\right) + \text{h.o.t}$$

To prove Theorem 2.1, it suffices to justify the above leading order expansion of $\widetilde{\Delta}$.

Proposition 3.12. *Take $\kappa = \frac{1}{2\lambda_3}|\log\varepsilon|$. There exists $M > 0$ independent of small ε such that, for any $y \in \mathcal{R}_\kappa$, it holds*

$$\begin{aligned}|\Xi_1(y)| &\leq \frac{M}{|y^2+\pi^2/4|^2} e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2} - |\operatorname{Im}(y)|\right)},\\ \left\|\Gamma(y,\tau) - \frac{2\lambda_3}{\varepsilon} C_{\text{in}} e^{-i2\sqrt{2}\frac{y-i\frac{\pi}{2}}{\varepsilon}} \sin 3\tau\right\|_{\ell_1} &\leq \frac{M}{\varepsilon|\log\varepsilon|} e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2} - |\operatorname{Im}(y)|\right)},\end{aligned}$$

for some constant M independent of ε. Moreover $\Theta(\bar{y},\tau) = \overline{\Gamma(y,\tau)}$ satisfies a similar estimate.

The proof of this proposition is deferred to Section 7. Recall that $\Xi_1(0) = \partial_y v^u(0) - \partial_y v^s(0) = 0$ (see Theorem 3.1). However, we also need to estimate this component for $y \in \mathcal{R}_\kappa$ to obtain the estimate of (Γ,Θ) due to the coupling (see Section 7). The definition of Γ and the above inequality imply inequality (2.5) except for the missing $\sin\tau$ mode, which easily follows from $\Xi_1(0) = 0$ and the estimate on $\Pi_1[\Delta]$ given by Lemma 3.10.

4. Estimates of the invariant manifolds: Proof of Theorem 3.1

4.1. **Banach Spaces and Linear Operators.**
In this section we prove Theorem 3.1 through a fixed point argument in some appropriate Banach spaces. We consider only the unstable case, since the stable one is completely analogous.

Given $\kappa \geq 1$ and a real-analytic function $h : D_\kappa^{\text{out},u} \to \mathbb{C}$ (see (3.5)), we define

$$\|h\|_{m,\alpha} = \sup_{y\in D_\kappa^{\text{out},u}\cap\{\operatorname{Re}(y)\leq -1\}} |\cosh(y)^m h(y)| + \sup_{y\in D_\kappa^{\text{out},u}\cap\{\operatorname{Re}(y)\geq -1\}} |(y^2+\pi^2/4)^\alpha h(y)|, \tag{4.1}$$

and given a function $\xi : D_\kappa^{\text{out},u} \times \mathbb{T} \to \mathbb{C}$ which is real analytic in $y \in D_\kappa^{\text{out},u}$, we define

$$\|\xi\|_{\ell_1,m,\alpha} = \sum_{n\geq 1} \|\Pi_n[\xi]\|_{m,\alpha}$$

and the Banach spaces

$$\begin{aligned}\mathcal{E}_{m,\alpha} &= \{\xi : D_\kappa^{\text{out},u} \to \mathbb{C};\ \xi \text{ is real-analytic in } y, \text{ and } \|\xi\|_{m,\alpha} < \infty\}\\ \mathcal{E}_{\ell_1,m,\alpha} &= \{\xi : D_\kappa^{\text{out},u} \times \mathbb{T} \to \mathbb{C};\ \xi(y,\tau) \text{ is real-analytic in } y \text{ and } \|\xi\|_{\ell_1,m,\alpha} < \infty\}.\end{aligned}$$

Lemma 4.1. *There exists $M > 0$ depending only on β such that, for any $g,h : D_\kappa^{\text{out},u} \times \mathbb{T} \to \mathbb{C}$, it holds*

(1) *If* $\alpha_2 \geq \alpha_1 \geq 0$*, then*

$$\|h\|_{\ell_1,m,\alpha_2} \leq M\|h\|_{\ell_1,m,\alpha_1} \quad \text{and} \quad \|h\|_{\ell_1,m,\alpha_1} \leq \frac{M}{(\kappa\varepsilon)^{\alpha_2-\alpha_1}}\|h\|_{\ell_1,m,\alpha_2}.$$

(2) *If* $\alpha_1, \alpha_2 \geq 0$*, and* $\|g\|_{\ell_1,m_1,\alpha_1}, \|h\|_{\ell_1,m_2,\alpha_2} < \infty$*, then*

$$\|gh\|_{\ell_1,m_1+m_2,\alpha_1+\alpha_2} \leq \|g\|_{\ell_1,m_1,\alpha_1}\|h\|_{\ell_1,m_2,\alpha_2}.$$

This lemma actually applies to general functions 2π-periodic in τ, not just to odd functions. The proof of this lemma is straightforward and we omit it.

Firstly to solve the linear equation $\mathcal{L}\xi = h$, we introduce the operator $\mathcal{G}(h)$ acting on the Fourier coefficients of h as

$$\mathcal{G}(h) = \sum_{n\geq 1} \mathcal{G}_n(h_n)\sin(n\tau), \quad \widetilde{\mathcal{G}}(h) = \sum_{n\geq 2} \mathcal{G}_n(h_n)\sin(n\tau) = \widetilde{\Pi}[\mathcal{G}(h)],$$

with

$$\mathcal{G}_1(h_1) = -\zeta_1(y)\int_0^y \zeta_2(s)h_1(s)ds + \zeta_2(y)\int_{-\infty}^y \zeta_1(s)h_1(s)ds \tag{4.2}$$

$$\mathcal{G}_n(h_n) = -\frac{i\varepsilon}{2\lambda_n}e^{i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{-i\frac{\lambda_n}{\varepsilon}s}h_n(s)ds + \frac{i\varepsilon}{2\lambda_n}e^{-i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{i\frac{\lambda_n}{\varepsilon}s}h_n(s)ds, \ n \geq 2. \tag{4.3}$$

where

$$\zeta_1(y) = -2\sqrt{2}\frac{\sinh(y)}{\cosh^2(y)} \quad \text{and} \quad \zeta_2(y) = -\frac{\sqrt{2}}{16}\frac{\sinh(y)}{\cosh^2(y)}(6y - 4\coth(y) + \sinh(2y)), \tag{4.4}$$

are linearly independent solutions of

$$\ddot{\zeta} - \zeta + \frac{3(v^h)^2}{4}\zeta = 0 \qquad \text{(see (3.1))}.$$

Remark 4.2. *When* $-\infty$ *is involved in the above integrals, it should be understood that the integral is along horizontal lines. As the integrands are analytic functions, integral paths may be modified to yield better estimates in certain cases.*

Proposition 4.3. *The following statements hold.*

(1) $\partial_y\Pi_1(\mathcal{G}(\xi))(0) = 0$.
(2) $\mathcal{G}\circ\mathcal{L}(\xi) = \mathcal{L}\circ\mathcal{G}(\xi) = \xi$.
(3) *For any* $m > 1$ *and* $\alpha \geq 5$*, there exists a constant* $M > 0$ *independent of* ε *and* κ *such that, for every* $h \in \mathcal{E}_{m,\alpha}$*,*

$$\|\mathcal{G}_1(h)\|_{1,\alpha-2} + \|\partial_y\mathcal{G}_1(h)\|_{1,\alpha-1} \leq M\|h\|_{m,\alpha}.$$

(4) *For any* $m \geq 1$*,* $\alpha \geq 0$*, there exists* $M > 0$ *such that for every* $n \geq 2$ *and* $h \in \mathcal{E}_{m,\alpha}$*,*

$$\|\mathcal{G}_n(h)\|_{m,\alpha} \leq M\frac{\varepsilon^2}{\lambda_n^2}\|h\|_{m,\alpha}, \quad \|\partial_y\mathcal{G}_n(h)\|_{m,\alpha} \leq M\frac{\varepsilon}{\lambda_n}\|h\|_{m,\alpha}, \quad \|\partial_y\mathcal{G}_n(h)\|_{m,\alpha} \leq M\frac{\varepsilon^2}{\lambda_n^2}\|\partial_y h\|_{m,\alpha}.$$

The proof of this proposition is deferred to Appendix A. In particular, the last item indicates a gain of an extra order regularity in τ for $\mathcal{G}(\xi)$ compared to general solutions to wave equations and an improvement in the estimate of $\partial_y\mathcal{G}_n(h)$ when $\partial_y h \in \mathcal{E}_{m,\alpha}$, which is a typical trading between the smoothness and the smallness in problems involving rapid oscillations.

4.2. **Fixed Point Argument.** Now, we use Proposition 4.3 to rewrite (3.3) as $\xi = \mathcal{G}\circ\mathcal{F}(\xi)$, where $\mathcal{F}$ is given in (3.2). We analyze the operator

$$\mathcal{F}^\sharp = \mathcal{G}\circ\mathcal{F}$$

defined on the closed ball

$$\mathcal{B}_0(R\varepsilon^2) = \left\{\xi \in \mathcal{E}_{\ell_1,1,3} \mid \|\xi\|_{\ell_1,1,3} + \|\partial_y\xi\|_{\ell_1,1,4} \leq R\varepsilon^2\right\}$$

for some $R > 0$.

Proposition 4.4. *There exists $M, \kappa_0, \varepsilon_0 > 0$, such that, if $\varepsilon \in (0, \varepsilon_0)$, $R > 0$, and $\kappa > \kappa_0 R^{\frac{1}{2}}$, then the operator*

$$\mathcal{F}^\sharp : \mathcal{E}_{\ell^1,1,3} \supset \mathcal{B}_0(R\varepsilon^2) \to \mathcal{E}_{\ell^1,1,3}$$

is well defined and satisfies

$$\begin{aligned}
&\|\partial_\tau^2 \mathcal{F}^\sharp(0)\|_{\ell_1,1,3} + \|\partial_\tau^2 \partial_y \mathcal{F}^\sharp(0)\|_{\ell_1,1,4} \leq M\varepsilon^2,\\
&\|\partial_\tau^2 \widetilde{\Pi}[\mathcal{F}^\sharp(\xi) - \mathcal{F}^\sharp(\xi')]\|_{\ell_1,1,3} + \|\partial_\tau^2 \partial_y \widetilde{\Pi}[\mathcal{F}^\sharp(\xi) - \mathcal{F}^\sharp(\xi')]\|_{\ell_1,1,4} \leq \frac{M}{\kappa^2}\big(\|\xi - \xi'\|_{\ell_1,1,3} + \|\partial_y \xi - \partial_y \xi'\|_{\ell_1,1,4}\big),\\
&\|\Pi_1[\mathcal{F}^\sharp(\xi) - \mathcal{F}^\sharp(\xi')]\|_{1,3} + \|\partial_y \Pi_1\big[\mathcal{F}^\sharp(\xi) - \widetilde{\mathcal{F}}(\xi')\big]\|_{1,4}\\
&\quad \leq M\frac{1+R}{\kappa^2}\left(\|\xi - \xi'\|_{\ell_1,1,3} + \|\partial_y \xi - \partial_y \xi'\|_{\ell_1,1,4}\right) + M\left(\|\widetilde{\Pi}[\xi - \xi']\|_{\ell_1,1,3} + \|\partial_y \widetilde{\Pi}[\xi - \xi']\|_{\ell_1,1,4}\right).
\end{aligned}$$

Notice that the above bounds on $\partial_\tau^2 \mathcal{F}^\sharp(\xi)$ immediately implies those on $\mathcal{F}^\sharp(\xi)$ as the zeroth mode is not included.

Proof. First, we rewrite the operator $\mathcal{F}$ given in (3.2), in order to make explicit some cancellations. Recall that $g(u) = u^3/3 + f(u)$ is given by (1.19). Then,

$$\begin{aligned}
\mathcal{F}(\xi) &= -\frac{1}{\varepsilon^3\omega^3} g(\varepsilon\omega(\xi + v^h \sin\tau)) + \left(\frac{(\xi_1 + v^h)^3}{4} - \frac{3v^h\xi_1^2}{4} - \frac{\xi_1^3}{4}\right)\sin\tau\\
&= -\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi + v^h\sin\tau))\right] + \left\{-\frac{1}{3}\Pi_1\left[\left((\xi_1 + v^h)\sin(\tau) + \widetilde{\Pi}(\xi)\right)^3\right]\right.\\
&\quad \left. -\frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(\xi + v^h\sin\tau))\right] + \frac{(\xi_1 + v^h)^3}{4} - \frac{3v^h\xi_1^2}{4} - \frac{\xi_1^3}{4}\right\}\sin\tau\\
&= -\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi + v^h\sin\tau))\right] + \left\{-\frac{1}{3}\Pi_1\left[(\xi_1 + v^h)^3\sin^3\tau + 3(\xi_1 + v^h)^2\sin^2\tau\widetilde{\Pi}[\xi]\right.\right.\\
&\quad \left. +3(\xi_1 + v^h)\sin\tau(\widetilde{\Pi}[\xi])^2 + (\widetilde{\Pi}[\xi])^3\right] - \frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(\xi + v^h\sin\tau))\right]\\
&\quad \left. + \frac{(\xi_1 + v^h)^3}{4} - \frac{3v^h\xi_1^2}{4} - \frac{\xi_1^3}{4}\right\}\sin\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{F}(\xi) = &-\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi + v^h\sin\tau))\right] + \left\{\Pi_1\left[-(\xi_1 + v^h)^2(\sin^2\tau)\widetilde{\Pi}[\xi] - (\xi_1 + v^h)(\sin\tau)(\widetilde{\Pi}[\xi])^2\right.\right.\\
&\left.\left. -\frac{1}{3}(\widetilde{\Pi}[\xi])^3\right] - \frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(\xi + v^h\sin\tau))\right] - \frac{3v^h\xi_1^2}{4} - \frac{\xi_1^3}{4}\right\}\sin\tau.
\end{aligned} \tag{4.5}$$

which implies

$$\mathcal{F}(0) = -\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(v^h\sin\tau))\right] - \frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(v^h\sin\tau))\right]\sin\tau.$$

Let g and f have the power series expansion

$$g(u) = \sum_{d=1}^\infty g_{2d+1}u^{2d+1}, \quad f(u) = \sum_{d=2}^\infty g_{2d+1}u^{2d+1}, \quad g_3 = \frac{1}{3},$$

with a positive radius of convergence. Using Lemma 4.1 and Proposition 4.3, one may estimate

$$\begin{aligned}
\|\partial_\tau^2 \widetilde{\mathcal{G}}\mathcal{F}(0)\|_{\ell_1,1,3} &\lesssim \varepsilon^2\|\widetilde{\Pi}\mathcal{F}(0)\|_{\ell_1,1,3} \lesssim \varepsilon^2\sum_{d=1}^\infty(\varepsilon\omega)^{2d-2}|g_{2d+1}|\|(v^h\sin\tau)^{2d+1}\|_{\ell_1,1,3}\\
&\lesssim \varepsilon^2\sum_{d=1}^\infty\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|(v^h\sin\tau)^{2d+1}\|_{\ell_1,2d+1,2d+1} \lesssim \varepsilon^2\sum_{d=1}^\infty\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|v^h\|_{1,1}^{2d+1} \lesssim \varepsilon^2,
\end{aligned}$$

for reasonably large κ. In particular, in the above the operator ∂_τ^2 creates a Fourier multiplier of n^2 to the mode of $\sin n\tau$, which is cancelled by the λ_n^{-2} in the estimate of $\mathcal{G}_n$ in Proposition 4.3. In order to obtain the desired estimate on $\|\partial_y \widetilde{\mathcal{G}}\mathcal{F}(0)\|_{\ell_1,1,4}$, we also need

$$\partial_y \mathcal{F}(0) = -\frac{1}{\varepsilon^2\omega^2} g'(\varepsilon\omega(v^h \sin\tau))(v^h)' \sin\tau = -\sum_{d=1}^{\infty}(2d+1)(\varepsilon\omega)^{2d-2} g_{2d+1}(v^h)^{2d}\partial_y v^h \sin^{2d+1}\tau$$

which implies

$$\begin{aligned}
\|\partial_y \mathcal{F}(0)\|_{\ell_1,1,4} &\lesssim \sum_{d=1}^{\infty}(2d+1)(\varepsilon\omega)^{2d-2}|g_{2d+1}|\|(v^h)^{2d}\partial_y v^h \sin^{2d+1}\tau\|_{\ell_1,1,4}\\
&\lesssim \sum_{d=1}^{\infty}(2d+1)\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|(v^h)^{2d}\partial_y v^h \sin^{2d+1}\tau\|_{\ell_1,2d+1,2d+2}\\
&\lesssim \sum_{d=1}^{\infty}(2d+1)\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|v^h\|_{1,1}^{2d}\|\partial_y v^h\|_{1,2} \lesssim 1,
\end{aligned}$$

for reasonably large κ. Hence the estimates related to $\|\cdot\|_{\ell_1,1,4}$ estimate related to $\partial_y\widetilde{\mathcal{G}}\mathcal{F}(0)$ follows from Proposition 4.3. Again, using Lemma 4.1 and Proposition 4.3, one may also estimate

$$\begin{aligned}
&\|\mathcal{G}_1\Pi_1\mathcal{F}(0)\|_{1,3} \lesssim \|\Pi_1\mathcal{F}(0)\|_{3,5} \lesssim \sum_{d=2}^{\infty}(\varepsilon\omega)^{2d-2}|g_{2d+1}|\|(v^h)^{2d+1}\|_{3,5}\\
&\lesssim (\varepsilon\omega)^2\sum_{d=2}^{\infty}\left(\frac{\omega}{\kappa}\right)^{2d-4}|g_{2d+1}|\|(v^h)^{2d+1}\|_{2d+1,2d+1} \lesssim (\varepsilon\omega)^2\sum_{d=2}^{\infty}\left(\frac{\omega}{\kappa}\right)^{2d-4}|g_{2d+1}|\|v^h\|_{1,1}^{2d+1} \lesssim \varepsilon^2,
\end{aligned}$$

for reasonably large κ. The estimate on $\partial_y\mathcal{G}\Pi_1[\mathcal{F}(0)]$ is obtained in a similar fashion. The sum of these inequalities imply the estimate on $\mathcal{F}^\sharp(0)$.

To estimate the Lipschitz constant of $\mathcal{F}^\sharp$, let $\xi,\xi' \in \mathcal{B}_0(R\varepsilon^2)$, we have

$$\begin{aligned}
\mathcal{F}(\xi)-\mathcal{F}(\xi') = &-\frac{1}{(\varepsilon\omega)^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi+v^h\sin\tau)) - g(\varepsilon\omega(\xi'+v^h\sin\tau))\right]\\
&+\Bigg\{-\Pi_1\left[(\xi_1+v^h)^2(\sin^2\tau)(\widetilde{\Pi}[\xi]-\widetilde{\Pi}[\xi']) - \left((\xi_1+v^h)^2-(\xi_1'+v^h)^2\right)(\sin^2\tau)\widetilde{\Pi}[\xi']\right]\\
&-\Pi_1\left[(\xi_1+v^h)(\sin\tau)(\widetilde{\Pi}[\xi]^2-\widetilde{\Pi}[\xi']^2) - (\xi_1-\xi_1')(\sin\tau)\widetilde{\Pi}[\xi']^2\right] - \frac{1}{3}\Pi_1\left[\widetilde{\Pi}[\xi]^3-\widetilde{\Pi}[\xi']^3\right]\\
&-\frac{1}{(\varepsilon\omega)^3}\Pi_1\left[f(\varepsilon\omega(\xi+v^h\sin\tau)) - f(\varepsilon\omega(\xi'+v^h\sin\tau))\right] - \frac{3v^h(\xi_1^2-(\xi_1')^2)}{4} - \frac{\xi_1^3-(\xi_1')^3}{4}\Bigg\}\sin\tau.
\end{aligned}$$

For any $d\geq 2$, $m\geq 0$, $\alpha\geq 0$, and $\zeta,\zeta' \in \mathcal{E}_{\ell_1,m,\alpha}$, it is straight forward to estimate

$$\|\zeta^d-(\zeta')^d\|_{\ell_1,dm,d\alpha} \lesssim d(\|\zeta\|_{\ell_1,m,\alpha}^{d-1}+\|\zeta'\|_{\ell_1,m,\alpha}^{d-1})\|\zeta-\zeta'\|_{\ell_1,m,\alpha}$$

where the constant is independent of d. Another useful inequality is

$$\|\xi\|_{\ell_1,1,1}+\|\partial_y\xi\|_{\ell_1,1,2} \lesssim (\kappa\varepsilon)^{-2}(\|\xi\|_{\ell_1,1,3}+\|\partial_y\xi\|_{\ell_1,1,4}) \lesssim \frac{R}{\kappa^2} \lesssim 1, \quad \xi\in\mathcal{B}_0(R\varepsilon^2).$$

Hence one may use Lemma 4.1 and Proposition 4.3 to estimate

$$\begin{aligned}
\|\partial_\tau^2\widetilde{\mathcal{G}}[\mathcal{F}(\xi)-\mathcal{F}(\xi')]\|_{\ell_1,1,3} &\lesssim \varepsilon^2\|\widetilde{\Pi}[\mathcal{F}(\xi)-\mathcal{F}(\xi')]\|_{\ell_1,1,3}\\
&\lesssim \varepsilon^2\sum_{d=1}^{\infty}(\varepsilon\omega)^{2d-2}|g_{2d+1}|\|(\xi+v^h\sin\tau)^{2d+1}-(\xi'+v^h\sin\tau)^{2d+1}\|_{\ell_1,1,3}\\
&\lesssim \varepsilon^2\sum_{d=1}^{\infty}\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|(\xi+v^h\sin\tau)^{2d+1}-(\xi'+v^h\sin\tau)^{2d+1}\|_{\ell_1,2d+1,2d+1}\\
&\lesssim \varepsilon^2\sum_{d=1}^{\infty}d\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|(1+\|\xi\|_{\ell_1,1,1}^{2d}+\|\xi'\|_{\ell_1,1,1}^{2d})\|\xi-\xi'\|_{\ell_1,1,1}\\
&\lesssim \kappa^{-2}\sum_{d=1}^{\infty}d\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|\xi-\xi'\|_{\ell_1,1,3}\lesssim\kappa^{-2}\|\xi-\xi'\|_{\ell_1,1,3}
\end{aligned}$$

for $\kappa\geq R$ reasonably large. To estimates $\partial_y\widetilde{\mathcal{G}}[\mathcal{F}(\xi)-\mathcal{F}(\xi')]$, in a similar fashion one needs to compute

$$\begin{aligned}
&\|\partial_y\big(\mathcal{F}(\xi)-\mathcal{F}(\xi')\big)\|_{\ell_1,1,4}\\
&\quad\lesssim\sum_{d=1}^{\infty}d(\varepsilon\omega)^{2d-2}|g_{2d+1}|\Big\|(\xi+v^h\sin\tau)^{2d}(\partial_y\xi+\partial_y\sin\tau)-(\xi'+v^h\sin\tau)^{2d}(\partial_y\xi'+\partial_y\sin\tau)\Big\|_{\ell_1,1,4}\\
&\quad\lesssim\sum_{d=1}^{\infty}d\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\Big\|(\xi+v^h\sin\tau)^{2d}(\partial_y\xi+\partial_yv^h\sin\tau)\\
&\qquad\qquad-(\xi'+v^h\sin\tau)^{2d}(\partial_y\xi'+\partial_yv^h\sin\tau)\Big\|_{\ell_1,2d+1,2d+2}\\
&\quad\lesssim\|\xi-\xi'\|_{\ell_1,1,1}+\|\partial_y\xi-\partial_y\xi'\|_{\ell_1,1,2}\lesssim(\kappa\varepsilon)^{-2}(\|\xi-\xi'\|_{\ell_1,1,3}+\|\partial_y\xi-\partial_y\xi'\|_{\ell_1,1,4}),
\end{aligned}$$

where in the derivation of the third $\lesssim$ we applied $\|\cdot\|_{\ell_1,1,1}$ norm to all ξ, ξ', and v^h and $\|\cdot\|_{\ell_1,1,2}$ norm to all $\partial_y\xi$, $\partial_y\xi'$, and ∂_yv^h. Along with Proposition 4.3 this inequality yields the desired estimate on $\partial_y\widetilde{\mathcal{G}}[\mathcal{F}(\xi)-\mathcal{F}(\xi')]$. The $\mathcal{G}_1$ component can be estimated much as in the above. In fact,

$$\begin{aligned}
\|\mathcal{G}_1\Pi_1[\mathcal{F}(\xi)-\mathcal{F}(\xi')]\|_{1,3}+\|\partial_y\mathcal{G}_1\Pi_1[\mathcal{F}(\xi)-\mathcal{F}(\xi')]\|_{1,4}&\lesssim\|\Pi_1[\mathcal{F}(\xi)-\mathcal{F}(\xi')]\|_{3,5}\\
&\lesssim\frac{1}{(\varepsilon\omega)^3}\|f(\varepsilon\omega(\xi+v^h\sin\tau))-f(\varepsilon\omega(\xi'+v^h\sin\tau))\|_{\ell_1,3,5}+\|\widetilde{\Pi}[\xi]-\widetilde{\Pi}[\xi']\|_{\ell_1,1,3}\\
&\quad+(\|\xi\|_{\ell_1,1,1}+\|\xi\|_{\ell_1,1,1}^2+\|\xi'\|_{\ell_1,1,1}+\|\xi'\|_{\ell_1,1,1}^2)\|\xi-\xi'\|_{\ell_1,1,3}
\end{aligned}$$

where all the ξ, ξ', and $v^h\sin\tau$ in front of $\xi-\xi'$ were taken the $\|\cdot\|_{\ell_1,1,1}$ norm. The f terms can be estimated much as in the above

$$\begin{aligned}
&\frac{1}{(\varepsilon\omega)^3}\|f(\varepsilon\omega(\xi+v^h\sin\tau))-f(\varepsilon\omega(\xi'+v^h\sin\tau))\|_{\ell_1,3,5}\\
&\qquad\lesssim\sum_{d=2}^{\infty}(\varepsilon\omega)^{2d-2}|g_{2d+1}|(\kappa\varepsilon)^{-(2d-4)}\|(\xi+v^h\sin\tau)^{2d+1}-(\xi'+v^h\sin\tau)^{2d+1}\|_{\ell_1,2d+1,2d+1}\\
&\qquad\lesssim\kappa^{-2}\sum_{d=1}^{\infty}d(\frac{\omega}{\kappa})^{2d-4}|g_{2d+1}|\|\xi-\xi'\|_{\ell_1,1,3}\lesssim\kappa^{-2}\|\xi-\xi'\|_{\ell_1,1,3}
\end{aligned}$$

for $\kappa\geq R$ reasonably large. Summarizing the above estimates, the proposition follows. □

With the above preparations, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We claim that, if κ is sufficiently large, then $\mathcal{F}^\sharp$ is a contraction on the set

$$\begin{aligned}
S=\{\xi\in\mathcal{E}_{\ell_1,1,3}\ |&\|\Pi_1[\xi]\|_{1,3}+\|\partial_y\Pi_1[\xi]\|_{1,4}\leq(1+M)^2\varepsilon^2,\\
&\|\widetilde{\Pi}[\xi]\|_{\ell_1,1,3}+\|\partial_y\widetilde{\Pi}[\xi]\|_{\ell_1,1,4}\leq(1+M)\varepsilon^2\}\subset\mathcal{B}_0(R\varepsilon^2),\quad R=(1+M)(2+M),
\end{aligned}$$

equipped with the metric

$$|\xi|_M := \|\Pi_1[\xi]\|_{1,3} + \|\partial_y \Pi_1[\xi]\|_{1,4} + (1+M)\big(\|\widetilde{\Pi}[\xi]\|_{\ell_1,1,3} + \|\partial_y \widetilde{\Pi}[\xi]\|_{\ell_1,1,4}\big),$$

where M is the constant from Proposition 4.4. In fact, using Proposition 4.4 it is straight forward to estimate that, for any $\xi \in S$,

$$\begin{aligned}\|\Pi_1[\mathcal{F}^\sharp(\xi)]\|_{1,3} + \|\partial_y \Pi_1[\mathcal{F}^\sharp(\xi)]\|_{1,4} \leq & \|\mathcal{F}^\sharp(0)\|_{\ell_1,1,3} + \|\partial_y \mathcal{F}^\sharp(0)\|_{\ell_1,1,4} + M\frac{1+R}{\kappa^2}R\varepsilon^2 + M(1+M)\varepsilon^2 \\ \leq & \left(M + M\frac{1+R}{\kappa^2}R + M(1+M)\right)\varepsilon^2 \leq (1+M)^2\varepsilon^2, \\ \|\widetilde{\Pi}[\mathcal{F}^\sharp(\xi)]\|_{\ell_1,1,3} + \|\partial_y \widetilde{\Pi}[\mathcal{F}^\sharp(\xi)]\|_{\ell_1,1,4} \leq & \|\mathcal{F}^\sharp(0)\|_{\ell_1,1,3} + \|\partial_y \mathcal{F}^\sharp(0)\|_{\ell_1,1,4} + \frac{M}{\kappa^2}R\varepsilon^2 \\ \leq & \left(M + \frac{M}{\kappa^2}R\right)\varepsilon^2 \leq (1+M)\varepsilon^2,\end{aligned}$$

and for any $\xi, \xi' \in S$,

$$\begin{aligned}|\mathcal{F}^\sharp(\xi) - \mathcal{F}^\sharp(\xi')|_M \leq & \left(M\frac{1+R}{\kappa^2} + (1+M)\frac{M}{\kappa^2}\right)\big(\|\Pi_1[\xi-\xi']\|_{1,3} + \|\widetilde{\Pi}[\xi-\xi']\|_{\ell_1,1,3} + \|\partial_y \Pi_1[\xi-\xi']\|_{1,4} \\ & + \|\partial_y \widetilde{\Pi}[\xi-\xi']\|_{\ell_1,1,4}\big) + M(\|\widetilde{\Pi}[\xi-\xi']\|_{\ell_1,1,3} + \|\partial_y \widetilde{\Pi}[\xi-\xi']\|_{\ell_1,1,4}) \\ \leq & \left(M\frac{1+R}{\kappa^2} + (1+M)\frac{M}{\kappa^2} + \frac{M}{1+M}\right)|\xi-\xi'|_M.\end{aligned}$$

Therefore our above claim holds if κ is large and $\mathcal{F}^\sharp$ has a unique fixed point $\xi^u \in S \subset \mathcal{B}_0(R\varepsilon^2)$. It clearly satisfies all desired properties in Theorem 3.1. Using that g given in (1.19) is an odd function, a straightforward computation shows that the operator $\mathcal{F}$ in (3.2) leaves invariant the subspace of functions $\xi : D_\kappa^{\mathrm{out},u} \times \mathbb{T} \to \mathbb{C}$ satisfying $\Pi_{2l}[\xi] = 0$, $\forall l \geq 0$. Consequently, ξ^u satisfies that $\Pi_{2l}[\xi^u] = 0$, $\forall l \geq 0$ which completes the proof of Theorem 3.1.

5. The Inner equation: Proof of Theorem 3.3

We look for solutions odd in τ of the inner equation (3.9) as

$$\phi^0 = \sum_{n \geq 1} \phi_n^0 \sin(n\tau). \tag{5.1}$$

Substituting (5.1) into (3.9), we obtain that

$$(\partial_z^2 + (n^2-1))\phi_n^0 + \Pi_n\left[\frac{1}{3}(\phi^0)^3 + f(\phi^0)\right] = 0, \qquad n \geq 1. \tag{5.2}$$

As explained in Section 3, we look for solutions of the form

$$\phi^0(z,\tau) = \frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi(z,\tau) \qquad \text{with} \qquad \psi = \mathcal{O}\left(\frac{1}{z^3}\right). \tag{5.3}$$

Then, by (5.2), $\psi(z,\tau) = \sum_{n\geq 1} \psi_n(z)\sin(n\tau)$ must satisfy

$$\begin{cases} \partial_z^2 \psi_1 - \dfrac{6}{z^2}\psi_1 = -\Pi_1\left[-\dfrac{8}{z^2}\sin^2(\tau)\widetilde{\Pi}\,[\psi] - \dfrac{2\sqrt{2}i}{z}\sin(\tau)\psi^2 + \dfrac{1}{3}\psi^3 + f\left(\dfrac{-2\sqrt{2}i}{z}\sin(\tau) + \psi\right)\right], \\ \partial_z^2 \psi_n + \mu_n^2 \psi_n = -\Pi_n\left[\dfrac{1}{3}\left(\dfrac{-2\sqrt{2}i}{z}\sin(\tau) + \psi\right)^3 + f\left(\dfrac{-2\sqrt{2}i}{z}\sin(\tau) + \psi\right)\right], \ n \geq 2, \end{cases} \tag{5.4}$$

where $' = d/dz$, and $\mu_n = \sqrt{n^2-1}$.

We observe that the nonlinearity $f(u) = \mathcal{O}(|u|^5)$ in (5.2) does not have to be a real analytic function, so we complexify the space $\mathcal{F}_r$, $r > 0$, in (1.7), into a complex Banach space

$$\mathcal{F}_r^c = \Big\{f : \{u \in \mathbb{C} : |u| < r\} \to \mathbb{C},\ f \text{ is odd, analytic, and } f(u) = \sum_{k \geq 2} f_k u^{2k+1},\ \|f\|_r < \infty\Big\}, \tag{5.5}$$

where

$$\|f\|_r := \sum_{k=0}^{\infty} |f_k| r^k, \quad \text{for} \quad f(u) = \sum_{k=0}^{\infty} f_k u^k. \tag{5.6}$$

We define the operators

$$\mathcal{I}(\psi) = \left(\partial_z^2 \psi_1 - \frac{6}{z^2}\psi_1\right)\sin(\tau) + \sum_{n\geq 2}\left(\partial_z^2\psi_n + \mu_n^2\psi_n\right)\sin(n\tau) \tag{5.7}$$

$$\begin{aligned}\mathcal{W}(f,\psi) = &- \Pi_1\left[-\frac{8}{z^2}\sin^2(\tau)\widetilde{\Pi}\left[\psi\right] - \frac{2\sqrt{2}i}{z}\sin(\tau)\psi^2 + \frac{1}{3}\psi^3\right]\sin(\tau) \\ &- \widetilde{\Pi}\left[\frac{1}{3}\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)^3\right] - f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)\end{aligned} \tag{5.8}$$

and notice that, for $\star = u, s$, to find a solution $\phi^{0,\star}$ of (3.9) satisfying (5.3) is equivalent to find a solution $\psi^\star$ of the functional equation

$$\mathcal{I}(\psi) = \mathcal{W}(f,\psi), \tag{5.9}$$

which satisfies $\psi^\star \sim \mathcal{O}(z^{-3})$ for $z \in D^{\star,\text{in}}_{\theta,\kappa}$, $\star = u, s$, as defined in (3.10). In the remainder of this section, we look for solutions of (5.9) with such asymptotics through a fixed point argument and analyze their dependence on $f \in \mathcal{F}^c_r$. As before, we consider only the unstable case, since the stable one is completely analogous.

5.1. **Banach Spaces and Linear Operators.** Given $\alpha \geq 0$ and an analytic function $h : D^{u,\text{in}}_{\theta,\kappa} \to \mathbb{C}$, where $D^{u,\text{in}}_{\theta,\kappa}$ is given in (3.10), consider the norm

$$\|h\|_\alpha = \sup_{z\in D^{u,\text{in}}_{\theta,\kappa}} |z^\alpha h(z)|,$$

and the Banach space

$$\mathcal{X}_\alpha = \{h : D^{u,\text{in}}_{\theta,\kappa} \to \mathbb{C};\ h \text{ is an analytic function and } \|h\|_\alpha < \infty\}.$$

Moreover, for $h : D^{u,\text{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C}$, analytic in the variable z, we define

$$\|h\|_{\ell_1,\alpha} = \sum_{n\geq 1} \|h_n\|_\alpha,$$

and the Banach space

$$\mathcal{X}_{\ell_1,\alpha} = \left\{h : D^{u,\text{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C};\ h \text{ is an analytic function in the variable } z \text{ and } \|h\|_{\ell_1,\alpha} < \infty\right\}.$$

Lemma 5.1. *Let $r > 0$. Given an analytic function $f : \{u \in \mathbb{C} : |u| < r\} \to \mathbb{C}$ and $g, h : D^{u,\text{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C}$, the following statements hold for some M depending only on θ and r,*

(1) *If $\alpha \geq \beta \geq 0$, then*
$$\|h\|_{\ell_1,\alpha-\beta} \leq \frac{M}{\kappa^\beta}\|h\|_{\ell_1,\alpha}.$$
(2) *If $\alpha, \beta \geq 0$, and $\|g\|_{\ell_1,\alpha}, \|h\|_{\ell_1,\beta} < \infty$, then*
$$\|gh\|_{\ell_1,\alpha+\beta} \leq \|g\|_{\ell_1,\alpha}\|h\|_{\ell_1,\beta}.$$
(3) *If $\alpha \geq 0$, $g, h \in \mathcal{X}_{\ell_1,\alpha}$ and $\|g\|_{\ell_1,0}$, $\|h\|_{\ell_1,0} < r/2$, then*
$$\|f(g) - f(h)\|_{\ell_1,\alpha} \leq M\|f\|_r\|g - h\|_{\ell_1,\alpha}.$$
(4) *Given $n \geq 0$, if $f^{(k)}(0) = 0$, for every $0 \leq k \leq n-1$, and $\|g\|_{\ell_1,0} < r/2$, then*
$$\|f(g)\|_{\ell_1,n\alpha} \leq M\|f\|_r(\|g\|_{\ell_1,\alpha})^n.$$
M also depends on n.

(5) *If $h \in \mathcal{X}_{\ell_1,\alpha}$ (with respect to the inner domain $D^{u,\text{in}}_{\theta,\kappa}$), then $\partial_z h \in \mathcal{X}_{\ell_1,\alpha+1}$ (with respect to the inner domain $D^{u,\text{in}}_{2\theta,4\kappa}$), and*

$$\|\partial_z h\|_{\ell_1,\alpha+1} \leq M\|h\|_{\ell_1,\alpha}.$$

Proof. Items (1)(2)(5) of this lemma are proved as Lemma 4.3 in [3]. To prove (3) and (4), let $f(u) = \sum_{k=0}^{\infty} f_k u^k$. One may estimate using item (2),

$$\begin{aligned}\|f(g)-f(h)\|_{\ell_1,\alpha} =& \Big\|(g-h)\sum_{k=0}^{\infty} f_{k+1}\sum_{j=0}^{k} g^{k-j}h^j\Big\|_{\ell_1,\alpha} \leq \sum_{k=0}^{\infty}|f_{k+1}|\sum_{j=0}^{k}\|g\|_{\ell_1,0}^{k-j}\|h\|_{\ell_1,0}^{j}\|g-h\|_{\ell_1,\alpha}\\ \leq& \sum_{k=0}^{\infty}(k+1)|f_{k+1}|\Big(\frac{r}{2}\Big)^k\|g-h\|_{\ell_1,\alpha} = \sum_{k=0}^{\infty}\frac{k+1}{2^k r}|f_{k+1}|r^{k+1}\|g-h\|_{\ell_1,\alpha},\end{aligned}$$

which implies item (3). Again based on item (2), the proof of item (4) is similar

$$\|f(g)\|_{\ell_1,n\alpha} = \Big\|\sum_{k=n}^{\infty} f_k g^k\Big\|_{\ell_1,n\alpha} \leq \sum_{k=n}^{\infty}|f_k|\|g\|_{\ell_1,0}^{k-n}\|g\|_{\ell_1,\alpha}^n \leq \sum_{k=n}^{\infty}|f_k|r^k r^{-n}\|g\|_{\ell_1,\alpha}^n$$

and thus item (4) follows. □

Now, define the linear operator acting on the Fourier coefficients of ψ

$$\mathcal{J}(\psi) = \sum_{n\geq 1}\mathcal{J}_n(\psi_n)\sin(n\tau),$$

where

$$\begin{aligned}\mathcal{J}_1(\psi_1)(z) &= \frac{z^3}{5}\int_{-\infty}^{z}\frac{\psi_1(s)}{s^2}ds - \frac{1}{5z^2}\int_{-\infty}^{z}s^3\psi_1(s)ds\\ \mathcal{J}_n(\psi_n)(z) &= \frac{1}{2i\mu_n}\int_{-\infty}^{z}e^{-i\mu_n(s-z)}\psi_n(s)ds - \frac{1}{2i\mu_n}\int_{-\infty}^{z}e^{i\mu_n(s-z)}\psi_n(s)ds,\ \ n\geq 2.\end{aligned} \tag{5.10}$$

See Remark 4.2 regarding the integral paths.

Proposition 5.2. *Consider $\kappa \geq 1$ big enough. Given $\alpha > 2$, the operator $(\partial_\tau^2)\circ\mathcal{J} : \mathcal{X}_{\ell_1,\alpha+2} \to \mathcal{X}_{\ell_1,\alpha}$ is well defined and the following statements hold.*

(1) *$\mathcal{J}\circ\mathcal{I}(\psi) = \mathcal{I}\circ\mathcal{J}(\psi) = \psi$.*

(2) *For any $\alpha > 2$, there exists a constant $M > 0$ independent of κ such that, for every $h \in \mathcal{X}_{\alpha+2}$,*

$$\|\mathcal{J}_1(h)\|_\alpha \leq M\|h\|_{\alpha+2}.$$

(3) *For any $\alpha > 1$, there exists a constant $M > 0$ independent of κ and n such that, for every $h \in \mathcal{X}_\alpha$,*

$$\|\mathcal{J}_n(h)\|_\alpha \leq \frac{M}{\mu_n^2}\|h\|_\alpha.$$

Again the above estimates represent the gain of one more order of derivative in τ. The assumption $\alpha > 1$ in the above last inequality ensures the convergence of the integral in the definition of $\mathcal{J}_n$ and also allows one to adjust the path of the integral in certain ways.

Proof. The proof of item (1) is straightforward. For $\mathcal{J}_n$, $n \geq 2$ and $\alpha > 1$, one can use the same trick as in the proof of Lemma 4.6 in [3], by using the Cauchy Integral Theorem to move the integral paths to the rays $\{z - se^{\pm i\theta} : s > 0\}$ for the two integrals respectively, to obtain that for $h \in \mathcal{X}_\alpha$ and $z \in D^{u,\text{in}}_{\theta,\kappa}$,

$$\begin{aligned}|z^\alpha\mathcal{J}_n(h)(z)| =& \left|\frac{z^\alpha e^{i\theta}}{2i\mu_n}\int_0^{\infty}e^{i\mu_n se^{i\theta}}h(z-se^{i\theta})ds - \frac{z^\alpha e^{-i\theta}}{2i\mu_n}\int_0^{\infty}e^{-i\mu_n se^{-i\theta}}h(z-se^{-i\theta})ds\right|\\ \leq& \frac{1}{2\mu_n}\left(\int_0^{\infty}e^{-\mu_n(\sin\theta)s}|z|^\alpha|h(z-se^{i\theta})|ds + \int_0^{\infty}e^{-\mu_n(\sin\theta)s}|z|^\alpha|h(z-se^{-i\theta})|ds\right) \leq \frac{M}{\mu_n^2}\|h\|_\alpha.\end{aligned}$$

For $\mathcal{J}_1$, taking $h \in \mathcal{X}_{\alpha+2}$, $\alpha > 2$ and $z \in D^{u,\mathrm{in}}_{\theta,\kappa}$,

$$\begin{aligned}|z^\alpha \mathcal{J}_1(h)(z)| =& \left| \frac{z^{\alpha+3}}{5} \int_{-\infty}^{z} \frac{h(s)}{s^2} ds - \frac{z^{\alpha-2}}{5} \int_{-\infty}^{z} s^3 h_1(s) ds \right| \\ \leq& M\|h\|_{\alpha+2} \left(\int_{-\infty}^{z} \frac{|z|^{\alpha+3}}{|s|^{\alpha+4}} ds + \int_{-\infty}^{z} \frac{|z|^{\alpha-2}}{|s|^{\alpha-1}} ds \right) \leq M\|h\|_{\alpha+2}.\end{aligned}$$

The proof of the proposition is complete. □

5.2. **The fixed point argument.** By Proposition 5.2, we rewrite (5.9) as

$$\psi = \mathcal{W}^\sharp(f,\psi), \quad \mathcal{W}^\sharp = \mathcal{J} \circ \mathcal{W}, \qquad f \in \mathcal{F}^c_r, \ \|\psi\|_{\ell_1,0} \leq r,$$

where $\mathcal{W}$ is given by (5.8). In the following proposition we study some properties of the operator $\mathcal{W}^\sharp$.

Proposition 5.3. *Given $r > 0$, for big enough $\kappa \geq \max\{1, 100/r\}$ and $R < \min\{\kappa^2, r\kappa^3/100\}$, the operator $\mathcal{W}^\sharp : \mathcal{F}^c_r \times \mathcal{B}_0(R) \to \mathcal{X}_{\ell_1,3}$ (where $\mathcal{B}_0(R) \subset \mathcal{X}_{\ell_1,3}$ is the ball of radius R) is analytic in both f and ψ and the following statements hold.*

1. *There exists a constant $M_1 > 0$ depending only on θ and r such that $\|\partial^2_\tau \mathcal{W}^\sharp(f,0)\|_{\ell_1,3} \leq M_1(1+\|f\|_r)$.*
2. *There exists a constant $M_2 > 1$ depending only on θ and r such that, for every $\psi, \psi' \in \mathcal{B}_0(R) \subset \mathcal{X}_{\ell_1,3}$,*

$$\left\|\mathcal{W}^\sharp(f,\psi) - \mathcal{W}^\sharp(f,\psi')\right\|_{\ell_1,3} \leq M_2 \left(\frac{1}{\kappa^2}(1 + R + \|f\|_r)\|\psi - \psi'\|_{\ell_1,3} + \|\widetilde{\Pi}[\psi] - \widetilde{\Pi}[\psi']\|_{\ell_1,3} \right).$$

Furthermore,

$$\left\|\partial^2_\tau \left(\widetilde{\Pi}[\mathcal{W}^\sharp(f,\psi)] - \widetilde{\Pi}[\mathcal{W}^\sharp(f,\psi')] \right) \right\|_{\ell_1,3} \leq \frac{M_2}{\kappa^2} \left(1 + R + \|f\|_r\right) \|\psi - \psi'\|_{\ell_1,3}.$$

Proof. $\mathcal{W}(f,0)$ is given by

$$\mathcal{W}(f,0) = -\widetilde{\Pi} \left[\frac{1}{3} \left(\frac{-2\sqrt{2}i}{z} \sin(\tau) \right)^3 \right] - f \left(\frac{-2\sqrt{2}i}{z} \sin(\tau) \right).$$

Thus, since $f(z) = \mathcal{O}(z^5)$, it follows from Lemma 5.1(4) that

$$\|\Pi_1[\mathcal{W}(f,0)]\|_5 \leq M\|f\|_r \left\| \frac{-2\sqrt{2}i}{z} \sin(\tau) \right\|^5_{\ell_1,1} \leq M\|f\|_r,$$

$$\left\|\widetilde{\Pi}[\mathcal{W}(f,0)]\right\|_{\ell_1,3} \leq M \left(\left\| \frac{-2\sqrt{2}i}{z} \sin(\tau) \right\|^3_{\ell_1,1} + \frac{\|f\|_r}{\kappa^2} \left\| \frac{-2\sqrt{2}i}{z} \sin(\tau) \right\|^5_{\ell_1,1} \right) \leq M \left(1 + \kappa^{-2}\|f\|_r \right).$$

Hence, from Lemma 5.1 and Proposition 5.2, there exists $M_1 > 0$ such that

$$\begin{aligned}\left\|\partial^2_\tau W^\sharp(f,0)\right\|_{\ell_1,3} \leq& \left\|\partial^2_\tau \mathcal{J}\left(\Pi_1[\mathcal{W}(f,0)]\sin(\tau)\right)\right\|_{\ell_1,3} + \left\|\partial^2_\tau \mathcal{J}\left(\widetilde{\Pi}[\mathcal{W}(f,0)]\right)\right\|_{\ell_1,3} \\ \leq& M \left(\|\Pi_1[\mathcal{W}(f,0)]\|_5 + \left\|\widetilde{\Pi}[\mathcal{W}(f,0)]\right\|_{\ell_1,3} \right) \leq M_1\left(1 + \|f\|_r\right).\end{aligned}$$

To prove item (2) on the Lipschitz property, assume that $\|\psi\|_{\ell_1,3}, \|\psi'\|_{\ell_1,3} \leq R$, and notice that

$$\begin{aligned}\mathcal{W}(f,\psi) - \mathcal{W}(f,\psi') =& -\Pi_1 \left[-\frac{8}{z^2} \sin^2(\tau) \left(\widetilde{\Pi}\left[\psi - \psi'\right] \right) - \frac{2\sqrt{2}i}{z} \sin(\tau) \left(\psi^2 - (\psi')^2 \right) \right. \\ &\left. + \frac{1}{3}\left(\psi^3 - (\psi')^3 \right) \right] \sin(\tau) - \frac{1}{3}\widetilde{\Pi} \left[\left(\frac{-2\sqrt{2}i}{z} \sin(\tau) + \psi \right)^3 - \left(\frac{-2\sqrt{2}i}{z} \sin(\tau) + \psi' \right)^3 \right] \\ &- f \left(\frac{-2\sqrt{2}i}{z} \sin(\tau) + \psi \right) + f \left(\frac{-2\sqrt{2}i}{z} \sin(\tau) + \psi' \right).\end{aligned}$$

Thus, again from Lemma 5.1,

$$
\begin{aligned}
\left\|\Pi_1\left[\mathcal{W}(f,\psi)-\mathcal{W}(f,\psi')\right]\right\|_5 \leq & \left\|\frac{8}{z^2}\sin^2(\tau)\right\|_{\ell_1,2}\left\|\widetilde{\Pi}\left[\psi-\psi'\right]\right\|_{\ell_1,3} \\
& + \left(\left\|\frac{2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}\|\psi+\psi'\|_{\ell_1,1}+\left\|\psi^2+\psi\psi'+(\psi')^2\right\|_{\ell_1,2}\right)\|\psi-\psi'\|_{\ell_1,3} \\
& + \left\|\int_0^1 f'\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+s\psi+(1-s)\psi'\right)ds\right\|_{\ell_1,2}\|\psi-\psi'\|_{\ell_1,3} \\
\leq & M\left(\left\|\widetilde{\Pi}\left[\psi-\psi'\right]\right\|_{\ell_1,3}+\frac{1}{\kappa^2}(R+\|f'\|_{\frac{r}{2}})\|\psi-\psi'\|_{\ell_1,3}\right) \\
\leq & M\left(\left\|\widetilde{\Pi}\left[\psi-\psi'\right]\right\|_{\ell_1,3}+\frac{1}{\kappa^2}(R+\|f\|_r)\|\psi-\psi'\|_{\ell_1,3}\right),
\end{aligned}
$$

and, recalling from (1.19) that $g(z)=\frac{z^3}{3}+f(z)=\mathcal{O}(z^3)$, we have that

$$
\begin{aligned}
\left\|\widetilde{\Pi}\left[\mathcal{W}(f,\psi)-\mathcal{W}(f,\psi')\right]\right\|_{\ell_1,3} \leq & \left\|\int_0^1 g'\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+s\psi+(1-s)\psi'\right)ds\right\|_{\ell_1,0}\|\psi-\psi'\|_{\ell_1,3} \\
\leq & \frac{M}{\kappa^2}\|g'\|_{\frac{r}{2}}\|\psi-\psi'\|_{\ell_1,3}\leq\frac{M}{\kappa^2}(1+\|f\|_r)\|\psi-\psi'\|_{\ell_1,3}.
\end{aligned}
$$

Item (2) follows from the estimates above and Proposition 5.2.

Finally we prove the analyticity of $\mathcal{W}^\sharp(f,\psi)$. Since $\mathcal{J}$ is linear, it suffices to show that $\mathcal{W}(f,\psi)$ is analytic in $f\in\mathcal{F}_r^c$ and $\psi\in\mathcal{B}_0(R)\subset\mathcal{X}_{\ell_1,3}$, which is equivalent to the analyticity of $(f,\psi)\to f\big(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\big)$ as the analyticity of the other terms is obvious. For any $\psi_0\in\mathcal{B}_0(R)$, let us denote

$$\varphi_0=\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi_0,$$

which, due to Lemma 5.1, satisfies

$$\|\varphi_0\|_{\ell_1,1}\leq 3+\kappa^{-2}\|\psi_0\|_{\ell_1,3}\leq 3+\kappa^{-2}R\leq 4.$$

Consider also $f\in\mathcal{F}_r^c$,

$$f(u)=\sum_{k=0}^{\infty}f_k u^k,\quad f_k\in\mathbb{C},\quad f_k=0,\ \forall k\in\{j\in\mathbb{N}:2|j \text{ or } j<5\},\quad \|f\|_r<\infty,$$

where the coefficient sequence (f_k) can be viewed as the coordinates of f. Near ψ_0, one may compute

$$
\begin{aligned}
f\Big(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi_0+\psi\Big)=&f(\varphi_0+\psi)=\sum_{k=0}^{\infty}f_k(\varphi_0+\psi)^k=\sum_{k=0}^{\infty}f_k\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}\varphi_0^{k-j}\psi^j \\
=&\sum_{j=0}^{\infty}\Big(\sum_{k=j}^{\infty}\frac{k!}{j!(k-j)!}f_k\varphi_0^{k-j}\Big)\psi^j\triangleq\sum_{j=0}^{\infty}A_j(\varphi_0)(f,\psi),
\end{aligned}
$$

where

$$A_j(\varphi_0)(f,\psi)=\sum_{k=j}^{\infty}\frac{k!}{j!(k-j)!}f_k\varphi_0^{k-j}\psi^j.$$

The $(j+1)$-linear transformation $A_j(\varphi_0)$ of f and ψ can be estimated by Lemma 5.1 as, for $j=0$,

$$\|A_0(\varphi_0)(f,\psi)\|_{\ell_1,3}\leq\sum_{k=5}^{\infty}|f_k|\|\varphi_0^k\|_{\ell_1,3}\leq\sum_{k=5}^{\infty}|f_k|r^k r^{-k}\|\varphi_0\|_{\ell_1,0}^{k-3}\|\varphi_0\|_{\ell_1,1}^3\leq M\sum_{k=5}^{\infty}|f_k|r^k\big(\frac{r\kappa}{4}\big)^{3-k}\leq M\|f\|_r,$$

and for $j \geq 1$,

$$\begin{aligned}\|A_j(\varphi_0)(f,\psi)\|_{\ell_1,3} \leq& \sum_{k=\max\{j,5\}}^{\infty} \frac{k!}{j!(k-j)!}|f_k|\|\varphi_0^{k-j}\psi^j\|_{\ell_1,3}\\ \leq& \|\psi\|_{\ell_1,3} \sum_{k=\max\{j,5\}}^{\infty} |f_k| r^k \frac{k!}{j!(k-j)!} r^{-k}\|\varphi_0\|_{\ell_1,0}^{k-j}\|\psi\|_{\ell_1,0}^{j-1}\\ \leq& \|\psi\|_{\ell_1,3}^j \sum_{k=\max\{j,5\}}^{\infty} |f_k| r^k \frac{k!4^{k-j}}{j!(k-j)!} r^{-k}\kappa^{-k-2j+3}.\end{aligned}$$

Since, using $\frac{k!}{(k-j)!} \leq k^j$, for any $k \geq \max\{j,5\}$,

$$\begin{aligned}\frac{k!4^{k-j}}{j!(k-j)!} r^{-k}\kappa^{-k-2j+3} \leq& \frac{1}{j!4^j\kappa^{2j-3}} k^j \Big(\frac{4}{r\kappa}\Big)^k = \frac{j^j}{j!4^j\kappa^{2j-3}}\Big(\log\frac{r\kappa}{4}\Big)^{-j}\Big(\Big(\frac{k}{j}\log\frac{r\kappa}{4}\Big)e^{-\frac{k}{j}\log\frac{r\kappa}{4}}\Big)^j\\ \leq& \frac{j^j\kappa^3}{j!e^j}\Big(4\kappa^2\log\frac{r\kappa}{4}\Big)^{-j} \leq Mj^{-\frac{1}{2}}\kappa^3\Big(4\kappa^2\log\frac{r\kappa}{4}\Big)^{-j},\end{aligned}$$

where the Stirling's approximation was used in the last step. Hence, $A_j(\varphi_0)$ is a bounded multi-linear transformation, which satisfies

$$\|A_j(\varphi_0)(f,\psi)\|_{\ell_1,3} \leq Mj^{-\frac{1}{2}}\kappa^3\Big(4\kappa^2\log\frac{r\kappa}{4}\Big)^{-j}\|f\|_r\|\psi\|_{\ell_1,3}^j.$$

This estimate also implies the analyticity of $f\big(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\big)$ in f and ψ. □

Proof of Theorem 3.3(1). Much as in the proof of Theorem 3.1, we use an equivalent norm on $\mathcal{X}_{\ell_1,3}$

$$\|\psi\|_* := \|\psi_1\|_3 + 2M_2\|\widetilde{\Pi}[\psi]\|_{\ell_1,3},$$

where M_2 is the constant resulted in Proposition 5.3(2). Let $R_0 > 0$ and consider $f \in \mathcal{F}_r^c$ with $\|f\|_r \leq R_0$. Using Proposition 5.3, it is straight forward to verify that, with in the above norm $\|\cdot\|_*$ for sufficiently large $\kappa > 0$, $\mathcal{W}^\sharp$ is a contraction on the closed ball of $\mathcal{X}_{\ell_1,3}$ with radius $R = 3M_1(1+2M_2)(1+R_0)$ with Lipschitz constant $\kappa^{-2}(1+R+R_0)M_2(1+2M_2)+\frac{1}{2} < \frac{2}{3}$. The unique fixed point ψ^u depends on $f \in \mathcal{F}_r^c$ analytically and gives the unstable solution $\phi^{0,u}$ in the form of (5.3) which satisfies the desired estimates. Using the same arguments in the proof of Theorem 3.1, one can conclude $\Pi_{2l}[\psi^u] \equiv 0$, $\forall l \geq 0$. □

5.3. **The difference between the solutions of the Inner Equation.** This section is devoted to prove the second and third statement of Theorem 3.3. We consider the two solutions $\phi_0^{u,s}$ of the inner equation (3.9) which are given by (3.11) and we study the difference

$$\Delta\psi(z,\tau) = \phi^{0,u}(z,\tau) - \phi^{0,s}(z,\tau) = \psi^u(z,\tau) - \psi^s(z,\tau),$$

for $z \in \mathcal{R}_{\theta,\kappa}^{\text{in},+} = D_{\theta,\kappa}^{u,\text{in}} \cap D_{\theta,\kappa}^{s,\text{in}} \cap \{z :\ z \in i\mathbb{R} \text{ and } \operatorname{Im}(z) < 0\}$ and $\tau \in \mathbb{T}$. For this purpose, we actually work on (3.9) as an ill-posed dynamical system of real independent variable along $\mathcal{R}_{\theta,\kappa}^{\text{in},+}$.

Remark 5.4. *We are interested in the behavior of the difference in the connected component* $\mathcal{R}_{\theta,\kappa}^{\text{in},+}$ *of* $D_{\theta,\kappa}^{u,\text{in}} \cap D_{\theta,\kappa}^{s,\text{in}} \cap i\mathbb{R}$ *because the change* $z = \varepsilon^{-1}(y - i\pi/2)$ *brings the origin* $y = 0$ *into* $z = -i\varepsilon^{-1}\pi/2 \in \mathcal{R}_{\theta,\kappa}^{\text{in},+}$.

Let $r \gg 1$. We define the change of variables

$$z = -ir, \quad \Psi_1(r) = \phi_1^0(-ir), \quad \Psi_{n\pm}(r) = \partial_r\big(\phi_n^0(-ir)\big) \pm \sqrt{n^2-1}\,\phi_n^0(-ir), \ \ n \geq 3. \tag{5.11}$$

That is

$$\phi^0(-ir,\tau) = \Psi_1(r)\sin\tau + \sum_{n\geq 3}\frac{1}{2\sqrt{n^2-1}}(\Psi_{n+}(r) - \Psi_{n-}(r))\sin n\tau, \tag{5.12}$$

$$\partial_r\big(\phi^0(-ir,\tau)\big) = \partial_r\Psi_1(r)\sin\tau + \sum_{n\geq 3}\frac{1}{2}(\Psi_{n+}(r) + \Psi_{n-}(r))\sin n\tau. \tag{5.13}$$

Then, equation (3.9) takes the form

$$\begin{cases} \partial_r^2 \Psi_1 - \frac{1}{4}\Psi_1^3 = F_1(\Psi) \\ \partial_r \Psi_{n\pm} = \pm\sqrt{n^2-1}\Psi_{n\pm} + F_n(\Psi), \end{cases} \tag{5.14}$$

where

$$F = (F_n)_{n=1}^{\infty}, \qquad F_1(\Psi) = \Pi_1\Big[\frac{1}{3}(\phi^0)^3 + f(\phi^0)\Big] - \frac{1}{4}\Psi_1^3, \qquad F_n(\Psi) = \Pi_n\Big[\frac{1}{3}(\phi^0)^3 + f(\phi^0)\Big], \quad n \geq 3.$$

Since $\frac{1}{4}\Psi_1^3$ in the nonlinearity is isolated into the left side of (5.14), the cubic terms in F_1 do not include Ψ_1^3. Note that, by item (1) of Theorem 3.3, we can restrict to the space of *odd* n's.

Let $\Psi^*(r)$, $* = u, s$, be the functions $\phi^{0,*}$, $* = u, s$, expressed in the coordinates introduced in (5.11). We are interested in $\Psi^s - \Psi^u$ as $r \to +\infty$ where, since we shall consider certain local invariant manifolds/foliation which are not necessarily analytic submanifolds, we work in the space ℓ_2 with the smooth norm

$$\|\Psi\|_{\ell_2}^2 := |\Psi_1|^2 + |\partial_r\Psi_1|^2 + \sum_{n=3,\text{odd}}^{\infty} n^2(|\Psi_{n+}|^2 + |\Psi_{n-}|^2)$$

and treat $\Psi_1, \partial_r\Psi_1, \Psi_{n\pm}$ as 2-dim real vectors. We also define

$$\Psi_c = (\Psi_1, \partial_r\Psi_1), \qquad \Psi_\pm = (\Psi_{n\pm})_{n=3}^{\infty}.$$

Part (1) of Theorem 3.3 implies that $\Psi^{u,s}$ do belong to the ℓ_2 space.

It is easy to see that F defines a smooth mapping on the ℓ_2 space. Due to both positively and negatively unbounded exponential growth rates caused by the linear parts, (5.14) is ill-posed both forward and backward in r. However, after multiplying a smooth cut-off function based on $\|\cdot\|_{\ell_2}$ to the nonlinearities F, the standard Lyapunov-Perron approach still yields smooth local invariant manifolds and foliations near $\Psi = 0$, including an infinite dimensional center-stable manifold W^{cs} where (5.14) is well-posed for $r > 0$ (see e. g. Theorem 4.4 in [16]), the 4-dim center manifold $W^c \subset W^{cs}$ (again see Theorem 4.4 in [16]), and stable fibers inside W^{cs} transverse to W^c (see e. g. Theorem 4.3 in [15]).[5] We shall outline a framework to derive of W^{cs} and W^c and the stable foliation inside W^{cs} for (5.14).

Following the standard cut-off technique, take $\gamma \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying $\mathrm{supp}(\gamma) \subset (-2, 2)$ and $\gamma|_{[-1,1]} = 1$. Let $\delta > 0$ and

$$F^{\#}(\Psi) = \left(F_n^{\#}(\Psi)\right)_{n=1}^{\infty}, \quad F_1^{\#} = \gamma\left(\frac{\|\Psi\|_{\ell_2}^2}{\delta^2}\right)\left(F_1(\Psi) + \frac{1}{4}\Psi_1^3\right), \quad F_n^{\#} = \gamma\left(\frac{\|\Psi\|_{\ell_2}^2}{\delta^2}\right)(F_n(\Psi)),\ n \geq 3.$$

Consider

$$\begin{aligned} \partial_r^2\Psi_1 &= F_1^{\#}(\Psi) \\ \partial_r\Psi_{n\pm} &= \pm\sqrt{n^2-1}\Psi_{n\pm} + F_n^{\#}(\Psi), \end{aligned} \tag{5.15}$$

whose nonlinearity has small Lipschitz constants for $\delta \ll 1$. We shall work on the global center-stable and center manifolds and stable foliations of (5.15). It is clear that solutions of (5.14) and (5.15) coincide in the δ-ball of ℓ_2 and thus we obtain local invariant manifolds and foliations of (5.14) containing $\Psi^{u,s}(r)$.

Center-stable manifold. The center-stable mainfold $W^{cs} = \{\Psi_+ = h^{cs}(\Psi_c, \Psi_-)\}$ of (5.15) is represented as a graph of a mapping h^{cs} satisfying

- $h^{cs} \in C^4$, $D^j h^{cs}(0) = 0$, $j = 0, 1, 2$, and h^{cs} is odd, i. e. $h^{cs}(-\Psi_c, -\Psi_-) = -h^{cs}(\Psi_c, \Psi_-)$.
- *Invariance*: if $\Psi_* \in W^{cs}$, then there exists a unique solution $\Psi(r) \in W^{cs}$, $r \geq 0$, to (5.15) such that $\Psi(0) = \Psi_*$ and
$$\Psi(\cdot) \subset \mathcal{E}^{cs} := \left\{\psi \in C^0\left([0,\infty), \ell_2\right) \mid \sup_{r\geq 0} e^{-r}\|\psi(r)\|_{\ell_2} < \infty\right\}.$$
- Any solution $\Psi(\cdot) \in \mathcal{E}^{cs}$ to (5.15) satisfies $\Psi(r) \in W^{cs}$ for any $r \geq 0$.

[5]Even though the linear operators in [15, 16] are assumed to be sectorial operators generating analytic semigroups, which is not satisfied by wave type PDEs, the same proofs and results still hold when the nonlinearity are smooth mappings on the phase spaces (i. e. without loss of regularity), which is the case of (5.14) when posed in ℓ_2 space.

To outline its construction, one observes that a solution $\Psi(r)$, $r \geq 0$, to (5.15)belongs to $\mathcal{E}^{cs}$ iff

$$\Psi_1(r) = \Psi_1(0) + r\partial_r \Psi_1(0) + \int_0^r (r-\tau) F_1^{\#}(\Psi(\tau) d\tau,$$
$$\Psi_{n-}(r) = e^{-\sqrt{n^2-1}r}\Psi_{n-}(0) + \int_0^r e^{-\sqrt{n^2-1}(r-\tau)} F_n^{\#}(\Psi(\tau))d\tau,$$
$$\Psi_{n+}(r) = -\int_r^{+\infty} e^{\sqrt{n^2-1}(r-\tau)} F_n^{\#}(\Psi(\tau))d\tau.$$

Denote the above righthand side as $\mathcal{T}\big(\Psi_c(0), \Psi_-(0), \Psi(\cdot)\big)$. Following the proof of Theorem 4.4 (mostly consisting of Lemma 3.1 – 3.4) in [16] (or that of Theorem 4.2 in [15]), one may prove that, for $\delta \ll 1$,

a.) $\mathcal{T}$ is a contraction in $\Psi(\cdot) \in \mathcal{E}^{cs}$ possessing a unique fixed point $\Psi\big(\cdot, \Psi_c(0), \Psi_-(0)\big) \in \mathcal{E}^{cs}$ depending on parameters $\Psi_c(0)$ and $\Psi_-(0)$;

b.) the mapping h^{cs}, defined by

$$h^{cs}(\Psi_c(0), \Psi_-(0)) = \Psi_+\big(r, \Psi_c(0), \Psi_-(0)\big)\big|_{r=0}$$

from this fixed point, gives the smooth center-stable manifold W^{cs} invariant under (5.15).

The oddness of h^{cs} is obtained from the fact $\Psi(\cdot) \in \mathcal{E}^{cs}$ is a solution iff so is $-\Psi(\cdot) \in \mathcal{E}^{cs}$ due to the oddness of (5.15).

The property $Dh^{cs}(0) = 0$ always holds and corresponds to the tangency of W^{cs} to the center-stable subspace. Here the extra $D^2 h^{cs}(0) = 0$ is a natural consequence of the oddness of h^{cs} from that of (5.15). More essentially, it is implied by the lack of the quadratic nonlinearity in (5.15).

Inside W^{cs}: the center manifold W^c. Inside the center-stable invariant manifold there is the 4-dimensional center manifold $W^c = \{\Psi \in W^{cs} \mid \Psi_- = h^c(\Psi_c)\}$ of (5.15), which is represented as a graph of a mapping h^c satisfying

- $h^c \in C^4$ is odd, $D^j h^c(0) = 0$, $j = 0, 1, 2$.
- *Invariance*: if $\Psi_* \in W^c$, then there exists a unique solution $\Psi(r) \in W^c$, $r \in \mathbb{R}$, to (5.15) such that $\Psi(0) = \Psi_*$ and
$$\Psi(\cdot) \subset \mathcal{E}^c := \big\{\psi \in C^0\big(\mathbb{R}, \ell_2\big) \mid \sup_{r\in\mathbb{R}} e^{-|r|}\|\psi(r)\|_{\ell_2} < \infty\big\}.$$
- Any solution $\Psi(\cdot) \in \mathcal{E}^c$ to (5.15) satisfies $\Psi(r) \in W^c$ for any $r \in \mathbb{R}$.

Due to the invariance of W^{cs}, when restricted to W^{cs}, (5.15) is equivalent to

$$\begin{cases} \partial_r^2 \Psi_1 = F_1^{cs}(\Psi_c, \Psi_-) \\ \partial_r \Psi_{n-} = -\sqrt{n^2-1}\Psi_{n-} + F_n^{cs}(\Psi_c, \Psi_-), \end{cases} \tag{5.16}$$

where

$$F^{cs}(\Psi_c, \Psi_-) = \big(F_n^{cs}(\Psi_c, \Psi_-)\big)_{n=1}^{\infty} := F^{\#}\big(\Psi_c, \Psi_-, h^{cs}(\Psi_c, \Psi_-)\big).$$

The construction of W^c is essentially that of an unstable manifold in W^{cs} and thus again can follow from Theorem 4.4 in [16] as illustrated in the above framework for W^{cs}.

Inside W^{cs}: stable foliation and fiber coordinates. The invariant foliation theorem (e. g. Theorem 4.3 in [15]) implies that, for $\delta \ll 1$, there exist $h^s(\tilde\Psi_c, \tilde\Psi_-) \in \mathbb{R}^4$ (which we call the stable foliation mapping) and $\Psi = \Gamma(\tilde\Psi_c, \tilde\Psi_-)$ on W^{cs} (which we call the stable fiber coordinate system) such that

- $h^s \in C^4$ is odd, $D^j h^s(0) = 0$, $j = 0, 1, 2$, and $h^s(\tilde\Psi_c, 0) = 0$ for any $\tilde\Psi_c$.
- $\Psi = (\Psi_c, \Psi_-, \Psi_+) = \Gamma(\tilde\Psi_c, \tilde\Psi_-)$ is defined as

$$\Psi_c = \tilde\Psi_c + h^s(\tilde\Psi_c, \tilde\Psi_-), \quad \Psi_- = h^c(\tilde\Psi_c) + \tilde\Psi_-, \quad \Psi_+ = h^{cs}(\Psi_c, \Psi_-). \tag{5.17}$$

- Let $\Psi_j(r) = \Gamma(\tilde\Psi_{c,j}(r), \tilde\Psi_{-,j}(r))$, $j = 1, 2$, $r > 0$, be solutions to (5.15) and $\tilde\Psi_{c,1}(0) = \tilde\Psi_{c,2}(0)$, then
  - $\tilde\Psi_{c,1}(r) = \tilde\Psi_{c,2}(r)$ for all $r \geq 0$ (*invariance*), and
  - there exists $M > 0$ depending only on f such that
$$\|\tilde\Psi_{-,1}(r) - \tilde\Psi_{-,2}(r)\|_{\ell_2} \leq Me^{-2r}\|\tilde\Psi_{-,1}(0) - \tilde\Psi_{-,2}(0)\|_{\ell_2}, \quad \forall r \geq 0.$$

- Consequently $W^c = \{\Gamma(\tilde\Psi_c, 0) : \tilde\Psi_c \in \mathbb{R}^4\}$ and, if $\Psi(r) = \Gamma(\tilde\Psi_c(r), \tilde\Psi_-(r)) \in W^{cs}$, $r > 0$, is a solution to (5.15), then $\Psi_b(r) = \Gamma(\tilde\Psi_c(r), 0) \in W^c$ is also a solution to (5.15), called the base solution of $\Psi(r)$.

For each $\tilde\Psi_c$, the submanifold given by the image $\Gamma(\tilde\Psi_c, \cdot)$ is often referred to as a stable fiber.

Note that the functions h^{cs} and h^c have been already obtained. Therefore, to construct Γ we only need to show the existence of h^s. To this end, we only need to work with (5.16). Let $(\Psi^c_c(r), \Psi^c_-(r) = h^c(\Psi^c_c(r)) \in W^c$ be solution to (5.16). One may compute that $(\Psi^c_c(r), \Psi^c_-(r)) + (\tilde\Psi_c(r), \tilde\Psi_-(r))$ where

$$(\tilde\Psi_c(\cdot), \tilde\Psi_-(\cdot)) \in \mathcal{E}^{ss} = \{(\tilde\psi_c, \tilde\psi_-) \in C^0([0,+\infty), \ell_2) \mid \sup_{r\geq 0} e^{2r}\|(\tilde\psi_c, \tilde\psi_-)\|_{\ell_2} < \infty\},$$

is also a solution to (5.16) iff, for all $r \geq 0$,

$$\tilde\Psi_1(r) = -\int_r^{+\infty}(r-\tau)\Big(F_1^{cs}(\Psi^c_c(\tau) + \tilde\Psi_c(r), \Psi^c_-(\tau) + \tilde\Psi_-(r)) - F_1^{cs}(\Psi^c_c(\tau), \Psi^c_-(\tau))\Big)\,d\tau,$$

$$\tilde\Psi_{n-}(r) = e^{-\sqrt{n^2-1}r}\tilde\Psi_-(0) + \int_0^r e^{-\sqrt{n^2-1}(r-\tau)}\Big(F_n^{cs}(\Psi^c_c(\tau) + \tilde\Psi_c(r), \Psi^c_-(\tau) + \tilde\Psi_-(r)) - F_n^{cs}(\Psi^c_c(\tau), \Psi^c_-(\tau))\Big)\,d\tau.$$

Following the proof of Theorem 4.3 (mostly contained in Section 3) in [15]), for $\delta \ll 1$,

a.) the above right side is a contraction in $\mathcal{E}^{ss}$ possessing a unique fixed point $\big(\tilde\Psi_c(\cdot, \tilde\Psi_-(0)), \tilde\Psi_-(\cdot, \tilde\Psi_-(0)\big) \in \mathcal{E}^{ss}$ depending on the parameter $\tilde\Psi_-(0)$;

b.) the desired mapping h^s is given by $h^s(\tilde\Psi_-(0)) = \tilde\Psi_c\big(r, \tilde\Psi_-(0)\big)|_{r=0}$. The oddness of h^s is also obtained from the oddness of (5.16).

Splitting estimates. By item (1) of Theorem 3.3 and (5.11), the stable/unstable solutions $\Psi^{u,s}$ to (5.14) satisfy $\lim_{r\to+\infty}\Psi^{u,s}(r,\tau) = 0$ and therefore they belong to the center-stable manifold W^{cs}. Thus, we can express them in the stable fiber coordinates,

$$\Psi^{u,s}(r) = \Gamma\big(\tilde\Psi^{u,s}_c(r), \tilde\Psi^{u,s}_-(r)\big), \quad \tilde\Psi^{u,s}_c(r) = \big(\tilde\Psi^{u,s}_1(r), \partial_r\tilde\Psi^{u,s}_1(r)\big), \tag{5.18}$$

and let $\Psi_b^{u,s}$ be their base points

$$\Psi_b^{u,s}(r) = \Gamma(\tilde\Psi^{u,s}_c(r), 0) = \Big(\tilde\Psi^{u,s}_c(r), h^c\big(\tilde\Psi^{u,s}_c(r)\big), h^{cs}\big(\tilde\Psi^{u,s}_c(r), h^c\big(\tilde\Psi^{u,s}_c(r)\big)\big)\Big) \in W^c,$$

which are solutions to (5.14) themselves and satisfy

$$\|\Psi^{u,s}(r) - \Psi_b^{u,s}(r)\|_{\ell_2} \leq \mathcal{O}(e^{-2r}), \quad \text{as } r \to +\infty.$$

Lemma 5.5. $\tilde\Psi^u_c(r) = \tilde\Psi^s_c(r)$.

Proof. From item (1) of Theorem 3.3 and Lemma 5.1, $\partial_\tau\Psi^{u,s}(r)$ have exactly the same leading order term proportional to $r^{-1}\sin\tau$ with remainders of $\mathcal{O}(r^{-3})$ in ℓ_2 metric and $\partial_r\Psi^{u,s}(r)$ with remainders of $\mathcal{O}(r^{-4})$. Since $Dh^c(0) = 0$ and $Dh^{cs}(0) = 0$, we have, for $r \gg 1$,

$$\mathcal{O}(r^{-3}) \geq \|\Psi^u(r) - \Psi^s(r)\|_{\ell_2} \geq \|\tilde\Psi^u_b(r) - \tilde\Psi^s_b(r)\|_{\ell_2} - \mathcal{O}(e^{-2r}) \geq \frac{1}{2}|\tilde\Psi^u_c(r) - \tilde\Psi^s_c(r)| - \mathcal{O}(e^{-2r}),$$

and thus

$$|\tilde\Psi^u_c(r) - \tilde\Psi^s_c(r)| \leq \mathcal{O}(r^{-3}). \tag{5.19}$$

Let

$$\big(\tilde\beta_1(r), \partial_r\tilde\beta_1(r)\big) = \tilde\Psi^u_c(r) - \tilde\Psi^s_c(r), \quad B(r) = \sup_{r'\geq r}(r')^3|\tilde\Psi^u_c(r') - \tilde\Psi^s_c(r')| < \infty,$$

where (5.19) is also used. Recall that $\Psi_b^{u,s}(r)$ are solutions to (5.14) contained in the center manifold W^c, governed by the dynamics of their center coordinates $\tilde\Psi^{u,s}_c(r)$. Substituting h^c and h^{cs} into the term $F_1(\Psi)$ in (5.14), using $Dh^c(0) = 0$ and $Dh^{cs}(0) = 0$ along with the leading order expansion of $\Psi^{u,s}(r)$ corresponding to (3.11), and observing that the cubic nonlinearity $F_1(\Psi)$ does not contain the term Ψ_1^3 in its Taylor expansion, we have

$$\partial_r^2\tilde\beta_1 - \frac{6}{r^2}\tilde\beta_1 = \tilde G(r) = \mathcal{O}\left(\frac{1}{r^3}(|\tilde\beta_1| + |\partial_r\tilde\beta_1|)\right) \leq \mathcal{O}\left(\frac{B(r)}{r^6}\right).$$

As in the definition of $\mathcal{J}_1$ in (5.10), a fundamental set of solutions of $\partial_r^2\tilde\beta_1 - \frac{6}{r^2}\tilde\beta_1 = 0$ are given by r^{-2} and r^3. Therefore the general solutions of the above equation is

$$\tilde\beta_1(r) = c_1 r^{-2} + c_2 r^3 + \frac{r^3}{5}\int_{+\infty}^{r} \frac{\tilde G(s)}{s^2} ds - \frac{1}{5r^2}\int_{+\infty}^{r} s^3 \tilde G(s) ds,$$

which implies

$$|\tilde\beta_1(r) - c_1 r^{-2} - c_2 r^3| \le \mathcal{O}\big(r^{-4}B(r)\big).$$

In the view of (5.19), we conclude $c_1 = c_2 = 0$ and thus $|\tilde\beta_1(r)| \le \mathcal{O}\big(r^{-4}B(r)\big)$. In turn it also implies $|\partial_r\tilde\beta_1(r)| \le \mathcal{O}\big(r^{-5}B(r)\big)$ and leads to a contradiction to the definition of $B(r)$ for $r \gg 1$, unless $B \equiv 0$. The lemma is proved. □

Finally we are ready to prove the estimate on the difference between $\Psi^{u,s}(r)$.

Proof of item (2) Theorem 3.3. Due to Lemma 5.5, to complete the proof of the theorem, we need to estimate

$$\tilde\beta_- = (\tilde\beta_{n-})_{n=3}^{+\infty} := \Psi_-^u(r) - \Psi_-^s(r) = \tilde\Psi_-^u(r) - \tilde\Psi_-^s(r) = \big(\tilde\Psi_{n-}^u(r) - \tilde\Psi_{n-}^s(r)\big)_{n=3}^{+\infty},$$

where the second equal sign is due to the definition (5.17) of the stable fiber coordinate system Γ. From (5.14) and using $\|DF(\Psi)\|_{L(\ell_2)} = \mathcal{O}(\|\Psi\|_{\ell_2}^2)$ for $\|\Psi\|_{\ell_2} \ll 1$ and $\|\Psi^{u,s}(r)\|_{\ell_2} = \mathcal{O}(\frac{1}{r})$ for $r \gg 1$, we have

$$\partial_r\tilde\beta_- = A\tilde\beta_- + \widetilde\Pi\big[F\big(\Gamma(\tilde\Psi_c^{u,s}(r), \tilde\Psi_-^u(r))\big) - F\big(\Gamma(\tilde\Psi_c^{u,s}(r), \tilde\Psi_-^s(r))\big)\big] \triangleq A\tilde\beta_- + \widetilde A_-(r)\tilde\beta_-,$$

where

$$A\Psi_- = \big(-\sqrt{n^2-1}\Psi_{n-}\big)_{n=3}^{+\infty}$$

$$\big(\widetilde A_-(r)\Psi_-\big)_n = \Big(\int_0^1 \widetilde\Pi\big[((DF_n)\circ\Gamma)D_{\tilde\Psi_-}\Gamma\big]\big(\tilde\Psi_c^{u,s}(r), (1-\tau)\tilde\Psi_-^s(r) + \tau\tilde\Psi_-^u(r)\big)d\tau\Big)\Psi_-,$$

which satisfies

$$\|\widetilde A_-(r)\|_{L(\ell_2)} = \mathcal{O}\left(r^{-2}\right).$$

Consequently,

$$\partial_r\big(e^{\sqrt8 r}\tilde\beta_-\big) = (A+\sqrt8)e^{\sqrt8 r}\tilde\beta_- + \widetilde A_-(r)e^{\sqrt8 r}\tilde\beta_-.$$

As $A+\sqrt8 \le 0$, $\|\widetilde A_-(r)\|_{L(\ell_2)} = \mathcal{O}(r^{-2})$ implies

$$\sup_{r\ge0}\Big\{e^{\sqrt8 r}\|\tilde\beta_-(r)\|_{\ell_2}\Big\} < +\infty.$$

Write $e^{\sqrt8 r}\tilde\beta_-$ using the variation of constants formula,

$$e^{\sqrt8 r}\tilde\beta_-(r) = e^{r(A+\sqrt8)}\tilde\beta_-(0) + \int_0^r e^{(r-r')(A+\sqrt8)}\widetilde A_-(r')e^{\sqrt8 r'}\tilde\beta_-(r')dr'.$$

Now, since $\Pi_3(A+\sqrt8) = 0$ and $(I-\Pi_3)(A+\sqrt8) \le -\sqrt{24}+\sqrt8 < -2$, one may estimate, for $r \gg 1$,

$$\begin{aligned}\|(I-\Pi_3)e^{\sqrt8 r}\tilde\beta_-(r)\|_{\ell_2} \le& e^{-2r}\|\tilde\beta_-(0)\|_{\ell_2} + \sup_{r''\ge0}\|e^{\sqrt8 r''}\tilde\beta_-(r'')\|_{\ell_2}\int_0^r e^{-2(r-r')}\|\widetilde A_-(r')\|_{L(\ell_2)}dr' \\ \le& e^{-2r}\|\tilde\beta_-(0)\|_{\ell_2} + \int_0^{\frac r2}\mathcal{O}(e^{-2(r-r')})dr' + \int_{\frac r2}^r \mathcal{O}\left(\frac{e^{-2(r-r')}}{r^2}\right)dr' \\ \le& \mathcal{O}\left(\frac{1}{r^2}\right).\end{aligned}$$

Defining

$$\tilde C_{\text{in}} = \tilde\beta_{3-}(0) + \int_0^{+\infty}\Pi_3\left[\widetilde A_-(r')e^{\sqrt8 r'}\tilde\beta_-(r')\right]dr',$$

which converges since $\|\widetilde A_-(r)\|_{L(\ell_2)} = \mathcal{O}(r^{-2})$, then we obtain

$$\|e^{\sqrt8 r}\tilde\beta_{3-}(r) - \tilde C_{\text{in}}\|_{\ell_2} \le \sup_{r''\ge0}\|e^{\sqrt8 r''}\tilde\beta_-(r'')\|_{\ell_2}\int_r^{+\infty}\|A_-(r')\|_{L(\ell_2)}dr' \le \mathcal{O}\left(\frac1r\right).$$

To complete the proof of Theorem 3.3, we need to estimate $e^{\sqrt{8}r}(\phi^{0,u}-\phi^{0,s})$ and $\partial_r\big(e^{\sqrt{8}r}(\phi^{0,u}-\phi^{0,s})\big)$. From (5.12) and Lemma 5.5,

$$\begin{aligned}e^{\sqrt{8}r}\Delta\phi^0(-ir) =& e^{\sqrt{8}r}(\phi^{0,u}(-ir)-\phi^{0,s}(-ir))\\ =& e^{\sqrt{8}r}(\Psi_1^u(r)-\Psi_1^s(r))\sin\tau+\sum_{n\geq 3}\frac{e^{\sqrt{8}r}}{2\sqrt{n^2-1}}\Big(\Psi_{n+}^u(r)-\Psi_{n+}^s(r)-\tilde{\beta}_{n-}(r)\Big)\sin n\tau.\end{aligned}$$

Therefore, for $C_{\rm in}=-\frac{\sqrt{2}}{8}\tilde{C}_{\rm in}$, from the definition of $\tilde{\beta}_-(r)$ and the cubic leading order of $h^{c,s,cs}$ in the stable fiber coordinates $\Gamma(\tilde{\Psi}_c,\tilde{\Psi}_-)$, we have

$$\begin{aligned}\|\partial_\tau\big(e^{\sqrt{8}r}\Delta\phi^0(-ir)-C_{\rm in}\sin 3\tau\big)\|_{\ell_1}\leq& e^{\sqrt{8}r}\big|\big(h^s(\tilde{\Psi}_c^{u,s}(r),\tilde{\Psi}_-^u(r))-h^s(\tilde{\Psi}_c^{u,s}(r),\tilde{\Psi}_-^s(r)\big)\big|\\ &+\|e^{\sqrt{8}r}\big(\Psi_+^u(r)-\Psi_+^s(r)\big)\|_{\ell_1}+\|e^{\sqrt{8}r}(\tilde{\beta}_{n-})_{n>3}\|_{\ell_1}+|e^{\sqrt{8}r}\tilde{\beta}_{3-}(r)-\tilde{C}_{\rm in}|\\ \leq& M\big(r^{-2}e^{\sqrt{8}r}\|\tilde{\beta}_-(r)\|_{\ell_2}+\|e^{\sqrt{8}r}(\tilde{\beta}_{n-})_{n>3}\|_{\ell_2}\big)+\mathcal{O}(r^{-1})\leq\mathcal{O}(r^{-1}).\end{aligned}$$

Similarly from (5.13)

$$\begin{aligned}\|\partial_r\big(e^{\sqrt{8}r}\Delta\phi^0(-ir)\big)\|_{\ell_1}=&\Big\|e^{\sqrt{8}r}\sum_{n\geq 3}\Big(\frac{\sqrt{n^2-1}+\sqrt{8}}{2\sqrt{n^2-1}}(\Psi_{n+}^u(r)-\Psi_{n+}^s(r))+\frac{\sqrt{n^2-1}-\sqrt{8}}{2\sqrt{n^2-1}}\tilde{\beta}_{n-}(r)\Big)\sin n\tau\\ &+\partial_r\big(e^{\sqrt{8}r}(\Psi_1^u(r)-\Psi_1^s(r))\big)\sin\tau\Big\|_{\ell_1}\\ \leq& M\big(r^{-2}e^{\sqrt{8}r}\|\tilde{\beta}_-(r)\|_{\ell_2}+\|e^{\sqrt{8}r}(\tilde{\beta}_{n-})_{n>3}\|_{\ell_2}\big)\leq\mathcal{O}(r^{-2}).\end{aligned}$$

Since $\Delta\phi^0(z)$ is analytic, the estimate on $\partial_r\big(e^{\sqrt{8}r}\Delta\phi^0(-ir)\big)$ implies the same estimate on $\partial_z\big(e^{i\sqrt{8}z}\Delta\phi^0(z)\big)$ and this completes the proof of Theorem 3.3. □

Finally, we prove that the Stokes constant is analytic with respect to $f\in\mathcal{F}_r^c$

Proof of item (3) of Theorem 3.3. This proof is based on the analytic dependence of $\phi^{0,\star}(z,\tau)$, $\star=u,s$, on $f\in\mathcal{F}_r^c$ in Theorem 3.3(1) and the asymptotics (3.13) in Theorem 3.3(2) just proven. For $s\in(-\infty,-\kappa)$, let

$$\widetilde{C}(s,f)=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-\mu_3 s}\Delta\phi^0(is,\tau)\sin 3\tau d\tau,$$

which is complex analytic in $f\in\mathcal{F}_r^c$. From (3.13) we have $C_{\rm in}(f)=\lim_{s\to-\infty}\widetilde{C}(s,f)$ uniformly in f with $\|f\|_r\leq R_0$. So we obtain the analyticity of $C_{\rm in}(f)$ in $f\in\mathcal{F}_r^c$, which also implies its analyticity in $f\in\mathcal{F}_r$. □

6. Complex matching estimates: Proof of Theorem 3.6

As usual, we consider only the unstable case, and in order to simplify the notation, we omit the superscript "u" of the solutions. Moreover, in this section, we use the domain $D_{+,\kappa}^{\rm mch,u}$ instead of $D_{\theta,\kappa}^{u,\rm in}$ (see (3.10) and (3.14)) but we work on the same notation for the norms and Banach spaces introduced in Section 5.1.

Proposition 6.1. *Let $\phi(z,\tau)$ and $\phi^0(z,\tau)$ be solutions to* (3.8) *and* (3.9)*, respectively. The function $\varphi: D_{+,\kappa}^{\rm mch,u}\times\mathbb{T}\to\mathbb{C}$ defined as*

$$\varphi(z,\tau)=\phi(z,\tau)-\phi^0(z,\tau). \tag{6.1}$$

satisfies the following differential equation

$$\mathcal{I}(\varphi)(z,\tau)=\Big(L(\varphi)(z)+\widehat{L}(\widetilde{\Pi}[\varphi])(z)\Big)\sin(\tau)+K(\varphi)(z,\tau)+\mathcal{C}_{\rm mch}(z,\tau), \tag{6.2}$$

where $\mathcal{I}$ is the operator given by (5.7)*, $L:\mathcal{X}_{\ell_1,\alpha}\to\mathcal{X}_{\alpha+4}$, $\widehat{L}:\mathcal{X}_{\ell_1,\alpha}\to\mathcal{X}_{\alpha+2}$, and $K:\mathcal{X}_{\ell_1,\alpha}\to\mathcal{X}_{\ell_1,\alpha+2}$ are linear operators and $\mathcal{C}_{\rm mch}:D_{+,\kappa}^{\rm mch,u}\times\mathbb{T}\to\mathbb{C}$ is an analytic function in the variable z. Moreover, $\Pi_1\circ K\equiv 0$ and there exists a constant $M>0$ independent of ε and κ such that, for $0<\gamma<1$, ε sufficiently small and κ big enough*

1. $\|\Pi_1[\mathcal{C}_{\rm mch}]\|_4\leq M\varepsilon^{3\gamma-1}$ *and* $\Big\|\partial_\tau^2\widetilde{\Pi}[\mathcal{C}_{\rm mch}]\Big\|_{\ell_1,3}\leq M\varepsilon^2$;
2. $\|L(\varphi)\|_{\alpha+4}\leq M\|\varphi\|_{\ell_1,\alpha}$;
3. $\|\widehat{L}(\varphi)\|_{\alpha+2}\leq M\|\varphi\|_{\ell_1,\alpha}$;

(4) $\|K(\varphi)\|_{\ell_1,\alpha+2} \leq M\|\varphi\|_{\ell_1,\alpha}$, $j = 0, 1, 2$.

Proof. Since ϕ and ϕ^0 satisfy (3.8) and (3.9), respectively, we have that $\varphi(z,\tau)$ satisfies

$$\partial_z^2\varphi - \partial_\tau^2\varphi - \varphi = \varepsilon^2\phi - \frac{1}{3}(\phi^3 - (\phi^0)^3) - \frac{1}{\omega^3}f(\omega\phi) + f(\phi^0), \quad \omega = (1+\varepsilon^2)^{-\frac{1}{2}}. \tag{6.3}$$

Now, recall that $\phi(z,\tau) = \varepsilon v(i\pi/2 + \varepsilon z, \tau)$, where $v(y,\tau) = v^h(y)\sin(\tau) + \xi(y,\tau)$, v^h is given by (1.30) and ξ is given by Theorem 3.1. An easy computation shows that

$$\varepsilon v^h(i\pi/2 + \varepsilon z) = -\frac{2\sqrt{2}i}{z} + l_1(z),$$

where l_1 is an analytic function such that $|l_1(z)| \leq M\varepsilon^2|z|$, for each $z \in D^{\mathrm{mch},u}_{+,\kappa}$. Thus,

$$\phi(z,\tau) = -\frac{2\sqrt{2}i}{z}\sin(\tau) + l_1(z)\sin(\tau) + \varepsilon\xi(i\pi/2 + \varepsilon z, \tau). \tag{6.4}$$

Using Theorem 3.1 and $y = i\pi/2 + \varepsilon z$, we have

$$\|\varepsilon\partial_\tau^2\xi(i\pi/2 + \varepsilon z, \tau)\|_{\ell_1,3} \leq \frac{1}{\varepsilon^2}\|\partial_\tau^2\xi(y,\tau)\|_{\ell_1,1,3} \leq M,$$

where $\|\cdot\|_{\ell_1,1,3}$ is the norm introduced in Section 4.1.

Since $M\kappa \leq |z| \leq M\varepsilon^{\gamma-1}$ for every $z \in D^{\mathrm{mch},u}_{+,\kappa}$, it holds

$$\left\|\partial_\tau^2\left(\phi^0(z,\tau) + \frac{2\sqrt{2}i}{z}\sin(\tau)\right)\right\|_{\ell_1,3} \leq M, \tag{6.5}$$

and using that $\|\phi\|_{\ell_1,1}, \|\phi^0\|_{\ell_1,1} \leq M$, $f(z) = \mathcal{O}(z^5)$, we obtain from the Mean Value Theorem that

$$\begin{aligned} -\frac{1}{3}\left(\phi^3 - (\phi^0)^3\right) - f(\phi) + f(\phi^0) &= -\frac{1}{3}\left(\phi^2 + \phi\phi^0 + (\phi^0)^2\right)\varphi - \varphi\int_0^1 f'(s\phi + (1-s)\phi^0)ds \\ &= \frac{6}{z^2}\Pi_1[\varphi]\sin(\tau) - \frac{2}{z^2}\Pi_1[\varphi]\sin(3\tau) + l_2(\varphi) + l_3(\widetilde{\Pi}[\varphi]), \end{aligned} \tag{6.6}$$

where $l_2 : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\ell_1,\alpha+4}$ and $l_3 : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\ell_1,\alpha+2}$ are linear operators such that,

$$\|l_2(\varphi)\|_{\ell_1,\alpha+4} \leq M\|\varphi\|_{\ell_1,\alpha} \quad \textit{and} \quad \|l_3(\varphi)\|_{\ell_1,\alpha+2} \leq M\|\varphi\|_{\ell_1,\alpha}.$$

The proof of the proposition follows from (6.3), (6.4), and (6.6) and by taking,

- $\mathcal{C}_{\mathrm{mch}} = \varepsilon^2\phi + f(\phi) - \omega^{-3}f(\omega\phi)$,
- $L(\varphi) = \Pi_1[l_2(\varphi)]$,
- $\widehat{L}(\varphi) = \Pi_1\left[l_3(\widetilde{\Pi}[\varphi])\right]$,
- $K(\varphi) = \widetilde{\Pi}\left[-\frac{2}{z^2}\Pi_1[\varphi]\sin(3\tau) + l_2(\varphi) + l_3(\widetilde{\Pi}[\varphi])\right]$.

□

Let $z_j = \varepsilon^{-1}(y_j - i\pi/2)$, $j = 1, 2$, where y_1 and y_2 are the vertices of the matching domain $D^{\mathrm{mch},u}_{+,\kappa}$ given by (3.14). Consider the following linear operator acting on the Fourier coefficients of $h(z,\tau) = \sum_{k\geq 0} h_{2k+1}(z)\sin((2k+1)\tau)$.

$$\mathcal{T}(h) = \sum_{k\geq 0}\mathcal{T}_{2k+1}(h_{2k+1})\sin((2k+1)\tau), \tag{6.7}$$

where

$$\begin{aligned}\mathcal{T}_1(h_1) &= \frac{z^3}{5}\int_{z_1}^{z}\frac{h_1(s)}{s^2}ds - \frac{1}{5z^2}\int_{z_2}^{z}h_1(s)s^3ds\\ &\quad - \frac{1}{5(z_2^5-z_1^5)}\left[\left(z^3-\frac{z_2^5}{z^2}\right)\int_{z_2}^{z_1}h_1(s)s^3ds + \left(z^3z_2^5-\frac{(z_1z_2)^5}{z^2}\right)\int_{z_1}^{z_2}\frac{h_1(s)}{s^2}ds\right]\\ \mathcal{T}_{2k+1}(h_{2k+1}) &= \int_{z_2}^{z}\frac{h_{2k+1}(s)e^{-i\mu_{2k+1}(s-z)}}{2i\mu_{2k+1}}ds - \int_{z_1}^{z}\frac{h_{2k+1}(s)e^{i\mu_{2k+1}(s-z)}}{2i\mu_{2k+1}}ds,\\ &\quad + \frac{\sin(\mu_{2k+1}(z_2-z))}{\sin(\mu_{2k+1}(z_1-z_2))}\int_{z_2}^{z_1}\frac{h_{2k+1}(s)e^{-i\mu_{2k+1}(s-z_1)}}{2i\mu_{2k+1}}ds\\ &\quad + \frac{\sin(\mu_{2k+1}(z_1-z))}{\sin(\mu_{2k+1}(z_1-z_2))}\int_{z_1}^{z_2}\frac{h_{2k+1}(s)e^{-i\mu_{2k+1}(s-z_2)}}{2i\mu_{2k+1}}ds, \qquad \text{for } k\geq 1.\end{aligned}$$

Observe that $\mathcal{T}$ is chosen such that $\mathcal{I}\circ\mathcal{T}=\mathrm{Id}$ and $\mathcal{T}(h)(z_j,\tau)=0$, $j=1,2$.

Moreover, consider the analytic in z function $\mathcal{Q}: D^{\mathrm{mch},u}_{+,\kappa}\times\mathbb{T}\to\mathbb{C}$ given by

$$\mathcal{Q}(z,\tau)=\sum_{k\geq 0}\mathcal{Q}_{2k+1}(z)\sin((2k+1)\tau), \tag{6.8}$$

which is defined using φ in (6.1) as follows, where $k\geq 1$,

$$\begin{aligned}\mathcal{Q}_1(z) &= \frac{1}{z_2^5-z_1^5}\left(z^3(z_2^2\varphi_1(z_2)-z_1^2\varphi_1(z_1)) - \frac{1}{z^2}\left(z_1^5z_2^2\varphi_1(z_2)-z_1^2z_2^5\varphi_1(z_1)\right)\right),\\ \mathcal{Q}_{2k+1}(z) &= \frac{\sin(\mu_{2k+1}(z-z_2))}{\sin(\mu_{2k+1}(z_1-z_2))}\varphi_{2k+1}(z_1) - \frac{\sin(\mu_{2k+1}(z-z_1))}{\sin(\mu_{2k+1}(z_1-z_2))}\varphi_{2k+1}(z_2).\end{aligned}$$

Observe that $\mathcal{Q}$ satisfies $\mathcal{I}\mathcal{Q}=0$ and $\mathcal{Q}_{2k+1}(z_j)=\varphi_{2k+1}(z_j)$, $j=1,2$.

In conclusion, observe that if $h,\widehat{\varphi}: D^{\mathrm{mch},u}_{+,\kappa}\times\mathbb{C}\to\mathbb{C}$ are analytic in z functions such that

$$\mathcal{I}(\widehat{\varphi})=h,\qquad \widehat{\varphi}(z_j)=\varphi(z_j),\quad j=1,2,$$

where φ is given in (6.1), then, we have that

$$\widehat{\varphi}(z,\tau)=\mathcal{Q}(z,\tau)+\mathcal{T}(h)(z,\tau),$$

where $\mathcal{T}$ and $\mathcal{Q}$ are given by (6.7) and (6.8). In particular, as the function φ satisfies (6.2) by Proposition 6.1, it can be written as

$$\varphi(z,\tau)=\mathcal{Q}(z_1,z_2)(z,\tau)+\mathcal{T}\left(\mathcal{C}_{\mathrm{mch}}(z,\tau)+\left(L(\varphi)(z)+\widehat{L}(\widetilde{\Pi}[\varphi])(z)\right)\sin(\tau)+K(\varphi)(z,\tau)\right). \tag{6.9}$$

We use this expression for φ to obtain estimates of this function for $z\in D^{\mathrm{mch},u}_{+,\kappa}$.

The next lemma gives estimates for the operators $\mathcal{T}$ and $\mathcal{Q}$ given in (6.7) and (6.8).

Lemma 6.2. *There exists $\delta>0$ depending only on $\beta_{1,2}$ (see (3.14)), such that, for $\kappa\varepsilon^{1-\gamma}\leq\delta$, the following statements hold.*

(1) *The linear operator $\mathcal{T}_1:\mathcal{X}_\alpha\to\mathcal{X}_{\alpha-2}$ is well defined and*

$$\|\mathcal{T}_1(h)\|_{\alpha-2}\leq M\|h\|_\alpha,\ \ \alpha>4;\quad \|\mathcal{T}_1(h)\|_2\leq M|\log\varepsilon|\,\|h\|_4,\ \ \alpha=4.$$

(2) *For $k\geq 1$ and $h\in\mathcal{X}_\alpha$, with $\alpha\geq 0$,*

$$\|\mathcal{T}_{2k+1}(h)\|_\alpha\leq\frac{M}{k^2}\|h\|_\alpha.$$

(3) *$\mathcal{Q}$ satisfies*

$$\|\mathcal{Q}_1\|_\alpha\leq M\left(\varepsilon^{(\alpha-3)(\gamma-1)}+\varepsilon^{2+(\alpha+1)(\gamma-1)}\right),\ \ \alpha\geq 2;\qquad \left\|\partial_\tau^2\widetilde{\Pi}[\mathcal{Q}]\right\|_\alpha\leq M\varepsilon^{(\alpha-3)(\gamma-1)},\ \ \alpha\geq 0.$$

Proof. Due to the assumption $e^{5(\pi-\beta_1)}-e^{-5\beta_2}\neq 0$, when δ is small, it holds

$$\frac{1}{M}\varepsilon^{\gamma-1}\leq|z_1|,\,|z_2|,\,|z_1^5-z_2^5|^{\frac{1}{5}}\leq M\varepsilon^{\gamma-1};\quad \kappa\leq|z|\leq M\varepsilon^{\gamma-1},\ \ \forall z\in D^{\mathrm{mch,u}}_{+,\kappa}. \tag{6.10}$$

Therefore,

$$\left|\frac{1}{5z^2}\int_{z_2}^{z} h(s)s^3 ds\right| \leq \frac{M\|h\|_\alpha}{|z|^2}\int_{z_2}^{z} |s|^{3-\alpha}ds \leq \begin{cases} M\|h\|_\alpha |z|^{2-\alpha}, & \alpha > 4, \\ M|\log\varepsilon|\|h\|_\alpha |z|^{2-\alpha}, & \alpha = 4, \end{cases}$$

and, for $\alpha \geq 4$,

$$\begin{aligned}
&\left|\frac{z^3}{5}\int_{z_1}^{z}\frac{h(s)}{s^2}ds\right| \leq M\|h\|_\alpha |z|^3 \int_{z_1}^{z}\frac{1}{|s|^{2+\alpha}}ds \leq M\|h\|_\alpha|z|^{2-\alpha} \\
&\left|\frac{1}{5(z_2^5 - z_1^5)}\left(z^3 - \frac{z_2^5}{z^2}\right)\int_{z_2}^{z_1} h(s)s^3 ds\right| \leq \frac{M\|h\|_\alpha}{|z|^2}\int_{z_2}^{z_1}|s|^{3-\alpha}ds \leq \frac{M\|h\|_\alpha \varepsilon^{(\gamma-1)(4-\alpha)}}{|z|^2} \leq M\|h\|_\alpha|z|^{2-\alpha} \\
&\left|\frac{1}{5(z_2^5 - z_1^5)}\left(z^3 z_2^5 - \frac{(z_1 z_2)^5}{z^2}\right)\int_{z_1}^{z_2}\frac{h(s)}{s^2}ds\right| \leq M\|h\|_\alpha\left(|z|^3 + \frac{|z_2|^5}{|z|^2}\right)\int_{z_1}^{z_2}\frac{1}{|s|^{2+\alpha}}ds \\
&\qquad\qquad \leq M\|h\|_\alpha\left(|z|^3 + \frac{|z_2|^5}{|z|^2}\right)\varepsilon^{(\gamma-1)(-1-\alpha)} \leq M\|h\|_\alpha|z|^{2-\alpha},
\end{aligned}$$

where the integral $\int_{z_1}^{z_2}$ was simply taken along the arc of the circle centered at $-i\kappa\varepsilon$. Hence, we finish the proof of item (1) of the theorem.

To deal with the higher modes, we will see that

$$\left|\frac{\sin(\mu_{2k+1}(z_j - z))}{\sin(\mu_{2k+1}(z_1 - z_2))}\right| \leq M,\ j = 1, 2,\ \forall\ z \in D^{\mathrm{mch},u}_{+,\kappa},\ \forall\ k \geq 1. \tag{6.11}$$

In fact, recalling that $|\sin^2(z)| = \frac{1}{2}(\cosh(2\operatorname{Im}(z)) - \cos(2\operatorname{Re}(z)))$, we have

$$\left|\frac{\sin(\mu_{2k+1}(z_j - z))}{\sin(\mu_{2k+1}(z_1 - z_2))}\right|^2 \leq \frac{\cosh(2\mu_{2k+1}\operatorname{Im}(z_j - z)) + 1}{\cosh(2\mu_{2k+1}\operatorname{Im}(z_1 - z_2)) - 1}.$$

Since $\operatorname{Im}(z_1 - z_2) = K\varepsilon^{\gamma-1}$ and $|\operatorname{Im}(z_j - z)| \leq |\operatorname{Im}(z_1 - z_2)|$, we obtain (6.11).

Assume that $\alpha \geq 0$. For each $z \in D^{\mathrm{mch},u}_{+,\kappa}$, there exist β_1^*, β_2^* (depending on z) between β_1 and β_2 and $t_2^*, t_1^* > 0$ (depending on z) such that $z_2 = z + e^{-i\beta_2^*}t_2^*$ and $z_1 = z + e^{i(\pi-\beta_1^*)}t_1^*$. Thus, we have that

$$\begin{aligned}
\left|\int_{z_2}^{z} h_{2k+1}(s)e^{-i\mu_{2k+1}(s-z)}ds\right| &\leq \int_0^{t_2^*}\left|h_{2k+1}\left(z + e^{-i\beta_2^*}t\right)\right| e^{-\mu_{2k+1}\sin(\beta_2^* t)}dt \\
&\leq \|h_{2k+1}\|_\alpha \int_0^{t_2^*}\frac{e^{-\mu_{2k+1}\sin(\beta_2^* t)}}{|z + e^{-i\beta_2^*}t|^\alpha}dt \leq \frac{\|h_{2k+1}\|_\alpha}{|z|^\alpha}\int_0^\infty e^{-\mu_{2k+1}\sin(\beta_2^* t)}dt \\
&\leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z|^\alpha}.
\end{aligned}$$

Analogously, we prove that

$$\left|\int_{z_1}^{z} h_{2k+1}(s)e^{i\mu_{2k+1}(s-z)}ds\right| \leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z|^\alpha},$$

and in particular, using that $|z_j| \geq M|z|$, $j = 1, 2$,

$$\begin{aligned}
\left|\int_{z_2}^{z_1} h_{2k+1}(s)e^{-i\mu_{2k+1}(s-z_1)}ds\right| &\leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z_1|^\alpha} \leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z|^\alpha} \\
\left|\int_{z_1}^{z_2} h_{2k+1}(s)e^{i\mu_{2k+1}(s-z_2)}ds\right| &\leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z_2|^\alpha} \leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z|^\alpha}.
\end{aligned}$$

Hence,

$$\|\mathcal{T}_{2k+1}(h_{2k+1})\|_\alpha \leq \frac{M}{\mu_{2k+1}^2}\|h_{2k+1}\|_\alpha,\ k \geq 1,\ \alpha \geq 0. \tag{6.12}$$

Items (2) follows (6.12).

To estimate $\mathcal{Q}$, observe that using (6.4) and (6.5), one has

$$\varphi(z,\tau) = l_1(z)\sin\tau + b(z,\tau), \qquad \text{with } b(z,\tau) = \varepsilon\xi(i\pi/2 + \varepsilon z, \tau) - \left(\phi_0(z,\tau) + \frac{2\sqrt{2}i}{z}\sin\tau\right),$$

where l_1 is given in (6.4). Then, $\|\partial_\tau^2 b\|_{\ell_1,3}\leq M$ and $|l_1(z)|\leq M\varepsilon^2|z|$, for each $z\in D^{\mathrm{mch},u}_{+,\kappa}$. Thus, from (6.10), we can see that

$$\begin{aligned}|\mathcal{Q}_1(z_1,z_2)(z)| &= \left|\frac{1}{z_2^5-z_1^5}\left(z^3(z_2^2\varphi_1(z_2)-z_1^2\varphi_1(z_1))-\frac{1}{z^2}\left(z_1^5z_2^2\varphi_1(z_2)-z_1^2z_2^5\varphi_1(z_1)\right)\right)\right|\\ &\leq M\left(|\varphi_1(z_1)|+|\varphi_1(z_2)|+\frac{|z_1^2|}{|z|^2}\,|\varphi_1(z_1)|+\frac{|z_2^2|}{|z|^2}\,|\varphi_1(z_2)|\right)\\ &\leq M\left(\frac{1}{|z_2||z|^2}+\varepsilon^2|z_2|+\frac{\varepsilon^2|z_2|^3}{|z|^2}\right).\end{aligned}$$

Therefore for $\alpha\geq 2$,

$$\|\mathcal{Q}_1(z_1,z_2)\|_\alpha\leq M\left(\varepsilon^{(\alpha-3)(\gamma-1)}+\varepsilon^{2+(\alpha+1)(\gamma-1)}\right).$$

Finally, from (6.11) and (6.8), we can see that, for $\alpha\geq 0$ and $k\geq 1$,

$$\begin{aligned}|z^\alpha\partial_\tau^2\mathcal{Q}_{2k+1}(z_1,z_2)(z)| &= \left|\frac{\sin(\mu_{2k+1}(z-z_2))}{\sin(\mu_{2k+1}(z_1-z_2))}z^\alpha\partial_\tau^2\varphi_{2k+1}(z_1)-\frac{\sin(\mu_{2k+1}(z-z_1))}{\sin(\mu_{2k+1}(z_1-z_2))}z^\alpha\partial_\tau^2\varphi_{2k+1}(z_2)\right|\\ &\leq Mk^2\|\Pi_{2k+1}[b]\|_3\frac{|z|^\alpha}{|z_2|^3}\leq Mk^2\|\Pi_{2k+1}[b]\|_3\varepsilon^{(\alpha-3)(\gamma-1)},\end{aligned}$$

and thus

$$\|\partial_\tau^2\mathcal{Q}_{2k+1}(z_1,z_2)\|_\alpha\leq M\varepsilon^{(\alpha-3)(\gamma-1)},\ \ \alpha\geq 0,\ \ k\geq 1,$$

which completes the proof of item (3). □

End of the proof of Theorem 3.6. To obtain the estimates for φ stated in the theorem, we just need to estimate $\|\varphi\|_{\ell^1,2}$. From (6.9), and Propositions 6.1 and 6.2, we have that

$$\begin{aligned}\|\varphi_1\|_2 &= \left\|\mathcal{Q}_1(z_1,z_2)+\mathcal{T}_1\left(\Pi_1\left[\mathcal{C}_{\mathrm{mch}}\right]+L(\varphi)+\widehat{L}(\widetilde{\Pi}[\varphi])\right)\right\|_2\\ &\leq \|\mathcal{Q}_1(z_1,z_2)\|_2+M|\log\varepsilon|\left(\|\Pi_1\left[\mathcal{C}_{\mathrm{mch}}\right]\|_4+\|L(\varphi)\|_4+\left\|\widehat{L}(\widetilde{\Pi}[\varphi])\right\|_4\right)\\ &\leq M(\varepsilon^{1-\gamma}+\varepsilon^{2+3(\gamma-1)})+M|\log\varepsilon|\left(\varepsilon^{3\gamma-1}+\|\varphi\|_{\ell_1,0}+\left\|\widetilde{\Pi}[\varphi]\right\|_{\ell_1,2}\right)\\ &\leq M\left(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1}|\log\varepsilon|+\frac{|\log\varepsilon|}{\kappa^2}\,\|\varphi\|_{\ell_1,2}+|\log\varepsilon|\left\|\widetilde{\Pi}[\varphi]\right\|_{\ell_1,2}\right).\end{aligned}$$

Moreover, since $\Pi_1\circ K\equiv 0$, we have that

$$\begin{aligned}\left\|\partial_\tau^2\widetilde{\Pi}[\varphi]\right\|_{\ell_1,2} &= \left\|\partial_\tau^2\widetilde{\Pi}\circ\mathcal{Q}(z_1,z_2,\varphi)+\partial_\tau^2\mathcal{T}\left(\widetilde{\Pi}\left[\mathcal{C}_{\mathrm{mch}}\right]+K(\varphi)\right)\right\|_{\ell_1,2}\\ &\leq \left\|\partial_\tau^2\widetilde{\Pi}\circ\mathcal{Q}(z_1,z_2,\varphi)\right\|_{\ell_1,2}+M\left(\left\|\widetilde{\Pi}\left[\mathcal{C}_{\mathrm{mch}}\right]\right\|_{\ell_1,2}+\|K(\varphi)\|_{\ell_1,2}\right)\\ &\leq M(\varepsilon^{1-\gamma}+\varepsilon^{2+3(\gamma-1)})+M\left(\frac{\varepsilon^2}{\kappa}+\|\varphi\|_{\ell_1,0}\right)\\ &\leq M\left(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1}+\frac{1}{\kappa^2}\,\|\varphi\|_{\ell_1,2}\right).\end{aligned}$$

Since $\kappa^{-2}|\log\varepsilon|$ is assumed to be small, it follows from multiplying the second inequality by $2M|\log\varepsilon|$ and adding it to the first one that

$$\|\varphi_1\|_2+M|\log\varepsilon|\|\partial_\tau^2\widetilde{\Pi}[\varphi]\|_{\ell_1,2}\leq 2M|\log\varepsilon|\left(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1}\right).$$

Finally, the estimate on $\partial_z\varphi$ could be derived by differentiating the formula of φ with respect to z. Alternatively, from Lemma 8.1 of [8], reducing the domain $D^{\mathrm{mch},u}_{+,\kappa}$ (see (3.14)), with vertices y_1 and y_2 such that $|y_j-i(\pi/2-\kappa\varepsilon)|=\varepsilon^\gamma$, $j=1,2$, to $D^{\mathrm{mch},u}_{+,2\kappa}\subset D^{\mathrm{mch},u}_{+,\kappa}$ having vertices $\widetilde{y}_1$ and $\widetilde{y}_2$ such that $|\widetilde{y}_j-i(\pi/2-2\kappa\varepsilon)|=\widetilde{c}\varepsilon^\gamma$, $j=1,2$, and $0<\widetilde{c}<1$, we obtain that

$$\|\partial_\tau^2\partial_z\varphi\|_{\ell_1,2}\leq\frac{M}{\kappa}|\log\varepsilon|(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1}).$$

It completes the proof of this theorem. In order to simplify the notation, we make no distinction between $D^{\mathrm{mch},u}_{+,\kappa}$ and $D^{\mathrm{mch},u}_{+,2\kappa}$. □

7. The distance between the manifolds: Proof of Proposition 3.12

7.1. **Banach Space and Operators.** We devote this section to prove Proposition 3.12. We start by defining the functional setting. Given an analytic function $f:\mathcal{R}_\kappa\to\mathbb{C}$ (see Figure 9), we define the norm

$$\|f\|_{\alpha,\exp}=\sup_{y\in\mathcal{R}_\kappa}\left|(y^2+\pi^2/4)^\alpha e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}f(y)\right|,$$

and the Banach space

$$\mathcal{X}_{\alpha,\exp}=\{f:\mathcal{R}_\kappa\to\mathbb{C};\ f\ \text{ analytic},\|f\|_{\alpha,\exp}<\infty\}.$$

Moreover, given an analytic function $f:\mathcal{R}_\kappa\times\mathbb{T}\to\mathbb{C}$ odd in $\tau\in\mathbb{T}$, we define the corresponding norm and the associated Banach space

$$\begin{aligned}\|f\|_{\ell_1,\alpha,\exp}=&\sum_{k\geq 1}\|\Pi_{2k+1}[f]\|_{\alpha,\exp}\\ \mathcal{X}_{\ell_1,\alpha,\exp}=&\{f:\mathcal{R}_\kappa\times\mathbb{T}\to\mathbb{C};\ f\text{ is an analytic function in the variable } y \text{ such that}\\ &\Pi_1[f]=\Pi_{2l}[f]=0,\forall l\geq 0\text{ and }\|f\|_{\ell_1,\alpha,\exp}<\infty\}.\end{aligned}$$

Finally, we consider the product Banach space

$$\mathcal{Y}_{\ell_1,2,\exp}=\mathcal{X}_{2,\exp}\times\mathcal{X}_{\ell_1,0,\exp}\times\mathcal{X}_{\ell_1,0,\exp},$$

endowed with the weighted norm

$$[\![(f,g,h)]\!]_{\ell_1,2,\exp}=\frac{1}{\varepsilon}\|f\|_{2,\exp}+\kappa\|g\|_{\ell_1,0,\exp}+\kappa\|h\|_{\ell_1,0,\exp}.$$

The next lemmas give estimates for the operators and functions given in Section 3.3.

Lemma 7.1. *The components of the operator $\mathcal{P}$ in* (3.25) *have the following properties.*

(1) *For $\alpha=2,5$, the operator $\mathcal{P}^W:\mathcal{X}_{\alpha,\exp}\to\mathcal{X}_{2,\exp}$ is well defined. Moreover, there exists a constant $M>0$ independent of ε and κ such that,*
   - *For $h\in\mathcal{X}_{2,\exp}$, $\|\mathcal{P}^W(h)\|_{2,\exp}\leq M\varepsilon\|h\|_{2,\exp}$.*
   - *For $h\in\mathcal{X}_{5,\exp}$, $\|\mathcal{P}^W(h)\|_{2,\exp}\leq \dfrac{M}{\varepsilon^2\kappa^3}\|h\|_{5,\exp}$.*

(2) *For $\alpha>1$, the operators $\mathcal{P}^\Gamma,\mathcal{P}^\Theta:\mathcal{X}_{\ell_1,\alpha,\exp}\to\mathcal{X}_{\ell_1,0,\exp}$ are well-defined. Moreover, there exists a constant $M>0$ independent of ε and κ such that, for every $h\in\mathcal{X}_{\ell_1,\alpha,\exp}$,*

$$\|\mathcal{P}^\Gamma(h)\|_{\ell_1,0,\exp},\|\mathcal{P}^\Theta(h)\|_{\ell_1,0,\exp}\leq\frac{M}{(\kappa\varepsilon)^{\alpha-1}}\|h\|_{\ell_1,\alpha,\exp}.$$

Proof. We first prove item (1). We take $h\in\mathcal{X}_{\ell_1,2,\exp}$ and, recalling that $\ddot v^h$ has a a pole of order 3, we obtain the following estimate for $\operatorname{Im}(y)>0$,

$$\begin{aligned}\left|e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}|y^2+\pi^2/4|^2\ddot v^h(y)\int_0^y\frac{h(s)}{\ddot v^h(s)}ds\right|\leq&\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y\left|\frac{h(s)}{\ddot v^h(s)}\right|ds\\ \leq& M\|h\|_{2,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(s)|\right)}|s^2+\pi^2/4|ds\\ \leq& M\|h\|_{2,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\operatorname{Im}(y)\right)}}{|y-i\pi/2|}\int_0^{\operatorname{Im}(y)}|\sigma-\pi/2|e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\sigma\right)}d\sigma\\ \leq& M\|h\|_{2,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\operatorname{Im}(y)\right)}}{|y-i\pi/2|}\int_{\frac{\frac{\pi}{2}-\operatorname{Im}(y)}{\varepsilon}}^{\frac{\pi}{2\varepsilon}}\varepsilon re^{-\lambda_3 r}\varepsilon dr\\ \leq&\frac{M\varepsilon\|h\|_{2,\exp}}{|y-i\pi/2|}\left(\frac{\varepsilon}{\lambda_3}+\frac{\pi}{2}-\operatorname{Im}(y)-e^{-\frac{\lambda_3}{\varepsilon}\operatorname{Im}(y)}\left(\frac{\varepsilon}{\lambda_3}+\frac{\pi}{2}\right)\right)\\ \leq& M\varepsilon\|h\|_{2,\exp}\left(\frac{1}{\kappa}+1\right)\leq\ M\varepsilon\|h\|_{2,\exp}.\end{aligned}$$

Analogously, one can obtain the same estimate for $\mathrm{Im}(y)<0$.

For $h\in\mathcal{X}_{\ell_1,5,\exp}$, one obtains,

$$\begin{aligned}
\left|e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}|y^2+\pi^2/4|^2\ddot{v}^h(y)\int_0^y\frac{h(s)}{\ddot{v}^h(s)}ds\right| &\leq \frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y\left|\frac{h(s)}{\ddot{v}^h(s)}\right|ds\\
&\leq M\|h\|_{5,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y\frac{e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(s)|\right)}}{|s^2+\pi^2/4|^2}ds\\
&\leq \frac{M\|h\|_{5,\exp}}{\kappa^3\varepsilon^3}e^{-\frac{\lambda_3}{\varepsilon}|\operatorname{Im}(y)|}\int_0^y e^{\frac{\lambda_3}{\varepsilon}|\operatorname{Im}(s)|}ds\\
&\leq \frac{M\|h\|_{5,\exp}}{\kappa^3\varepsilon^2}.
\end{aligned}$$

We prove item (2) only for the operator $\mathcal{P}^\Gamma$, since the result for $\mathcal{P}^\Theta$ follows analogously. Let $h(y,\tau)=\sum_{k\geq1}h_{2k+1}(y)\sin((2k+1)\tau)$. We bound each component of the operator $\mathcal{P}^\Gamma$ as

$$\begin{aligned}
\left|\mathcal{P}^\Gamma_{2k+1}(h_{2k+1})e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}\right| &\leq \|h_{2k+1}\|_{\alpha,\exp}\int_y^{y^+}\left|e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}\frac{e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(s)|\right)}}{|s^2+\pi^2/4|^\alpha}e^{i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)}\right|ds\\
&\leq \|h_{2k+1}\|_{\alpha,\exp}\int_{\operatorname{Im}(y)}^{\frac{\pi}{2}-\kappa\varepsilon}\frac{e^{\frac{1}{\varepsilon}(\lambda_3|\sigma|-\lambda_{2k+1}\sigma-(\lambda_3|\operatorname{Im}(y)|-\lambda_{2k+1}\operatorname{Im}(y)))}}{|\sigma^2-\pi^2/4|^\alpha}d\sigma.
\end{aligned}$$

Now, since the functions $f_k(t)=\lambda_3|t|-\lambda_{2k+1}t$ are decreasing for $t\in\mathbb{R}$ and $k\geq1$, $\sigma>\operatorname{Im}(y)$, and recalling that $\alpha>1$, we obtain

$$\left|\mathcal{P}^\Gamma_{2k+1}(h_{2k+1})e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}\right|\leq\|h_{2k+1}\|_{\alpha,\exp}\int_{\operatorname{Im}(y)}^{\frac{\pi}{2}-\kappa\varepsilon}\frac{1}{|\sigma^2-\pi^2/4|^\alpha}d\sigma\leq\frac{M}{(\kappa\varepsilon)^{\alpha-1}}\|h_{2k+1}\|_{\alpha,\exp}.$$

□

In next proposition, we obtain estimates for the right hand side of equation (3.26).

Proposition 7.2. *There exists a constant M independent of ε and κ such that the following statements hold.*

(1) *The operator $\widetilde{\mathcal{M}}:\mathcal{Y}_{\ell_1,2,\exp}\to\mathcal{Y}_{\ell_1,2,\exp}$ introduced in (3.26) is well-defined and*

$$\left[\!\left[\widetilde{\mathcal{M}}\left(\Xi_1,\Gamma,\Theta\right)\right]\!\right]_{\ell_1,2,\exp}\leq\frac{M}{\kappa}\left[\!\left[(\Xi_1,\Gamma,\Theta)\right]\!\right]_{\ell_1,2,\exp}.$$

Moreover, denoting $\widetilde{\mathcal{M}}=(\widetilde{\mathcal{M}}_1,\widetilde{\mathcal{M}}_2,\widetilde{\mathcal{M}}_3)$, we have that

$$\begin{aligned}
\left\|\widetilde{\mathcal{M}}_1\left(\Xi_1,\Gamma,\Theta\right)\right\|_{2,\exp}&\leq\frac{M}{\kappa^3}\|\Xi_1\|_{2,\exp}+M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right),\\
\left\|\widetilde{\mathcal{M}}_j\left(\Xi_1,\Gamma,\Theta\right)\right\|_{\ell_1,0,\exp}&\leq\frac{M}{\kappa^2\varepsilon}\|\Xi_1\|_{2,\exp}+\frac{M}{\kappa}\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right),\ j=2,3.
\end{aligned}$$

(2) *The function $\widetilde{\Delta}$ defined in (3.19) satisfies*

$$\widetilde{\Delta}=(I-\widetilde{M})^{-1}\big(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d)\big)\quad\text{and}\quad\left[\!\left[\widetilde{\Delta}-\big(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d)\big)\right]\!\right]_{\ell_1,2,\exp}\leq\frac{M}{\kappa}\left[\!\left[\big(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d)\big)\right]\!\right]_{\ell_1,2,\exp},$$

where $\mathcal{I}_\Gamma(c)$, $\mathcal{I}_\Theta(d)$ are the functions defined in (3.24) and (3.27).

Proof. Assume that $(\Xi_1,\Gamma,\Theta)\in\mathcal{Y}_{\ell_1,2,\exp}$. To estimate the first component of $\mathcal{M}$, using the estimates for m_W and $\mathcal{M}_W$ in Proposition 3.11 and Lemma 7.1 for the estimates on $\mathcal{P}^W$,

$$\begin{aligned}
\left\|\widetilde{\mathcal{M}}_1\left(\Xi_1,\Gamma,\Theta\right)\right\|_{2,\exp}&\leq\left\|\mathcal{P}^W\left(m_W\Xi_1\right)\right\|_{2,\exp}+\left\|\mathcal{P}^W\left(M_W(\Gamma,\Theta)\right)\right\|_{2,\exp}\\
&\leq\frac{M}{\varepsilon^2\kappa^3}\left\|m_W\Xi_1\right\|_{5,\exp}+M\varepsilon\left\|M_W(\Gamma,\Theta)\right\|_{2,\exp}\\
&\leq\frac{M}{\kappa^3}\left\|\Xi_1\right\|_{2,\exp}+M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right).
\end{aligned}$$

Now we estimate $\widetilde{\mathcal{M}}_2$. The estimates for $\widetilde{\mathcal{M}}_3$ can be done analogously. Using as before Proposition 3.11 and Lemma 7.1,

$$\begin{aligned}\left\|\widetilde{\mathcal{M}}_2\left(\Xi_1,\Gamma,\Theta\right)\right\|_{\ell_1,0,\exp} &\le \left\|\mathcal{P}^\Gamma\left(m_{\rm osc}\Xi_1\right)\right\|_{\ell_1,0,\exp}+\left\|\mathcal{P}^\Gamma\left(M_{\rm osc}(\Gamma,\Theta)\right\|_{\ell_1,0,\exp}\\ &\le \frac{M}{\kappa^2\varepsilon^2}\left\|m_{\rm osc}\Xi_1\right\|_{\ell_1,3,\exp}+\frac{M}{\kappa\varepsilon}\left\|M_{\rm osc}(\Gamma,\Theta)\right\|_{\ell_1,2,\exp}\\ &\le \frac{M}{\kappa^2\varepsilon}\left\|\Xi_1\right\|_{\ell_1,2,\exp}+\frac{M}{\kappa}\left(\|\Gamma\|_{\ell_1,0,\exp}+\|\Theta\|_{\ell_1,0,\exp}\right).\end{aligned}$$

Item (2) of the proposition is simply a direct consequence of item (1) and (3.26). □

The rest of this section is devoted to estimating $\left(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d)\right)$.

Lemma 7.3. *Take $\kappa=\frac{1}{2\mu_3}|\log\varepsilon|$. There exist $\varepsilon_0>0$ and a constant $M>0$ independent of ε such that, for each $\varepsilon\in(0,\varepsilon_0)$,*

$$\left\|\mathcal{I}_\Gamma(c)-\frac{2\mu_3}{\varepsilon}C_{\rm in}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp},\quad\left\|\mathcal{I}_\Theta(d)-\frac{2\mu_3}{\varepsilon}\overline{C_{\rm in}}e^{i\frac{\lambda_3}{\varepsilon}(y+i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp}\le\frac{M}{\varepsilon|\log\varepsilon|}.$$

Proof. From Theorems 3.3 and 3.6 (see also (3.7)), the function Δ given in (3.15) can be written as

$$\begin{aligned}\Delta(y,\tau) &= \frac{1}{\varepsilon}\phi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\phi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\ &= \frac{1}{\varepsilon}\Delta\phi^0\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)+\frac{1}{\varepsilon}\varphi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\varphi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\ &= \frac{1}{\varepsilon}e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}\left(C_{\rm in}\sin(3\tau)+\chi\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\right)\\ &\quad+\frac{1}{\varepsilon}\varphi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\varphi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\ &= \frac{1}{\varepsilon}C_{\rm in}e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}\sin(3\tau)+E_1^+(y,\tau)+E_2^+(y,\tau),\end{aligned}$$

for every $y\in\mathcal{R}^+_{{\rm mch},\kappa}=D^{{\rm mch},u}_{+,\kappa}\cap D^{{\rm mch},s}_{+,\kappa}\cap i\mathbb{R}$ and κ satisfying assumptions in Theorems 3.3 and 3.6, where $E_1^+,E_2^+:\mathcal{R}_{{\rm mch},\kappa}\times\mathbb{T}\to\mathbb{C}$ are analytic functions in the variable y. It follows from Theorem 3.3 that

$$\|\partial_\tau E_1^+\|_{\ell_1}(y)\le\frac{M|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|}\quad\textit{and}\quad\|\partial_y E_1^+\|_{\ell_1}(y)\le\frac{M|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|^2},\tag{7.1}$$

and from Theorem 3.6, choosing $\gamma=1/2$, we obtain

$$\|\partial_\tau^2 E_2^+\|_{\ell_1}(y)\le\frac{M\varepsilon^{3/2}|\log\varepsilon|}{|y-i\pi/2|^2}\quad\textit{and}\quad\|\partial_\tau^2\partial_y E_2^+\|_{\ell_1}(y)\le\frac{M\varepsilon^{1/2}|\log\varepsilon|}{\kappa|y-i\pi/2|^2}.\tag{7.2}$$

Analogously, since Δ is real-analytic one can deduce that for $y\in\mathcal{R}^-_{{\rm mch},\kappa}=\{z:\bar z\in\mathcal{R}^+_{{\rm mch},\kappa}\}$,

$$\Delta(y,\tau)=\frac{1}{\varepsilon}\overline{C_{\rm in}}e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}\sin(3\tau)+E_1^-(y,\tau)+E_2^-(y,\tau),$$

where $E_j^-(y,\tau)=\overline{E_j^+(\bar y,\tau)}$, which satisfy

$$\begin{aligned}&\|\partial_\tau E_1^-\|_{\ell_1}(y)\le\frac{M|e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|}\quad\text{and}\quad\|\partial_y E_1^-\|_{\ell_1}(y)\le\frac{M|e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|^2},\\ &\|\partial_\tau^2 E_2^-\|_{\ell_1}(y)\le\frac{M\varepsilon^{3/2}|\log\varepsilon|}{|y+i\pi/2|^2}\quad\text{and}\quad\|\partial_y E_2^-\|_{\ell_1}(y)\le\frac{M\varepsilon^{1/2}|\log\varepsilon|}{\kappa|y+i\pi/2|^2}.\end{aligned}$$

Using (3.18) and recalling that $\lambda_3 = \mu_3 + \mathcal{O}(\varepsilon^2)$, we obtain that for $(y,\tau) \in \mathcal{R}^+_{\mathrm{mch},\kappa} \times \mathbb{T}$,

$$\begin{aligned}\Gamma(y,\tau) &= \sum_{k\geq 1}(\lambda_{2k+1}\Delta_{2k+1}(y) + i\varepsilon\partial_y\Delta_{2k+1}(y))\sin((2k+1)\tau)\\ &= \frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}(1+\mathcal{O}(\varepsilon^2))\sin(3\tau)\\ &\quad + \sum_{k\geq 1}\lambda_{2k+1}\Pi_{2k+1}\left[E_1^+ + E_2^+\right]\sin((2k+1)\tau) + i\varepsilon\widetilde{\Pi}\left[\partial_y E_1^+ + \partial_y E_2^+\right](y,\tau).\end{aligned}$$

Moreover, using (7.1) and (7.2), we have that

$$\begin{aligned}\left\|\sum_{k\geq 0}\lambda_{2k+1}\Pi_{2k+1}\left[E_1^+ + E_2^+\right]\sin((2k+1)\tau)\right\|_{\ell_1}(y) &\leq M\left(\|\partial_\tau E_1^+\|_{\ell_1}(y) + \|\partial_\tau E_2^+\|_{\ell_1}(y)\right)\\ &\leq M\left(\frac{|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|} + \frac{\varepsilon^{3/2}|\log\varepsilon|}{|y-i\pi/2|^2}\right)\\ \left\|i\varepsilon\widetilde{\Pi}\left[\partial_y E_1^+ + \partial_y E_2^+\right]\right\|_{\ell_1}(y) &\leq M\left(\frac{\varepsilon|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|^2} + \frac{\varepsilon^{3/2}|\log\varepsilon|}{\kappa|y-i\pi/2|^2}\right).\end{aligned}$$

Then, for $(y,\tau) \in \mathcal{R}^+_{\mathrm{mch},\kappa} \times \mathbb{T}$, Γ satisfies

$$\Gamma(y,\tau) = \frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}\sin(3\tau) + E_\Gamma^+(y,\tau),$$

where $E_\Gamma^+ : \mathcal{R}^+_{\mathrm{mch},\kappa} \times \mathbb{T} \to \mathbb{C}$ is an analytic function in the variable τ such that

$$\left\|E_\Gamma^+\right\|_{\ell_1}(y) \leq M\left(\frac{|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|} + \frac{\varepsilon^{3/2}|\log\varepsilon|}{|y-i\pi/2|^2}\right).$$

Proceeding in the same way for the function

$$\Theta(y,\tau) = \sum_{k\geq 0}(\lambda_{2k+1}\Delta_{2k+1}(y) - i\varepsilon\partial_y\Delta_{2k+1}(y))\sin((2k+1)\tau),$$

we conclude that there exists a function $E_\Theta^- : \mathcal{R}^-_{\mathrm{mch},\kappa} \times \mathbb{T} \to \mathbb{C}$ analytic in the variable y such that Θ can be written as

$$\Theta(y,\tau) = \frac{2\mu_3}{\varepsilon}\overline{C_{\mathrm{in}}}e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}\sin(3\tau) + E_\Theta^-(y,\tau), \qquad \text{for } (y,\tau) \in \mathcal{R}^-_{\mathrm{mch},\kappa} \times \mathbb{T}$$

and

$$\left\|E_\Theta^-\right\|_{\ell_1}(y) \leq M\left(\frac{|e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|} + \frac{\varepsilon^{3/2}|\log\varepsilon|}{|y+i\pi/2|^2}\right), \qquad \text{for } y \in \mathcal{R}^-_{\mathrm{mch},\kappa}.$$

Now that we have good estimates for the functions Γ and Θ in the domains $\mathcal{R}^\pm_{\mathrm{mch},\kappa}$, we analyze the functions $\mathcal{I}_\Gamma(c)$, $\mathcal{I}_\Theta(d)$. Recall that $\mathcal{I}_\Gamma(c)(y^+) = \Gamma(y^+)$. Therefore

$$\begin{aligned}\left\|\mathcal{I}_\Gamma(c) - \frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y^+) &= \left\|\Gamma - \frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y^+)\\ &= \left\|E_\Gamma^+\right\|_{\ell_1}(y^+)\\ &\leq M\left(\frac{|e^{-i\mu_3\frac{y^+-i\pi/2}{\varepsilon}}|}{|y^+-i\pi/2|} + \frac{\varepsilon^{3/2}|\log\varepsilon|}{|y^+-i\pi/2|^2}\right)\\ &\leq M\left(\frac{e^{-\mu_3\kappa}}{\kappa\varepsilon} + \frac{\varepsilon^{3/2}|\log\varepsilon|}{\kappa^2\varepsilon^2}\right),\end{aligned}$$

and notice that, from (3.24), we have that

$$\left\|\mathcal{I}_\Gamma(c) - \frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp} = e^{\lambda_3\kappa}\left\|\mathcal{I}_\Gamma(c) - \frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y^+),$$

and thus, taking $\kappa = \frac{1}{2\lambda_3}\log(\varepsilon^{-1})$, we have that

$$\left\| \mathcal{I}_\Gamma(c) - \frac{2\mu_3}{\varepsilon} C_{\rm in} e^{-i\frac{\lambda_3}{\varepsilon}(y - i\pi/2)} \sin(3\tau) \right\|_{\ell_1,0,\exp} \leq M\left(\frac{e^{(\lambda_3-\mu_3)\kappa}}{\kappa\varepsilon} + \frac{\varepsilon^{3/2}|\log\varepsilon| e^{\lambda_3\kappa}}{\kappa^2\varepsilon^2} \right) \leq \frac{M}{\varepsilon|\log\varepsilon|}.$$

The estimate on $\mathcal{I}_\Theta(d)$ follows analogously and it completes the proof of the lemma. □

Proposition 3.12 follows directly from Proposition 7.2 and Lemma 7.3.

8. Breathers with exponentially small tails: Proof of Proposition 2.2

To prove Proposition 2.2 we analyze the intersection of the center-stable manifold $W^{cs}(0)$ and center-unstable manifold $W^{cu}(0)$ of the zero solution which form a tube homoclinic to the center manifold $W^c(0)$ in the phase space. In the original coordinates, they correspond to an infinite dimensional family of waves of (1.2) which are ω-periodic in t (with ω given in (1.21) with $k = 1$) and of the order

$$u(x,t) = \mathcal{O}(\varepsilon e^{-\varepsilon|x|}) + \mathcal{O}(e^{-\frac{\sqrt{2}\pi}{\varepsilon}}).$$

In particular, the exponentially small oscillating tails do not decay as $|x| \to \infty$. The construction of such generalized breathers is largely based on the approach in [61, 44, 43], so we shall adapt the problem into the framework in [44].

We shall adopt a slightly different coordinate system and phase space in this section compared to that in Section 3. Let

$$q = (q_1, q_2)^T := (v_1, \partial_y v_1)^T, \ \ Q = \left(\varepsilon^{-1}\widetilde{\Pi}[v], \, (-\partial_\tau^2 - \omega^{-2})^{-\frac{1}{2}}\widetilde{\Pi}[\partial_y v]\right)^T. \tag{8.1}$$

In the above, the operator $(-\partial_\tau^2 - \omega^{-2})^{-\frac{1}{2}}$ is bounded uniformly in ε on $\ker \Pi_1$. In the (q, Q) variables, equation (1.2), or equivalently (1.23), takes the form

$$\begin{cases} \partial_y q = Aq + F(q, Q, \varepsilon) \\ \partial_y Q = \frac{J}{\varepsilon} Q + G(q, Q, \varepsilon), \end{cases} \tag{8.2}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = (-\partial_\tau^2 - \omega^{-2})^{\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : H^1(\mathbb{S}^1) \cap \ker \Pi_1 \to L^2(\mathbb{S}^1) \cap \ker \Pi_1,$$

and with $v = q_1 \sin\tau + \widetilde{\Pi}[v]$,

$$F(q, Q, \varepsilon) = \left(0, \, -\frac{1}{4}q_1^3 + \left(-\frac{1}{\varepsilon^3\omega^3}\Pi_1\left[g(\varepsilon\omega v)\right] + \frac{q_1^3}{4}\right)\right)^T,$$

$$G(q, Q, \varepsilon) = \left(0, \, -\frac{1}{\varepsilon^3\omega^3}(-\partial_\tau^2 - \omega^{-2})^{-\frac{1}{2}}\widetilde{\Pi}\left[g(\varepsilon\omega v)\right]\right)^T.$$

While $q \in X := \mathbb{R}^2$, we take $Y = L^2(\mathbb{S}^1) \cap \ker \Pi_1$. Apparently

$$J : Y \supset D(J) = Y_1 \to Y, \ \ Y_1 := H^1(\mathbb{S}^1) \cap \ker \Pi_1, \quad J^* = -J,$$

where L^2 and H^1 stand for the standard Sobolev space of square integrable functions and the subspace of L^2 functions with square integrable first order derivatives. It is straight forward to verify that $X_1 = X$, Y, Y_1, A, J, F, and G fit into the framework of [44] and satisfy all assumptions (A1–A5) in Section 2, (B1–B5) and (C1–C2) in Section 4, and (D1–D5) in Section 6 there. (In fact G satisfies a stronger estimates

$$\|G\|_{H^1} \leq M, \quad \|D_q^{l_1} D_Q^{l_2} G\|_{L\left((\mathbb{R}^2\times L^2)\otimes(\otimes^{l_1+l_2-1}(\mathbb{R}^2\times H^1)), \mathbb{R}^2\times L^2\right)} \leq M\varepsilon^{3l_2},$$

for some $M > 0$ independent of small $\varepsilon > 0$, on any bounded set in $X \times Y_1$.) Therefore smooth local invariant manifolds of 0, including the 1-dim stable and unstable manifolds analyzed in details in this current paper, exist with sizes and bounds (in (q, Q) variables) uniform in ε (Theorems 4.2, 4.9–4.11 in [44]).

In the following, we consider the homoclinic tube formed by the intersection of the center-stable manifold $W^{cs}(0)$ and the center-unstable manifold $W^{cu}(0)$. We also include the estimate of the minimal value of the Hamiltonian $\mathcal{H}$ on the homoclinic tube which in turn yields an estimate on the minimal amplitude of the oscillating tails of the corresponding generalized breathers.

• *Notation.* In this section all differentiation D are only with respect to the variables (q, Q) in the phase space, but never with respect to ε.

• **The local invariant manifolds and the restriction of the Hamiltonian $\mathcal{H}$ there.** Let q_u and q_s be the coordinates of q in the eigenvector expansion

$$q = q_u(1,1)^T + q_s(1,-1)^T$$

in term of the stable and unstable eigenvectors. According to Theorem 4.2 in [44], locally the center-unstable (or center-stable, center) manifold $W^{cu}(0) \in X \times Y_1$ (or $W^{cs}(0)$, $W^c(0)$) can be represented as the graph of a smooth mapping $h^{cu}(\cdot, \varepsilon) : \{|q_u|, \|Q\|_{Y_1} \le \delta\} \subset Y_1 \times \mathbb{R} \to \mathbb{R}$ (or h^{cs}, h^c):

$$W^{cu}(0) \cap \{|q_u|, |q_s|, \|Q\|_{Y_1} \le \delta\} = \{q_s = h^{cu}(q_u, Q, \varepsilon)\},$$
$$W^{cs}(0) \cap \{|q_u|, |q_s|, \|Q\|_{Y_1} \le \delta\} = \{q_u = h^{cs}(q_s, Q, \varepsilon)\},$$
$$W^{c}(0) \cap \{|q_u|, |q_s|, \|Q\|_{Y_1} \le \delta\} = \{(q_u, q_s) = h^{c}(Q, \varepsilon)\},$$

for some $\delta > 0$ independent of sufficiently small $\varepsilon > 0$. Moreover $h^{c\star}(q_\star, Q = 0, 0)$, $\star = u, s$, is well-defined and correspond to the 1-dim stable and unstable manifold of (1.29) with $k = 1$. They satisfy the following estimates[6]. For $l \ge 1$ and some $M > 0$ independent of ε, for $\star = u, s$,

$$|h^{c\star}(q_\star, 0, \varepsilon) - h^{c\star}(q_\star, 0, 0)| + |D_{q_\star} h^{c\star}(q_\star, 0, \varepsilon) - D_{q_\star} h^{c\star}(q_\star, 0, 0)| + \|D_Q h^{c\star}(q_\star, 0, \varepsilon)\|_{(H^1)^*} \le M\varepsilon,$$

$$Dh^{c\star}(0, 0, \varepsilon) = 0, \quad Dh^c(0, \varepsilon) = 0, \quad \|D^l h^{c\star}\| + \|D^l h^c\| \le M.$$

In the (q_u, q_s, w) variables the Hamiltonian $\mathcal{H}$ defined in (1.24) takes the form

$$\mathcal{H}(q_u, q_s, Q, \varepsilon) = -2\pi q_u q_s + \frac{1}{2} \left\| (-\partial_\tau^2 - \omega^{-2})^{\frac{1}{2}} Q \right\|_{L^2}^2 + \int_{\mathbb{T}} \left(\frac{v^4}{12} + \frac{F(\varepsilon \omega v)}{\varepsilon^4 \omega^4} \right) d\tau,$$

which is smooth in $(q, Q) \in \mathbb{R}^2 \times Y_1$ and ε due to $F(u) = \mathcal{O}(u^6)$ near $u = 0$. Since

$$D_Q^2 \mathcal{H}(0, 0, 0, \varepsilon)(Q, Q) = \|(-\partial_\tau^2 - \omega^{-2})^{\frac{1}{2}} Q\|_{L^2}^2 \ge \frac{1}{2} \|Q\|_{H^1}^2,$$

it is straight forward to obtain the uniform quadratic positivity of $\mathcal{H}$ restricted on the center manifold $W^c(0)$

$$D_Q^2 \big(\mathcal{H}\big(h^c(Q, \varepsilon), Q, \varepsilon\big) \big)(\tilde{Q}, \tilde{Q}) \ge \frac{1}{3} \|\tilde{Q}\|_{H^1}^2, \quad \mathcal{H}\big(h^c(Q, \varepsilon), Q, \varepsilon\big) \ge \frac{1}{6} \|Q\|_{H^1}^2, \quad \forall Q \in Y_1,\ \|Q\|_{H_1} \le \delta. \tag{8.3}$$

The quadratic positivity implies that the center manifold $W^c(0)$ is unique and 0 is stable both forward and backward in y on $W^c(0)$ for (1.23) (with $k = 1$). By the conservation of energy and the invariant foliation structure (Theorems 5.1, 5.3, and 5.4 in [44]), we have that $\mathcal{H} \ge 0$ on $W^{c\star}(0)$ and it achieves 0 exactly at $W^\star(0)$, $\star = u, s$. Therefore, at any $U \in W^\star(0)$, $\|U\|_{H^1} \le \delta$, $\star = u, s$,

$$T_U W^{c\star}(0) = \ker D\mathcal{H}(U, \varepsilon), \quad \ker \big(D^2 \mathcal{H}(U, \varepsilon)|_{T_U W^{c\star}(0)} \big) = T_U W^\star(0).$$

Here $\ker D\mathcal{H}(U, \varepsilon)$ is viewed as a linear functional on $\mathbb{R}^2 \times Y_1$ and $D^2 \mathcal{H}(U, \varepsilon)|_{T_U W^{c\star}(0)}$ a bounded linear operator on $T_U W^{c\star}(0)$ induced by the symmetric quadratic form on $T_U W^{c\star}(0)$. Moreover, for any hyperplane P in the tangent space $T_U W^{c\star}(0) \subset \mathbb{R}^2 \times Y_1$ transversal to $T_U W^\star(0)$, there exists $\sigma > 0$ such that

$$\|D^2 \mathcal{H}(U, \varepsilon)|_P\|_{L(P \otimes P, \mathbb{R})} \ge \sigma. \tag{8.4}$$

• **Analyzing $W^{cs}(0) \cap W^{cu}(0)$.** In terms of the $(q, Q) = (q_1, q_2, Q)$ coordinates, let

$$\Lambda = \{q_2 = 0\} \subset \mathbb{R}^2 \times Y_1$$

be the hyperplane perpendicular to the unperturbed homoclinic orbit

$$\Gamma^h := \big\{ (v^h(y), \partial_y v^h(y), 0) \mid y \in \mathbb{R} \big\} \subset \mathbb{R}^2 \times Y_1, \qquad \text{(see (1.30))},$$

at $U_0 = (v^h(0), 0, 0)$.

By Theorem 2.2 and 2.3 in [44], for any fixed time $T > 0$ the time–T map of (8.2) is smooth in the phase space $\mathbb{R}^2 \times Y_1$ with its derivative bounded uniformly in ε. (Even though only the first differentiation was carefully estimated in [44], the uniform in ε bounds of the higher order derivatives simply follow from a similar argument inductively.) Due to the uniform in ε sizes and bounds on $W^{cs}(0)$ and $W^{cu}(0)$, they can

[6] Actually some better estimates have been obtained in this current paper.

be extended to stripes along Γ^h. For $\star = u, s$, consider the following intersections with Λ for the first time after $W^{c\star}(0)$ are extended from a neighborhood of 0 by the flow of (8.2),

$$\widetilde{W}^{c\star}(0) = W^{c\star}(0) \cap \Lambda, \quad U_\star = (q_{1,\star}, 0, Q_\star) = W^\star(0) \cap \Lambda \in \widetilde{W}^{c\star}(0).$$

Clearly, here $U_\star$ corresponds to the values of the stable and unstable solutions $\big(v^\star(0), \partial_y v^\star(0)\big)$ analyzed in Theorem 2.1.

We shall start with the decomposition $\Lambda = \big(\mathbb{R}(1,0)^T\big) \oplus Y_1$ to set up a coordinate system to analyze $\widetilde{W}^{c\star}(0)$, $\star = u, s$. Here in particular we notice

$$\{q = 0\} \times Y_1 = \ker D\mathcal{H}(U_0, 0) \cap \Lambda, \quad (q_1, q_2, Q) = \nabla \mathcal{H}(U_0, 0) = \frac{10\sqrt{2}}{3}\big(1, 0, 0\big). \tag{8.5}$$

Clearly $\widetilde{W}^{c\star}(0)$ is a hypersurface in Λ. Due to the conservation of the Hamiltonian $\mathcal{H}$ by the flow map, it holds

$$\mathcal{H}(U_\star, \varepsilon) = 0, \quad T_{U_\star} W^{c\star}(0) = \ker D\mathcal{H}(U_\star, \varepsilon), \quad T_{U_\star} \widetilde{W}^{c\star}(0) = \ker D\mathcal{H}(U_\star, \varepsilon) \cap \Lambda,$$

which implies that locally $\{\mathcal{H}(\cdot, \varepsilon) = 0\} \cap \Lambda$ and $\widetilde{W}^{c\star}(0)$ can be expressed as the graphs of smooth mapping from $Y_1 \to \mathbb{R}$. In fact, due to the smoothness of $\mathcal{H}$ in ε and the uniform in ε bounds of $W^{c\star}(0)$ near 0 and the flow map, there exist $\delta > 0$ independent of ε and $\widetilde{h}^0, \widetilde{h}^{c\star} : Y_1 \to \mathbb{R}$, $\star = u, s$, such that inside the box $\{|q_1 - v^h(0)|, \|Q - Q_\star\|_{H^1} \le \delta\}$ in Λ,

$$\begin{aligned} \{\mathcal{H}(\cdot, \varepsilon) = 0\} \cap \Lambda = \{q_1 = \widetilde{h}^0(Q, \varepsilon)\}, \quad & \widetilde{h}^0(Q_\star, \varepsilon) = q_{1,\star}, \star = u, s \\ \widetilde{W}^{c\star}(0) = \{q_1 = \widetilde{h}^{c\star}(Q, \varepsilon)\}, \quad & \widetilde{h}^{c\star}(Q_\star, \varepsilon) = q_{1,\star}, \star = u, s, \end{aligned}$$

where $\widetilde{h}^0$ and $\widetilde{h}^{c\star}$ along with their derivatives are bounded uniformly in small ε.

Due to (8.5), it is clear

$$(q_1, 0, Q) \in \widetilde{W}^{cu}(0) \cap \widetilde{W}^{cs}(0) \Longleftrightarrow q_1 = \widetilde{h}^{cu}(Q, \varepsilon) = \widetilde{h}^{cs}(Q, \varepsilon) \Longleftrightarrow \mathcal{H}\big(\widetilde{h}^{cu}(Q, \varepsilon), 0, Q, \varepsilon\big) = \mathcal{H}\big(\widetilde{h}^{cs}(Q, \varepsilon), 0, Q, \varepsilon\big),$$

By (8.3) and (8.5), there exists $C > 0$ such that

$$0 \le \mathcal{H}\big(\widetilde{h}^{c\star}(Q, \varepsilon), 0, Q, \varepsilon\big) = \mathcal{H}\big(\widetilde{h}^{c\star}(Q, \varepsilon), 0, Q, \varepsilon\big) - \mathcal{H}\big(\widetilde{h}^0(Q, \varepsilon), 0, Q, \varepsilon\big) \le C(\widetilde{h}^{c\star}(Q, \varepsilon) - \widetilde{h}^0(Q, \varepsilon)), \tag{8.6}$$

which implies

$$\widetilde{h}^{c\star}(Q, \varepsilon) \ge \widetilde{h}^0(Q, \varepsilon), \quad \text{and `` = '' holds iff } Q = Q_\star, \quad \star = u, s. \tag{8.7}$$

Moreover, from (8.4), the conservation of $\mathcal{H}$, and the uniform in ε bound on the flow map, we have

$$C\|Q - Q_\star\|_{H^1}^2 \ge \mathcal{H}\big(\widetilde{h}^{c\star}(Q, \varepsilon), 0, Q, \varepsilon\big) \ge \frac{1}{C}\|Q - Q_\star\|_{H^1}^2, \quad \star = u, s, \tag{8.8}$$

Therefore if $Q_u = Q_s$, clearly $U_u = U_s$ and $W^s(0) = W^u(0)$ which gives rises to a homoclinic orbit to 0. In the case of $Q_u \ne Q_s$, (8.7) implies

$$\widetilde{h}^{cu}(Q_s, \varepsilon) > \widetilde{h}^0(Q_s, \varepsilon) = \widetilde{h}^{cs}(Q_s, \varepsilon) \ \text{ and } \ \widetilde{h}^{cs}(Q_u, \varepsilon) > \widetilde{h}^0(Q_u, e) = \widetilde{h}^{cu}(Q_u, \varepsilon).$$

Therefore, there exists $\widetilde{Q}$, e.g. on the segment connecting Q_u and Q_s, such that $\widetilde{h}^{cu}(\widetilde{Q}, \varepsilon) = \widetilde{h}^{cs}(\widetilde{Q}, \varepsilon)$ and thus

$$(q_1, q_2, Q) = \big(\widetilde{h}^{cu}(\widetilde{Q}, \varepsilon), 0, \widetilde{Q}\big) \in \widetilde{W}^{cs}(0) \cap \widetilde{W}^{cu}(0) \subset W^{cu}(0) \cap W^{cs}(0).$$

This completes the proof of $W^{cs}(0) \cap W^{cu}(0) \ne \emptyset$, which had been also obtained in [43]. Moreover, (8.6) and (8.8) imply that such that

$$D\widetilde{h}^{c\star}(Q_\star, \varepsilon) = 0, \quad D^2\widetilde{h}^{c\star}(Q_\star, \varepsilon) \ge \tfrac{1}{C} > 0, \qquad \star = u, s.$$

Since Q_u and Q_s are exponentially close and the derivatives of $h^{c\star}$ are bounded uniformly in ε, we obtain the transversality of the intersection of $W^{cs}(0) \cap W^{cu}(0)$ near the above mentioned $\widetilde{Q}$ on the segment connecting Q_u and Q_s if $Q_u \ne Q_s$. See Figure 10. This completes the proof of Proposition 2.2(3).

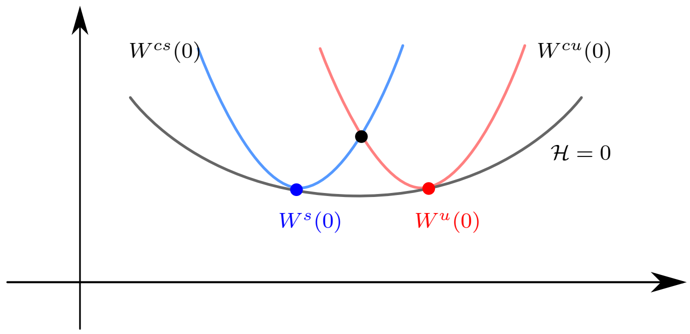

FIGURE 10. Intersection between $W^{cs}(0)$ and $W^{cu}(0)$ giving rise to a breather with an exponentially small tail.

Remark 8.1. *The above argument is carried out in the energy space $(v, \partial_x v) \in H^1_\tau(\mathbb{S}^1) \times L^2_\tau(\mathbb{S}^1)$ which heavily depends on the coercivity of the conserved energy. Hence $W^{cs}(0) \cap W^{cu}(0)$ is obtained in the energy space. In fact, locally it contains a dense a subset consisting of smooth functions of τ in the case of $Q_u \neq Q_s$. To see this, one observes that the transversality of the intersection implies that each nearby point in $W^{cs}(0) \cap W^{cu}(0)$ can be realized as a transversal intersection of $W^{cs}(0) \cap W^{cu}(0)$ and a smooth curve $\mathcal{C}$ connecting U_s and U_u. It is easy to see that the proof of Theorem 3.1 can be carried out in any Sobolev space of higher regularity in τ, hence U_s and U_s are smooth in τ as well. Approximating $\mathcal{C}$ by a curve in any higher order Sobolev space, we obtain a nearby point in $W^{cs}(0) \cap W^{cu}(0)$ of higher regularity in τ.*

Each orbit $\big(q(y), Q(y)\big)$ starting in $\widetilde{W}^{cs}(0) \cap \widetilde{W}^{cu}(0)$ is homoclinic to $W^c(0)$. Due to the invariant foliation structure within $W^{c\star}(0)$ (Theorems 5.1, 5.3, and 5.4 in [44]), as $y \to \pm\infty$ it converges to two orbits in $W^c(0)$, which in the original coordinates (see (8.1)) can be written as $\big(v_c^\pm(y), \partial_y v_c^\pm(y)\big) \subset W^c(0)$. Moreover, by (8.3),

$$\tfrac{1}{C}\mathcal{H}(q, Q) = \tfrac{1}{C}\mathcal{H}(v_c^\pm, \partial_y v_c^\pm) \leq \varepsilon^{-2}\|v_c^\pm\|_{H^1}^2 + \|\partial_y v_c^\pm\|_{L^2}^2 \leq C\mathcal{H}(v_c^\pm, \partial_y v_c^\pm) = C\mathcal{H}(q, Q).$$

According to (8.8), $\mathcal{H}\big(q(0), Q(0)\big)$ can be used as an equivalent measure between the $\big(q(0), Q(0)\big)$ and $\big(v^\star(0), \partial_y v^\star(0)\big)$, $\star = u, s$. For y on any finite interval, the square of the distance between $\big(q(y), Q(y)\big)$ and $\big(v^\star(y), \partial_y v^\star(y)\big)$ is proportional to $\mathcal{H}\big(q(0), Q(0)\big)$ simply due to the uniform-in-ε boundedness on the derivatives of the flow maps. When $\big(q(y), Q(y)\big)$ is close to 0 (within a small $\mathcal{O}(1)$ distance), the distance between $\big(q(y), Q(y)\big)$ and $\big(v^\star(y), \partial_y v^\star(y)\big)$, where $\star = u$ for $y \ll 0$ and $\star = s$ for $y \gg 1$, can be estimate using the stable/unstable foliations which along with their derivatives are bounded uniformly in ε (see Section 5 of [44]). Combined with

$$\|Q\|_{H^1}^2 + |q|^2 \sim \big\| \varepsilon^{-1}| - \partial_\tau^2 - \omega^{-2}|^{\frac{1}{2}} v \big\|_{L^2}^2 + \|\partial_y v\|_{L^2}^2$$

uniformly in ε, this finishes the proof of (2.6) and thus of Proposition 2.2(2).

Finally, we estimate $\inf \mathcal{H}$ on $\widetilde{W}^{cs}(0) \cap \widetilde{W}^{cu}(0)$. Let $Q \in Y_1$ such that $\|Q - Q_\star\|_{H^1} \leq \delta$ and $\big(q_1 = \widetilde{h}^{cu}(Q, \varepsilon), 0, Q\big) \in \widetilde{W}^{cs} \cap \widetilde{W}^{cu}$, then (8.8) implies

$$\|Q - Q_u\|_{H^1} \leq C\|Q - Q_s\|_{H^1} \quad \text{and} \quad \|Q - Q_s\|_{H^1} \leq C\|Q - Q_u\|_{H^1},$$

which further yields

$$C\|Q - Q_\star\|_{H^1} \geq \|Q_u - Q_s\|_{H^1}, \quad \star = u, s.$$

Taking into account (8.8) again, one obtains

$$\mathcal{H}\big(\widetilde{h}^{c\star}(Q, \varepsilon), 0, Q, \varepsilon\big) \geq \frac{1}{C}\|Q_u - Q_s\|_{H^1}^2.$$

Moreover, if such Q is on the segment connecting Q_u and Q_s, one has

$$\mathcal{H}\big(\widetilde{h}^{c\star}(Q, \varepsilon), 0, Q, \varepsilon\big) \leq C\|Q_u - Q_s\|_{H^1}^2$$

Therefore we obtain

$$C\|Q_u - Q_s\|_{H^1}^2 \geq \inf_{\widetilde{W}^{cu}(0) \cap \widetilde{W}^{cs}(0)} \mathcal{H} \geq \frac{1}{C}\|Q_u - Q_s\|_{H^1}^2,$$

and Proposition 2.2(1) follows from the above argument.

9. Non-existence of small breathers: strongly hyperbolic case $\omega \in J_k(\varepsilon_0)$

This section is devoted to proving statement (1) of Theorem 1.3, that is the results for the case $\omega \in J_k(\varepsilon_0)$, $k \geq 0$. The other case $\omega \in I_k(\varepsilon_0)$ will be proved in Section 10. The oddness of u in t is not assumed to start with in these two sections.

For any $\omega \in J_k(\varepsilon_0)$, $k > 0$, we adopt the rescaling $\tau = \omega t$ and the nonlinear Klein-Gordon equation (1.2) turns into the form of (1.17). Treating x as the dynamic variable and recalling that u is 2π-periodic in τ, the unknown $u(x, \tau)$ can be expanded in Fourier series

$$u(x,\tau) = \sum_{n=-\infty}^{\infty} u_n(x) e^{in\tau}, \quad u_{-n} = \overline{u_n}.$$

(Note this Fourier series is different from the rest of the paper by a ratio of $-\frac{i}{2}$. The latter was adapted so that u_n is the real coefficient of the Fourier sine series when u is odd in t.) The eigenvalues of the linearization of (1.17) at 0, that is

$$\partial_x^2 u - \omega^2 \partial_\tau^2 u - u = 0,$$

are $\pm\nu_n$, where

$$\nu_n = \sqrt{1 - n^2\omega^2}, \tag{9.1}$$

and their eigenfunctions can be calculated using the Fourier series. The hyperbolic eigenvalues correspond to $0 \leq n \leq k$ and

$$\nu_0 \geq \ldots \geq \nu_k \in \left(\frac{\varepsilon_0}{\sqrt{k+\varepsilon_0^2}}, \frac{\sqrt{2k+1}}{k+1} \right], \tag{9.2}$$

while the center eigenvalues correspond to $n \geq k+1$ and

$$\nu_n = i\vartheta_n, \quad 0 \leq \vartheta_{k+1} \leq \vartheta_{k+2} \leq \ldots. \tag{9.3}$$

Let $W_\omega^\star(0)$, $\star = c, s, u$, denote the locally invariant center, stable, and unstable manifolds of 0 for the equation (1.17) in the energy space $H_\tau^1 \times L_\tau^2$. Their existence and smoothness follow from standard arguments (see Theorem 4.4 in [16], for example) since the nonlinearity $g(u) : H_\tau^1 \to H_\tau^2 \hookrightarrow L_\tau^2$ is analytic in u. Due to the uniqueness, $W_\omega^\star(0)$, $\star = s, u$, are also obviously the local stable and unstable manifolds of 0 in the ℓ_1 based phase space $(u, \partial_x u) \in \mathbf{X}$ defined in (2.1). On such finite dimensional submanifolds, different metrics including $H_\tau^1 \times L_\tau^2$ and $\|\cdot\|_{\mathbf{X}}$, all induce the same equivalent topology. Clearly $\dim W_\omega^\star(0) = 2k+1$, $\star = s, u$, while $W_\omega^c(0)$ is of codim-$(4k+2)$. Statement (1) of Theorem 1.3 for the case of $\omega \in J_k(\varepsilon_0)$, $k \geq 0$, will be proved by showing a.) some uniform-in-k-and-ω estimates on the size of $W_\omega^\star(0)$ in ℓ_1, $\star = s, u$, where the norm is dominated by the energy norm, and b.) no solutions converging to 0 along $W_\omega^c(0) \subset H_\tau^1 \times L_\tau^2$.

• **Estimates on the local stable/unstable manifolds for $\omega \in J_k(\varepsilon_0)$.** Usually the sizes of the local stable/unstable manifolds in phase spaces are determined by the power nonlinearity and the minimal absolute value of the real parts of the stable/unstable eigenvalues, which is $\nu_k > \frac{\varepsilon_0}{\sqrt{k+\varepsilon_0^2}}$ according to (9.2). We prove the following proposition on a lower bound of the sizes of $W_\omega^\star(0)$ in $\mathbf{X}$.

Proposition 9.1. *There exists $\rho, M > 0$ such that, for any $\varepsilon_0 \in (0, 1/2)$, $\omega \in J_k(\varepsilon_0)$, $k \geq 0$, there exist $\Omega^u, \Omega^s : B_{\mathbb{R}^{2k+1}}(0, \rho\nu_k) \to \mathbf{X}$, where $B_{\mathbb{R}^{2k+1}}(0, \rho\nu_k)$ is the ball in $\mathbb{R}^{2k+1}$ centered at 0 and with radius $\rho\nu_k$, such that, the image $\Omega^\star\big(B_{\mathbb{R}^{2k+1}}(0, \rho\nu_k)\big)$ is an open subset of $W_\omega^\star(0)$, $\star = s, u$, and*

$$\Omega^\star(0,\tau) = 0, \quad \|\Omega_1^\star(a,\cdot) - \Omega_1^\star(\tilde a,\cdot)\|_{\ell_1} + \nu_k^{-1}\|\Omega_2^\star(a,\cdot) - \Omega_2^\star(\tilde a,\cdot)\|_{\ell_1} \leq M\nu_k^{-2}(|a|_1^2 + |\tilde a|_1^2)|a - \tilde a|_1$$

where

$$\Omega^\star(a,\tau) = \left(\sum_{n=-k}^{k} a_n e^{in\tau} + \Omega_1^\star(a,\tau), \ \sum_{n=-k}^{k} (\mp\nu_n) a_n e^{in\tau} + \Omega_2^\star(a,\tau) \right),$$

and a and $\tilde a$ are parameters of $(2k+1)$-dim (real) satisfying

$$a = (a_{-k}, \ldots, a_k), \ \ a_n \in \mathbb{C}, \ \ a_{-n} = \overline{a_n}, \ \ -k \leq n \leq k, \quad |a|_1 := \sum_{n=-k}^{k} |a_n| < \rho\nu_k. \tag{9.4}$$

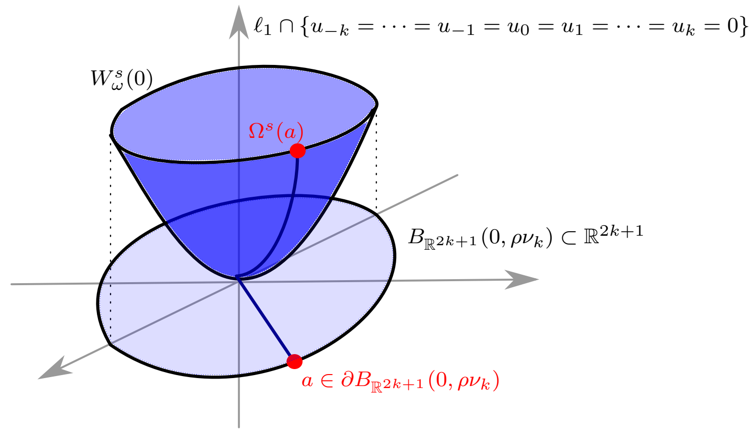

FIGURE 11. Parameterization of the local stable manifold $W^s_\omega(0)$ by Ω^s.

Here we identified complex numbers a_n with 2-dim real vectors. These $\Omega^\star$ can be viewed as coordinate mappings of $W^\star_\omega(0)$ (see Figure 11). They can actually be proved to be analytic in a, but our main focus here is the sizes of their domains and the error estimates.

We use the classical Perron method and will only outline the argument to prove the proposition for the stable manifold. Consider the following Banach space

$$\mathcal{E}_{\mathcal{S}} = \{h : [0, +\infty) \times \mathbb{T} \to \mathbb{R};\ h \text{ is analytic in } x, \text{ and } \|h\|_{\nu_k,\ell_1} < \infty\},$$

where

$$\|h\|_{\nu_k,\ell_1} = \sum_{n\geq 1} \|h_n\|_{\nu_k} \quad \textit{and} \quad \|h_n\|_{\nu_k} = \sup_{x\geq 0} |e^{\nu_k x} h_n(x)|,$$

and define the linear operator $\mathcal{S}$ acting on the Fourier modes of a function $h(x, \tau)$

$$\mathcal{S}\,(h) = \sum_{n\geq 1} \mathcal{S}_n(h_n) \sin(n\tau), \tag{9.5}$$

with

$$\mathcal{S}_n(h) = \frac{1}{2\nu_n}\left(\int_{+\infty}^{x} e^{\nu_n(x-s)} h(s)ds - \int_0^x e^{-\nu_n(x-s)} h(s)ds\right) \qquad \text{for } \ 1 \leq n \leq k,$$
$$\mathcal{S}_n(h) = \int_{+\infty}^{x} \frac{\sin(\vartheta_n(x-s))}{\vartheta_n} h(s)ds \qquad \text{for } \ n > k.$$

where we recall $\nu_n = i\vartheta_n$.

Note that we are including in $J_k(\varepsilon_0)$ the case $\omega = 1/(k+1)$. For this value of ω, one has that $\vartheta_{k+1} = 0$. In this case, one can take the limit $\vartheta_{k+1} \to 0$ in $\mathcal{S}_{k+1}(h)$ to obtain

$$\mathcal{S}_{k+1}(h) = \int_{+\infty}^{x} (x-s)h(s)ds.$$

We also define the function

$$\Xi(a, x, \tau) = \sum_{n=-k}^{k} a_n e^{-\nu_n x + in\tau},$$

where $a = (a_{-k}, \ldots, a_k)$ are parameters satisfying (9.4). One can check that a solution $u(x, \tau)$ of (1.17) belongs to the stable manifold of $u = 0$ if, and only if, it is a fixed point of the operator

$$\widetilde{\mathcal{S}}(a, u) = \Xi(a) + \mathcal{S}(g(u)),$$

for some a as in (9.4), where g is the nonlinearity introduced in (1.19).

The following lemma is a direct consequence of the particular form of the operator $\mathcal{S}$ in (9.5) and the fact that the function g is of order 3 near $u = 0$.

Lemma 9.2. *There exists $M, r_1 > 0$ independent of $k \geq 0$ and $\omega \in J_k(\varepsilon_0)$ such that, for any $0 < r \leq r_1$ and $a \in \mathbb{R}^{2k+1}$, the operator $\widetilde{\mathcal{S}} : B(0,r) \subset \mathcal{E}_{\mathcal{S}} \to \mathcal{E}_{\mathcal{S}}$ is a well-defined Lipschitz operator which satisfies*

$$\left\|\widetilde{\mathcal{S}}(a,u)\right\|_{\nu_k,\ell_1} \leq |a|_1 + \frac{Mr^3}{\nu_k^2}, \ \left\|\partial_x\widetilde{\mathcal{S}}(a,u)\right\|_{\nu_k,\ell_1} \leq |a|_1 + \frac{Mr^3}{\nu_k}, \qquad \forall\, u \in B(0,r) \subset \mathcal{E}_{\mathcal{S}}$$

and its Lipschitz constant on $B(0,r)$ satisfies

$$\mathrm{Lip}_u(\widetilde{\mathcal{S}}) \leq \frac{Mr^2}{\nu_k^2}, \quad \mathrm{Lip}_u(\partial_x\widetilde{\mathcal{S}}) \leq \frac{Mr^2}{\nu_k}.$$

Consequently, there exists $\rho > 0$ independent of $k \geq 0$ and $\omega \in J_k(\varepsilon_0)$ such that, for any $a \in B_{\mathbb{R}^{2k+1}}(0, \rho\nu_k)$, by taking $r = 2|a|_1$, there exists a unique fixed point $h_*(a) \in B(0,r) \subset \mathcal{E}_{\mathcal{S}}$ of $\widetilde{\mathcal{S}}(a,\cdot)$ which also satisfies $h_*(a=0) = 0$ and

$$\begin{aligned}&\left\|\big(h_*(a) - \Xi(a)\big) - \big(h_*(\tilde{a}) - \Xi(\tilde{a})\big)\right\|_{\nu_k,\ell_1} \leq M\nu_k^{-2}(|a|_1^2 + |\tilde{a}|_1^2)|a - \tilde{a}|_1 \\ &\left\|\partial_x\big(h_*(a) - \Xi(a)\big) - \partial_x\big(h_*(\tilde{a}) - \Xi(\tilde{a})\big)\right\|_{\nu_k,\ell_1} \leq M\nu_k^{-1}(|a|_1^2 + |\tilde{a}|_1^2)|a - \tilde{a}|_1.\end{aligned}$$

Let

$$\Omega^s(a,\tau) = \big(h_*(a,0,\tau), \partial_x h_*(a,0,\tau)\big).$$

The conclusions of Proposition 9.1 follow from standard and straight forward arguments.

• **Nonexistence of decaying solutions on the center manifold.** Recall that, when x is viewed as the dynamic variable, the nonlinear Klein-Gordon equation (1.17) conserves the Hamiltonian $\mathcal{H}(u, \partial_x u)$ where

$$\mathcal{H}(u_1,u_2) = \int_{-\pi}^{\pi} \left(\frac{1}{2}u_2^2 + \frac{\omega^2}{2}(\partial_\tau u_1)^2 - \frac{1}{2}u_1^2 + \frac{1}{12}u_1^4 + F(u_1)\right) d\tau$$

is smoothly defined on the energy space $H_\tau^1 \times L_\tau^2$. Let $W_\omega^c(0)$ be a center manifold of $(0,0)$ for (1.17). The following lemma holds for all $\omega > 0$, not just those in $I_k(\varepsilon_0)$ or $J_k(\varepsilon_0)$.

Lemma 9.3. *For any $\omega > 0$, $(0,0)$ is a strict local minimum of $\mathcal{H}$ restricted on its local center manifold .*

Proof. There exists unique $k \geq 0$ such that $\omega \in [1/(k+1), 1/k)$. Let $\mathbf{Y}^c, \mathbf{Y}^h \subset H_\tau^1 \times L_\tau^2$ denote the center and hyperbolic subspaces of the linearization of (1.17) at $(0,0)$

$$\begin{aligned}\mathbf{Y}^c &= \{(u_1,u_2) \in H_\tau^1 \times L_\tau^2 \mid u_j(\tau) = \sum_{|n|\geq k+1} u_{j,n}e^{in\tau}, \ u_{j,-n} = \overline{u_{j,n}}, \ j = 1,2\}, \\ \mathbf{Y}^h &= \{(u_1,u_2) \mid u_j(\tau) = \sum_{|n|\leq k} u_{j,n}e^{in\tau}, \ u_{j,-n} = \overline{u_{j,n}}. \ j = 1,2\}.\end{aligned}$$

Locally $W_\omega^c(0)$ can be represented as the graph of a smooth mapping $\gamma^c(u_1,u_2)$ from a small neighborhood of $(0,0)$ in $\mathbf{Y}^c$ to $\mathbf{Y}^h$. Due to the lack of quadratic nonlinear terms in (1.17), γ^c satisfies

$$\gamma^c(u_1,u_2) = \mathcal{O}\big(\|u_1\|_{H_\tau^1}^3 + \|u_2\|_{L_\tau^2}^3\big).$$

Due to $F(u) = \mathcal{O}(|u|^6)$ for small u and the orthogonality between $\mathbf{Y}^c$ and $\mathbf{Y}^h$, for small $(u_1,u_2) \in \mathbf{Y}^c$,

$$\begin{aligned}\mathcal{H}\big((u_1,u_2) + \gamma^c(u_1,u_2)\big) =& \int_{-\pi}^{\pi} \frac{1}{2}u_2^2 + \frac{\omega^2}{2}(\partial_\tau u_1)^2 - \frac{1}{2}u_1^2 + \frac{1}{12}u_1^4 d\tau + \mathcal{O}\big(\|u_1\|_{H_\tau^1}^6 + \|u_2\|_{L_\tau^2}^6\big) \\ \geq& \frac{1}{2}\|u_2\|_{L_\tau^2}^2 + \pi \sum_{|n|\geq k+1}^{\infty} \vartheta_n^2 u_{1,n}^2 + \frac{1}{24\pi}\|u_1\|_{L_\tau^2}^4 - \mathcal{O}\big(\|u_1\|_{H_\tau^1}^6 + \|u_2\|_{L_\tau^2}^6\big).\end{aligned}$$

If $\omega \neq \frac{1}{k+1}$, then there exists $\delta > 0$ such that

$$\frac{\vartheta_n^2}{1+n^2} \geq \delta, \ \forall |n| \geq k+1 \implies \sum_{n=k+1}^{\infty} \vartheta_n^2 u_{1,n}^2 \geq \frac{\delta}{2\pi}\|u_1\|_{H_\tau^1}^2.$$

Therefore in this case $(0,0)$ is clearly a non-degenerate local minimum of $\mathcal{H}$ on $W^c_\omega(0)$. If $\omega = \frac{1}{k+1}$, then $\vartheta_{\pm(k+1)} = 0$ and there exists $\delta > 0$ such that

$$\frac{\vartheta_n^2}{1+n^2} \geq \delta, \ \forall |n| \geq k+2.$$

Let

$$\tilde{u}_1 = \sum_{|n|\geq k+2} u_{1,n} e^{in\tau}$$

and then we have

$$\mathcal{H}\big((u_1,u_2)+\gamma^c(u_1,u_2)\big) \geq \frac{1}{2}\|u_2\|^2_{L^2_\tau} + \frac{\delta}{2}\|\tilde{u}_1\|^2_{H^1_\tau} + \frac{1}{24\pi}|u_{1,\pm(k+1)}|^4 - \mathcal{O}\big(\|\tilde{u}_1\|^6_{H^1_\tau} + |u_{1,\pm(k+1)}|^6 + \|u_2\|^6_{L^2_\tau}\big).$$

Again $(0,0)$ is clearly a strict local minimum of $\mathcal{H}$ on $W^c_\omega(0)$. □

Due the conservation of $\mathcal{H}$, we immediately obtain

Corollary 9.4. *For any $\omega > 0$, $(0,0)$ has a locally unique center manifold $W^c_\omega(0)$ and is stable on $W^c_\omega(0)$ both forward and backward in x. Moreover, except $(0,0)$ no solution on $W^c_\omega(0)$ converges to $(0,0)$ as $x \to +\infty$ or $-\infty$.*

Finally we are ready to complete the proof of statement (1) of Theorem 1.3.

Proof of statement (1) of Theorem 1.3. Let $\varepsilon_0 \in (0,1/2)$, $k \geq 0$, and $\omega \in J_k(\varepsilon_0)$. Since the $\|\cdot\|_{\ell_1}$ norm is invariant under a rescaling in τ, we can work on (1.17) equivalently. Without loss of generality, assume $u(x,\tau)$ is a solution such that $(u, \partial_x u)$ converging to $(0,0)$ in $H^1_\tau \times L^2_\tau$ as $x \to +\infty$. Such a solution must belong to the local center-stable manifold of $(0,0)$ for $x \gg 1$ (see Theorem 4.4 in [16] or the outline of the arguments in Section 5.3). It is well-known that the center-stable manifold is foliated into stable fibers based on the local center manifold $W^s_\omega(0)$ (for example, see Theorem 4.3 in [15]). The dynamics of all initial data on each fiber is shadowed by that of the based point on $W^c_\omega(0)$. According to Corollary 9.4, no non-trivial solutions on $W^c_\omega(0)$ converges to $(0,0)$ as $x \to +\infty$, the based point of the decaying solution $u(x,\tau)$ must be $(0,0) \in W^s_\omega(0)$ and thus it belongs to the stable manifold $W^s_\omega(0)$. From Proposition 9.1, locally the stable manifold $W^s_\omega(0) = \Omega^s\big(B_{\mathbb{R}^{2k+1}}(0,\rho\nu_k)\big)$ and Ω^s is a small perturbation of the isomorphism $\big(\Xi(a), \mathrm{diag}(\nu_{-k},\dots,\nu_k)\Xi(a)\big)$. In the coordinates $(a_{-k},\dots,a_k)$, the dynamics on $W^s_\omega(0)$ is governed by

$$\frac{da}{dx} = \big(D\Xi + D_a\Omega^s_1(a,\cdot)\big)^{-1}\big(-\mathrm{diag}(\nu_{-k},\dots,\nu_k)\Xi(a) + \Omega^s_2(a,\cdot)\big) = -\mathrm{diag}(\nu_{-k},\dots,\nu_k)a + \mathcal{O}(\nu_k\rho^2|a|_1),$$

where the estimates on Ω^s_1 and Ω^s_2 given in Proposition 9.1 were used. It is straightforward to prove that, as x evolves backwards, every solution on $W^s_\omega(0)$ must exit through its boundary where the u_1 component (corresponding to $u(x,\cdot)$ itself) satisfies $\|u_1\|_{\ell_1} \geq \frac{1}{2}\rho\nu_k$. Finally statement (1) of Theorem 1.3 follows from

$$\frac{\nu_k}{\omega^{\frac{1}{2}}} = \left(\frac{1}{\omega} - k^2\omega\right)^{\frac{1}{2}} \geq \left(\sqrt{k(k+\varepsilon_0^2)} - \frac{k^{\frac{3}{2}}}{\sqrt{k+\varepsilon_0^2}}\right)^{\frac{1}{4}} \geq \varepsilon_0\left(\frac{k}{k+\varepsilon_0^2}\right)^{\frac{1}{2}} \geq \frac{\varepsilon_0}{2}, \ \text{ if } k \geq 1,$$

and $\nu_0 = 1$ if $k = 0$. □

10. Bifurcation analysis for $\omega \in I_k(\varepsilon_0)$

We devote this section to the completion of the proof of Statement 2 of Theorem 1.3 and Proposition 1.5, that are the statements concerning $\omega \in I_k(\varepsilon_0)$. For such ω, there are two pair of (weakly) hyperbolic eigenvalues along with $2k-1$ pairs of stronger ones (see (10.2)). Our strategy is to reduce the problem to $\omega \in I_1(\varepsilon_0)$ and $u(x,t)$ odd in t.

We analyze the birth of small homoclinic loops taking

$$\omega = \sqrt{\frac{1}{k(k+\varepsilon^2)}} \qquad \text{with} \qquad k \geq 1, \ \ 0 < \varepsilon \leq \varepsilon_0 \leq \frac{1}{2}.$$

We expand the (real) solution $u(x,\tau)$ to the nonlinear Klein-Gordon equation (1.17) in Fourier series in τ as

$$u(x,\tau) = \sum_{n=-\infty}^{+\infty} \left(-\frac{i}{2}\right) u_n(y) e^{in\tau}, \ \ u_{-n} = -\overline{u_n},$$

where the $-i/2$ factor is simply for the technical convenience that, if $u(x,\tau)$ is odd in τ, then $u_n(y)$, $n>0$, coincides with the coefficient in its Fourier sine series expansion. Subsequently (1.17) is equivalent to a coupled system of equations in the form of

$$\partial_x^2 u_n = \nu_n^2 u_n - \Pi_n\left[g(u)\right], \quad n \in \mathbb{Z}, \tag{10.1}$$

where Π_n is the projection from $u(\tau)$ to the n-th mode u_n as in the above expansion and

$$\nu_n = \sqrt{1-n^2\omega^2}, \quad 1 = \nu_0 > \ldots > \nu_k = \varepsilon(k+\varepsilon^2)^{-\frac{1}{2}}, \ \ \nu_n = i\vartheta_n, \ \ \vartheta_{k+1} < \vartheta_{k+2} < \ldots, n \geq k+1, \tag{10.2}$$

are same as those in (9.1) and (9.3). In particular,

$$\nu_{k-1} = \sqrt{\frac{(2+\varepsilon^2)k-1}{k(k+\varepsilon^2)}} \geq \sqrt{\frac{1}{k}}, \quad \vartheta_{k+1} = \sqrt{\frac{(2-\varepsilon^2)k+1}{k(k+\varepsilon^2)}} \geq \sqrt{\frac{1}{k}}. \tag{10.3}$$

Linearizing at $u \equiv 0$, clearly $|n| \leq k$ corresponds to $2k+1$ pairs of hyperbolic directions, and $|n| \geq k+1$ to codim-$(4k+2)$ center directions. From the same argument as in the proof of statement (1) of Theorem 1.3 (see Section 9) based on Lemma 9.3, a solution $u(x,\tau)$ satisfies $\|(u,\partial_x u)\|_{\mathbf{X}} \to 0$ as $x \to \pm\infty$ if and only if $(u,\partial_x u) \in W_\omega^\star(0)$, $\star = s,u$. Hence we shall focus on the estimates of the sizes and the splitting distance between $W_\omega^u(0)$ and $W_\omega^s(0)$.

10.1. **Estimates on the local stable/unstable manifolds for** $\omega \in I_k(\varepsilon_0)$**.** For a semilinear PDE like (10.1), the standard theorems (see, for example, Theorem 4.4 in [16]) yield the existence of smooth local invariant manifolds $W_\omega^\star(0)$, $\star = s,u,c,cs,cu$, in the phase space $\mathbf{X}$ defined in (2.1). There are two issues, however. On the one hand, usually the sizes of the local invariant manifolds are generally determined by the gap between the real parts of the eigenvalues. While $\nu_n \geq k^{-\frac{1}{2}}$ for $|n| \leq k-1$, the weakest stable/unstable eigenvalues $\pm\nu_k = \mathcal{O}(\varepsilon k^{-\frac{1}{2}})$ of (10.1) are too small for the analysis of possible breathers of amplitude $\|u\|_{\ell_1} = O(k^{-\frac{1}{2}})$. On the other hand, the "angles" between the stable and unstable eigenfunctions in $\mathbf{X}$ of (10.1) can be rather small for $n \sim k$. In this subsection, we shall outline the construction of $W_\omega^\star(0)$, $\star = s,u$, with desired estimates based on the specific structure of (1.17), or equivalently (10.1). Essentially our strategy is to construct $W_\omega^s(0)$ as the union of strong stable fibers based on a weak stable manifold.

Observe

$$\mathcal{Z}_o = \left\{(u_1,u_2) \in \mathbf{X} \mid u_j(\tau) = \sum_{n\in\mathbb{Z}} \left(-\frac{i}{2}\right) u_{j,kn} e^{ikn\tau} = \sum_{n\in\mathbb{N}} u_{j,kn} \sin(kn\tau), \ u_{j,kn} \in \mathbb{R}, \ j=1,2\right\},$$

is an invariant subspace under (1.17), or equivalently (10.1). Any such solution $u(x,\cdot) \in \mathcal{Z}$ is odd and actually $\frac{2\pi}{k\omega}$-periodic in τ. Let

$$\tilde\varepsilon = k^{-\frac{1}{2}}\varepsilon \leq \varepsilon_0, \quad \tilde\tau = k\tau, \quad \tilde\omega = k\omega = (1+\tilde\varepsilon^2)^{-\frac{1}{2}}, \quad y = \tilde\varepsilon\tilde\omega x, \quad u = \tilde\varepsilon\tilde\omega v,$$

then $v(y,\tilde\tau)$ is 2π-periodic and odd in $\tilde\tau$ and satisfies

$$\partial_y^2 v - \frac{1}{\tilde\varepsilon^2}\partial_{\tilde\tau}^2 v - \frac{1}{\tilde\varepsilon^2\tilde\omega^2} v + \frac{1}{3}v^3 + \frac{1}{\tilde\varepsilon^3\tilde\omega^3} f(\tilde\varepsilon\tilde\omega v) = 0.$$

Note that this is in the form of (1.23) with $k=1$ (and note that $0 < \tilde\varepsilon \leq \varepsilon \leq \varepsilon_0$). Therefore for any $y_0 \in \mathbb{R}$, there exists $\varepsilon_0, M > 0$ (independent of k) such that, for $\varepsilon \in (0,\varepsilon_0]$, Theorem 2.1 applies to imply the existence of the unique odd-in-$\tilde\tau$ stable and unstable solutions $v_{\mathrm{wk}}^\star(y,\tilde\tau)$ of (1.17) such that $(v_{\mathrm{wk}}^\star, \partial_y v_{\mathrm{wk}}^\star) \in \mathbf{X}_o$ (see (2.2)) and

$$\left\|(1-\partial_{\tilde\tau}^2)\left(\begin{pmatrix} v_{\mathrm{wk}}^\star(y,\tilde\tau) \\ \partial_y v_{\mathrm{wk}}^\star(y,\tilde\tau)\end{pmatrix} - \begin{pmatrix} v^h(y) \\ (v^h)'(y)\end{pmatrix}\sin\tilde\tau\right)\right\|_{\ell_1} \leq M\tilde\varepsilon^2 v^h(y), \ \ \text{where } v^h(y) = \frac{2\sqrt{2}}{\cosh y}, \tag{10.4}$$

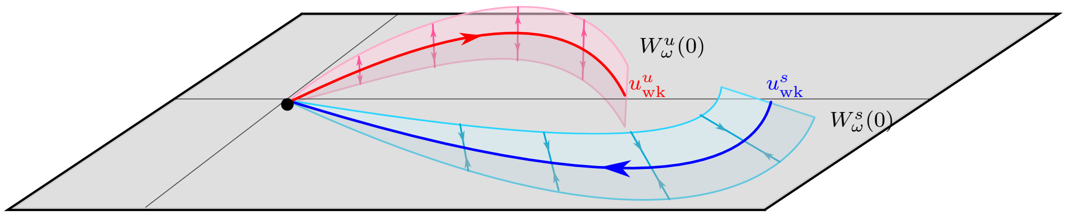

FIGURE 12. We construct the stable manifold $W^s_\omega(0)$ as the union of the strong stable fibers based on the weak stable manifold formed by the solution $u^s_{\rm wk}$ (and its τ-translations), which lives in the invariant subspace of $2\pi/k$-periodic-in-τ functions. We proceed analogously for the unstable manifold $W^u_\omega(0)$.

for $y \geq -y_0$ for $\star = s$ or $y \leq y_0$ for $\star = u$. One notices that we replaced $\partial^2_{\tilde\tau}$ in Theorem 2.1 which does not change the estimates as $v^\star_{\rm wk}(y,\tilde\tau)$ are odd in $\tilde\tau$. Moreover $\partial_y\Pi_1[v^\star_{\rm wk}(0,\cdot)] = 0$ and they satisfy the exponentially small splitting estimate at $y=0$

$$\Big\|\Big(|-\partial^2_{\tilde\tau} - \tfrac{1}{\tilde\omega^2}|^{\frac12}(v^u_{\rm wk} - v^s_{\rm wk}) + i\tilde\varepsilon\partial_y(v^u_{\rm wk} - v^s_{\rm wk})\Big)(0,\cdot) - \tfrac{4\sqrt2}{\tilde\varepsilon}C_{\rm in}e^{-\frac{\sqrt2\pi}{\tilde\varepsilon}}\sin 3\tilde\tau\Big\|_{\ell_1} \leq \frac{Me^{-\frac{\sqrt2\pi}{\tilde\varepsilon}}}{\tilde\varepsilon\log(\tilde\varepsilon)^{-1}}.$$

Proposition 2.2 there exist solutions $v(y,\tilde\tau)$ in $\mathbf{X}_o$ homoclinic to either 0 or its center manifolds, which have bounds in terms of the values of their Hamiltonian $\mathcal{H}$. From the estimates on the splitting and the inf $\mathcal{H}$ in Theorem 2.1, these orbits satisfy

$$\tilde\varepsilon^{-1}\big\||-\partial^2_{\tilde\tau} - \tfrac{1}{\tilde\omega^2}|^{\frac12}(v - v^\star_{\rm wk})\big\|_{L^2_{\tilde\tau}(-\pi,\pi)} + \|\partial_y(v - v^\star_{\rm wk})\|_{L^2_{\tilde\tau}(-\pi,\pi)} \leq M\tilde\varepsilon^{-2}e^{-\frac{\sqrt2\pi}{\tilde\varepsilon}}, \tag{10.5}$$

for $y \geq -y_0$ with $\star = s$ and $y \leq y_0$ with $\star = u$. When $C_{\rm in} \neq 0$, a lower bound of the same order also holds.

We rescale and obtain the unique stable and unstable solutions

$$u^\star_{\rm wk}(x,\tau) = \sqrt{k}\varepsilon\omega v^\star_{\rm wk}(\sqrt{k}\varepsilon\omega x, k\tau),$$

of (1.17) such that $(u^\star_{\rm wk}, \partial_x u^\star_{\rm wk}) \in \mathcal{Z}_o$ for any $x \in \mathbb{R}$. For any $x_0 \in \mathbb{R}$, there exists $\varepsilon_0, M > 0$ independent of k such that, for $\varepsilon \in (0,\varepsilon_0]$,

$$\left\|(1 - \tfrac{1}{k^2}\partial^2_\tau)\left(\begin{pmatrix} u^\star_{\rm wk}(x,\tau) \\ \frac{\partial_x u^\star_{\rm wk}(x,\tau)}{\sqrt{k}\varepsilon\omega}\end{pmatrix} - \sqrt{k}\varepsilon\omega\begin{pmatrix} v^h(\varepsilon\sqrt{k}\omega x) \\ (v^h)'(\varepsilon\sqrt{k}\omega x)\end{pmatrix}\sin k\tau\right)\right\|_{\ell_1} \leq Mk^{-\frac12}\varepsilon^3\omega v^h(\varepsilon\sqrt{k}\omega x), \tag{10.6}$$

for $x \geq -\frac{x_0}{\sqrt{k}\varepsilon\omega}$ for $\star = s$ or $x \leq \frac{x_0}{\sqrt{k}\varepsilon\omega}$ for $\star = u$. Moreover $\partial_x\Pi_k[u^\star_{\rm wk}(0,\cdot)] = 0$ and they satisfy the exponentially small splitting estimate

$$\Big\|\tfrac{1}{(k\omega)^2}\Big(|-\omega^2\partial^2_\tau - 1|^{\frac12}(u^u_{\rm wk} - u^s_{\rm wk}) + i(\partial_x u^u_{\rm wk} - \partial_x u^s_{\rm wk})\Big)(0,\cdot) - 4\sqrt2 C_{\rm in}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}}\sin 3k\tau\Big\|_{\ell_1} \leq \frac{Me^{-\frac{\sqrt{2k}\pi}{\varepsilon}}}{\frac12\log k - \log\varepsilon}.$$

Since $0 < 1 - (k\omega)^2 < \frac{\varepsilon^2}{k}$, the stable and unstable solutions prove *statements (2a-b) of Theorem 1.3*. The existence and estimates of breathers with exponentially small tails (in $\mathcal{Z}_o$) follow from the same rescaling and thus *Proposition 1.5* is also proved.

We shall prove statement (2c) of Theorem 1.3 in the rest of the section. The translations (in τ) of these solutions $(u^\star_{\rm wk}(x,\cdot+\theta), \partial_x u^\star_{\rm wk}(x,\cdot+\theta))$ form locally invariant 2-dim surfaces, parametrized by x and $\theta \in \mathbb{R}/(\frac{2\pi}{k}\mathbb{Z})$, of the nonlinear Klein-Gordon equation (1.17), or equivalently (10.1), where solutions grow or decay at weak exponential rates. It is worth pointing out that $(u^\star_{\rm wk}(x,\cdot), \partial_x u^\star_{\rm wk}(x,\cdot))$, $x \in \mathbb{R}$, corresponds to only one of the two branches of the 1-dim stable/unstable manifold of (1.17) in $\mathcal{Z}_o$, while the other branch corresponds to $(-u^\star_{\rm wk}(x,\cdot), -\partial_x u^\star_{\rm wk}(x,\cdot))$. When $x = \pm\infty$ is included, the 2-dim surface generated by the translation in τ does include 0 in the interior and the other branch (corresponding to $\theta = \frac{\pi}{k}$). Obviously they are submanifolds of the $(2k+1)$-dim stable/unstable manifolds and actually we shall construct the latter based on these weak ones (see Figure 12).

Proposition 10.1. *There exist $M > 1, \varepsilon_0, \rho_1$ independent of $k \geq 1$ and ω, and unique mappings for any $\varepsilon \in (0,\varepsilon_0)$ and ω given in* (1.13),

$$\zeta^\star = \big(\zeta^\star_1(r,\theta,\delta), \zeta^\star_2(r,\theta,\delta)\big) \in \mathbf{X}, \quad \theta \in \mathbb{R}/(\tfrac{2\pi}{k}\mathbb{Z}), \quad \delta = (\delta_{1-k},\ldots,\delta_{k-1}) \in \mathbb{C}^{2k-1}, \quad \delta_{-n} = -\overline{\delta_n}, \quad |\delta|_1 < \tfrac{\rho_1}{\sqrt{k}},$$

for $r \in \big[-\frac{y_0^s}{\varepsilon\sqrt{k}\omega}, +\infty\big]$, *if* $\star = s$ *and* $r \in \big[-\infty, \frac{y_0^u}{\varepsilon\sqrt{k}\omega}\big]$ *if* $\star = u$, *where* $y_0^\star$ *are any values satisfying*

$$\|u_{\rm wk}^u\|_{C^0_{y_0^u}} \triangleq \|u_{\rm wk}^u\|_{C^0\left(r\in\left[-\infty, \frac{y_0^u}{\varepsilon\sqrt{k}\omega}\right], |\tau|\le\pi\right)} \le \frac{\rho_1}{\sqrt{k}}, \quad \|u_{\rm wk}^s\|_{C^0_{y_0^s}} \triangleq \|u_{\rm wk}^s\|_{C^0\left(r\in\left[-\frac{y_0^s}{\varepsilon\sqrt{k}\omega}, +\infty\right], |\tau|\le\pi\right)} \le \frac{\rho_1}{\sqrt{k}},$$

such that

$$\Pi_n[\zeta_2^\star] \pm \nu_n \Pi_n[\zeta_1^\star] = 0,\ \forall\, |n| \le k-1,\ \ \star = u, s; \quad \Pi_{-n}[\zeta^\star] = \overline{\Pi_n[\zeta^\star]},\ \forall\, n \in \mathbb{Z}; \quad \zeta^\star(r,\theta,0) = 0,$$
$$\|\zeta_1^\star(r,\theta,\delta) - \zeta_1^\star(r,\theta,\tilde\delta)\|_{\ell_1} + \sqrt{k}\|\zeta_2^\star(r,\theta,\delta) - \zeta_2^\star(r,\theta,\tilde\delta)\|_{\ell_1} \le kM(\|u_{\rm wk}^\star\|^2_{C^0_{y_0^\star}} + |\delta|_1^2 + |\tilde\delta|_1^2)|\delta - \tilde\delta|_1,$$

and the images of $\xi^\star(r,\theta,\delta)$ *is an open subset of* $W_\omega^\star(0) \subset \mathbf{X}$ *where*

$$\xi^\star(r,\theta,\delta) = \big(\xi_1^\star(r,\theta,\delta), \xi_2^\star(r,\theta,\delta)\big) = \big(u_{\rm wk}^\star(r,\cdot+\theta), \partial_x u_{\rm wk}^\star(r,\cdot+\theta)\big) + \Xi^\star(\delta) + \zeta^\star(r,\theta,\delta),$$
$$\Xi^\star(\delta) = \sum_{|n|\le k-1} \left(-\tfrac{i}{2}\right)\delta_n e^{in\tau}(1, \pm\nu_n), \ \ \star = u, s.$$

Moreover, the orbits (of the dynamic variable x*) on* $W_\omega^\star(0)$ *takes the form* $\xi^\star\big(x + r, \theta, \delta(x)\big)$ *with*

$$\sum_{|n|\le k-1} \big|\partial_x\delta_n \mp \nu_n\delta_n\big| \le M\nu_{k-1}^{-1}(\|u_{\rm wk}^\star\|^2_{C^0_{y_0^\star}} + |\delta|_1^2)|\delta|_1, \ \ \star = u, s.$$

Remark 10.2. *By including* $r = \pm\infty$, *where* $u_{\rm wk}^\star(\pm\infty,\cdot) = \partial_x u_{\rm wk}^\star(\pm\infty,\cdot) = 0$ *for* $\star = s, u$, *the images of* $\xi^\star$ *do contain a whole open neighborhood of the zero solution in the stable/unstable manifolds* $W_\omega^\star(0) \subset \mathbf{X}$. *In fact,* $\xi^\star(\pm\infty,\theta,\delta)$ *become independent of* θ *and give the* $(2k-1)$*–dimensional strong stable/unstable manifolds corresponding to the eigenvalues* $\pm\nu_n$, $|n| \le k-1$. *In the* (r,δ) *coordinates on the invariant manifolds* $W_\omega^\star(0)$, *the PDE* (1.17) *corresponds to a vector field whose* r *component is always* 1 *and the* δ *components depend on* r *and* δ *which is a small perturbation to* $\pm\nu_n\delta_n$. *The following proof could be carried out in the spaces with high regularity in* τ *such as* $(1 + |\partial_\tau|)^{-N}\mathbf{X}$ *for any* $N \ge 0$ *and thus the local invariant manifolds* $W_\omega^\star(0) \subset (1 + |\partial_\tau|)^{-N}\mathbf{X}$ *enjoy the same properties. The smoothness of* $\zeta^\star$ *in* r *and* θ *is also true, for which we refer the readers to, for example, Theorem 4.3 in* [15] *for details, while we focus on the needed quantitative estimates on the sizes and the Lipschitz constant in* δ. *Alternatively, one may also work on the rescaled variables as in* (1.22) *and obtain equivalent estimates.*

Proof of Proposition 10.1. The proof follows the standard Lyapunov-Perron method which we shall only outline for the unstable case. Given parameters r and θ, we see solutions to (1.17) (or equivalently (10.1)) in the form of

$$u(x,\tau) = u_{\rm wk}^u(r + x, \tau + \theta) + U(x,\tau), \ \ x \le 0,$$

which decay to 0 as $x \to -\infty$. The equation satisfied by U takes the form

$$\mathcal{L}_k U = \mathcal{F}_k(U) \tag{10.7}$$

where

$$\mathcal{L}_k U = \sum_{n\in\mathbb{Z}} \big((\partial_x^2 - \nu_n^2)U_n\big)e^{in\tau}, \quad \mathcal{F}_k(r,\theta,U) = g(u_{\rm wk}^u + U) - g(u_{\rm wk}^u), \ \ \text{for } U(x,\tau) = \sum_{n\in\mathbb{Z}} U_n(x)e^{in\tau}.$$

Here we used the fact $u_{\rm wk}^s = u_{\rm wk}^u(r + x, \tau + \theta)$ is an exact solution. The decay property of $u(x,\tau)$ as $x \to +\infty$ is built into the Banach space which U belongs to

$$\mathcal{P} = \big\{U \in C^0\big((-\infty,0), \ell_1\big) \mid \|U\|_{\mathcal{P}} := \sup_{x\le 0} e^{-\frac{2}{3}\nu_{k-1}x}\|U(x)\|_{\ell_1} < \infty\big\}.$$

To set up the Lyapunov-Perron integral equation, define the linear transformation

$$\big(\mathcal{G}_k(h)\big)(x,\tau) = \sum_{n\in\mathbb{Z}} \big(\mathcal{G}_{k,n}(h_n)\big)(x)e^{in\tau}, \ \ \text{where } h(x,\tau) = \sum_{n\in\mathbb{Z}} h_n(x)e^{in\tau}, \quad x \le 0,$$

with

$$\big(\mathcal{G}_{k,n}(h_n)\big)(x) = \frac{1}{2\nu_n}e^{\nu_n x}\int_0^x e^{-\nu_n x'}h_n(x')dx' - \frac{1}{2\nu_n}e^{-\nu_n x}\int_{-\infty}^x e^{\nu_n x'}h_n(x')dx', \ |n| \le k-1,$$
$$\big(\mathcal{G}_{k,n}(h_n)(x)\big) = \frac{1}{\nu_n}\int_{-\infty}^x \sinh\big(\nu_n(x - x')\big)h_n(x')dx', \ |n| \ge k,$$

which serves as an inverse of $\mathcal{L}_k$. Here we note that for $|n| > k$, $\nu_n = i\vartheta_n$ and $\vartheta_n \geq k^{-\frac{1}{2}}$ and thus $\sinh\big(\nu_n(x-x')\big) = i\sin\big(\vartheta_n(x-x')\big)$. We also define

$$\widetilde{\Xi}(\delta, x, \tau) = \sum_{|n|\leq k-1} \left(-\tfrac{i}{2}\right)\delta_n e^{\nu_n x + in\tau}.$$

The desired solution U satisfies the fixed point equation

$$U = \widetilde{\mathcal{F}}(r,\theta,\delta,U) := \widetilde{\Xi}(\delta) + \mathcal{G}_k\big(\mathcal{F}_k(r,\theta,U)\big).$$

Using (10.2) and (10.3), it is straightforward to verify

$$\|\widetilde{\Xi}\|_{\mathcal{P}} \leq \frac{1}{2}|\delta|_1, \quad \|\mathcal{G}_k(h)\|_{\mathcal{P}} \leq \frac{100}{\nu_{k-1}^2}\|h\|_{\mathcal{P}},$$

$$\|\mathcal{F}_k(r,\theta,U) - \mathcal{F}_k(r,\theta,\tilde{U})\|_{\mathcal{P}} \leq M(\|u_{\rm wk}^u\|^2_{C^0_{y_0^u}} + \|U\|^2_{\mathcal{P}} + \|\tilde{U}\|^2_{\mathcal{P}})\|U-\tilde{U}\|_{\mathcal{P}}.$$

Therefore there exists $\rho_1 > 0$ independent of ε and $k \geq 1$, such that, for $|\delta|_1 \leq \frac{\rho_1}{\sqrt{k}}$, $\widetilde{\mathcal{F}}$ is a contraction on the ball of radius $\frac{2\rho_1}{\sqrt{k}}$ in $\mathcal{P}$. Let $U^u(r,\theta,\delta,x,\tau)$ be the unique fixed point of $\widetilde{\mathcal{F}}$,

$$\zeta_1^u(r,\theta,\delta) = U^u(r,\theta,\delta,0,\cdot) - \widetilde{\Xi}(\delta,0,\cdot), \quad \zeta_2^u(r,\theta,\delta) = \partial_x U^u(r,\theta,\delta,0,\cdot) - \partial_x\widetilde{\Xi}(\delta,0,\cdot),$$

and ξ^u accordingly. The desired estimates on ζ^u follow from straightforward calculations. The invariance of $W_\omega^u(0) = \text{image}(\xi^s)$ is a direct consequence of the uniqueness of the decaying solutions in $\mathcal{P}$, which implies that solutions on $W_\omega^u(0)$ are parametrized by $\delta(x)$ and take the following two forms

$$\begin{aligned} u(x,\cdot) =& u_{\rm wk}^u(r+x,\cdot+\theta) + U^u\big(r,\theta,\delta(0),x,\cdot\big) \\ =& \xi_1^u\big(r+x,\theta,\delta(x)\big) = u_{\rm wk}^u(r+x,\cdot+\theta) + U^u\big(r+x,\theta,\delta(x),0,\cdot\big). \end{aligned}$$

The invariance also allows us to obtain a more general identity along this solution $u(x)$ is

$$U^u\big(r,\theta,\delta(0),x+x',\cdot\big) = u(x+x',\cdot) - u_{\rm wk}^u(r+x+x',\cdot+\theta) = U^u\big(r+x,\theta,\delta(x),x',\cdot\big).$$

From the definition of $\mathcal{G}_k$, one may compute, for $|n| \leq k-1$,

$$\delta_n(x) = i\Pi_n\big[\big(I+\nu_n^{-1}\partial_{x'}\big)U^u\big(r+x,\theta,\delta(x),x',\cdot\big)\big]\big|_{x'=0} = i\Pi_n\big[\big(I+\nu_n^{-1}\partial_{x'}\big)U^u\big(r,\theta,\delta(0),x+x',\cdot\big)\big]\big|_{x'=0}.$$

Therefore, differentiating this identity and using (10.7),

$$\partial_x\delta_n(x) = \nu_n\delta_n(x) + i\nu_n^{-1}\Pi_n[\mathcal{F}_k(U^u)]\big|_{(r,\theta,\delta(0),x)} = \nu_n\delta_n(x) + i\nu_n^{-1}\Pi_n[\mathcal{F}_k(U^u)]\big|_{(r+x,\theta,\delta(x),0)}.$$

Letting $x=0$, we obtain the estimate on $\partial_x\delta$ straightforwardly and complete the proof of the proposition. □

The following corollary is direct consequence of the proposition and (10.6).

Corollary 10.3. *For any $y_0 \geq 0$, there exist $\varepsilon_0, \rho_1, M > 0$ independent of k and ω, and unique mappings $\zeta^\star$, $\star = u, s$, for any $\varepsilon \in (0,\varepsilon_0)$ such that the results in Proposition 10.1 along with $\|u_{\rm wk}^\star\|_{C^0_{y_0}} \leq M\frac{\varepsilon}{\sqrt{k}}$ hold for $r \in \big[-\frac{y_0}{\varepsilon\sqrt{k\omega}}, +\infty\big]$, if $\star = s$ and $r \in \big[-\infty, \frac{y_0}{\varepsilon\sqrt{k\omega}}\big]$ if $\star = u$.*

10.2. **Small homoclinic solutions.** We first show that, regarding small breathers, $u_{\rm wk}^\star$, $\star = u, s$, is the only object that matters.

Proposition 10.4. *There exist $\varepsilon_0 > 0, \rho_2 > 0$ independent of $k \geq 1$ and ω, such that, for any $\varepsilon \in (0,\varepsilon_0)$ and ω given in (1.13), a 2π-periodic-in-τ solution $u(x,\tau)$ to (1.17) exists satisfying*

$$\sup_{x\in\mathbb{R}}\|u(x,\cdot)\|_{\ell_1} \leq \tfrac{\rho_2}{\sqrt{k}}, \quad \textit{and} \quad \lim_{x\to-\infty}\|u(x,\cdot)\|_{H^1_\tau} + \|u(x,\cdot)\|_{L^2_\tau} = 0,$$

if and only if

$$\sup_{x\in\mathbb{R}}\|u_{\rm wk}^u(x,\cdot)\|_{\ell_1} \leq \tfrac{\rho_2}{\sqrt{k}} \quad \textit{and } u(x,\tau) = u_{\rm wk}^u(x+r,\tau+\theta)$$

for some $r,\theta \in \mathbb{R}$. Similarly $u(x,\tau) = u_{\rm wk}^s(x+x_0,\tau+\theta)$ for some $x_0,\theta \in \mathbb{R}$ if instead the above limit holds as $x \to +\infty$.

Proof. The "$\Longleftarrow$" direction is obvious by (10.6). We shall only consider the " $\Longrightarrow$ " direction. Let $M \geq 1, \varepsilon_0 \geq 0, \rho_1 \geq 0, \xi^\star = (\xi_1^\star, \xi_2^\star), \zeta^\star = (\zeta_1^\star, \zeta_2^\star), \Xi^\star = (\Xi_1^\star, \Xi_2^\star), \star = u, s$, be given by Proposition 10.1, and

$$\rho_2 = \min\{\rho_1, M^{-1}\}/20.$$

We shall work on the case of $\star = u$ only as the proof for the other case is verbatim. The convergence of $u(x,\tau)$ as $x \to -\infty$ implies that $u(x,\cdot) \in W_\omega^u(0)$ which is also the unstable manifold of 0 in the $\|\cdot\|_{\ell_1}$-based phase space $\mathbf{X}$. Hence, for all $x \ll -1$, there exists $r, \theta \in \mathbb{R}$ such that $u(x,\cdot) = \xi_1^u\big(r + x, \theta, \delta(x)\big)$ for some $\delta(x) \in \mathbb{C}^{2k-1}$ satisfying $\delta_{-n}(x) = -\overline{\delta_n}(x)$.

Let

$$x_1 = \sup\Big\{x \in \mathbb{R} \mid \forall x' \leq x,\ u(x',\cdot) = \xi_1^u\big(r + x', \theta, \delta(x')\big),\ \&\ |\delta(x')|_1, \|u_{\mathrm{wk}}^u\big(r + x', \cdot\big)\|_{\ell_1} \leq \tfrac{10\rho_2}{\sqrt{k}}\Big\} \leq +\infty.$$

Clearly $x_1 > -\infty$ since the image of ξ^u is a neighborhood of 0 in $W_\omega^u(0)$. The estimates on ζ^u given in Proposition 10.1 and the $\frac{2\pi}{k}$-periodicity-in-τ of $u_{\mathrm{wk}}^u(x,\tau)$ imply that, for any $x \leq x_1$,

$$\frac{\rho_2}{\sqrt{k}} \geq \Big\| \sum_{|n| \leq k-1} \Pi_n[u(x,\cdot)] \Big\|_{\ell_1} = \Big\| \Xi_1^u\big(\delta(x)\big) + \sum_{|n|\leq k-1} \Pi_n\big[\zeta_1^u\big(r + x, \theta, \delta(x)\big)\big] \Big\|_{\ell_1} \geq \frac{1}{2}|\delta(x)|_1.$$

In turn, along with Proposition 10.1 it yields

$$\|u_{\mathrm{wk}}^u(r + x, \cdot)\|_{\ell_1} = \|u(x,\cdot) - \Xi_1^u\big(\delta(x)\big) - \zeta_1^u(x + r, \theta, \delta(x))\|_{\ell_1} \leq \tfrac{\rho_2}{\sqrt{k}} + \tfrac{3}{2}|\delta(x)|_1 \leq \tfrac{4\rho_2}{\sqrt{k}}, \quad \forall x \leq x_1.$$

The definition of x_1 immediately implies $x_1 = +\infty$ and, in particular, $\|u_{\mathrm{wk}}^u(x,\cdot)\|_{\ell_1} \leq \frac{4\rho_2}{\sqrt{k}} < \frac{\rho_1}{\sqrt{k}}$ for all $x \in \mathbb{R}$. Again according to Proposition 10.1, $\xi^\star$ and $\zeta^\star$, $\star = u, s$, are well-defined near all $r \in \mathbb{R}$. Finally, from the estimate on the evolution $\partial_x \delta$ of $\delta(x)$, we have, for any $x \in \mathbb{R}$ and $|n| \leq k - 1$,

$$|\partial_x \delta_n(x) - \nu_n \delta_n(x)|_1 \leq \tfrac{20M\rho_2^2}{k\nu_{k-1}}|\delta(x)|_1 \leq 20M\rho_2^2\nu_{k-1}|\delta(x)|_1 \leq (\nu_n/2)|\delta(x)|_1.$$

Therefore

$$|\delta_n(x)|_1 \leq e^{-\frac{\nu_n}{2}N}|\delta_n(x + N)|_1 \leq 2e^{-\frac{\nu_n}{2}N}k^{-\frac{1}{2}}\rho_2,$$

which implies $\delta_n(x) = 0$ by taking $N \to +\infty$, and thus $u(x,\tau) = u_{\mathrm{wk}}^u(r + x, \tau + \theta)$. □

As an immediate corollary, there exists a small breather solution $u(x,\tau)$ to the nonlinear Klein-Gordon equation (1.17) satisfying (1.8) as $|x| \to \infty$ and $\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell_1} \leq \frac{\rho_2}{\sqrt{k}}$ iff $u_{\mathrm{wk}}^u(x+r,\tau) = u_{\mathrm{wk}}^s(x,\tau)$ for some $r \in \mathbb{R}$, namely $u_{\mathrm{wk}}^\star(x,\cdot)$, $\star = u, s$, are small and have the same orbits. The translation in τ is not needed since $u_{\mathrm{wk}}^\star$ are both primarily supported in the k-th mode if τ and odd in τ. This proves Theorem 1.3(2c).

11. The Stokes constant

We devote this section to analyze the Stokes constant C_{in} appearing in Theorems 1.3 and 1.4. As proved in Theorem 3.3, C_{in} depends on the nonlinearity $f \in \mathcal{F}_r$ analytically. In Section 11.1, we complete the proof of Theorem 1.4 by showing that $C_{\mathrm{in}} \neq 0$ in an open and dense set in $\mathcal{F}_r$. In Section 11.2, we give some more discussions on C_{in} and conjecture a formula for C_{in} in terms of a power series.

11.1. **Proof of Theorem 1.4.**

To complete the proof of Theorem 1.4, first we recall the inner equation introduced in Section 2.2

$$\partial_z^2\phi^0 - \partial_\tau^2\phi^0 - \phi^0 + \frac{1}{3}(\phi^0)^3 + f(\phi^0) = 0, \tag{11.1}$$

where f is a real-analytic odd function such that $f(u) = \mathcal{O}(u^5)$ for $|u| \ll 1$. More concretely $f \in \mathcal{F}_r$, where $\mathcal{F}_r$ is given in (1.7). Observe that $\mathcal{F}_r$ is a Banach space with the norm $\|\cdot\|_r$.

Notice that (11.1) can be rewritten as

$$\partial_\tau^2\phi_0 - \partial_z^2\phi_0 + \frac{1}{\sqrt{2}}\sin(\sqrt{2}\phi_0) + \Delta(\phi_0) = 0, \tag{11.2}$$

where Δ and f are related through

$$\Delta(u) = -\Big(\frac{1}{\sqrt{2}}\sin(\sqrt{2}u) - u + \frac{1}{3}u^3 + f(u)\Big). \tag{11.3}$$

From Theorem 3.3, equation (11.1), and therefore equation (11.2), admit two solutions $\phi^{0,\star} : D^{\star,\mathrm{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C}$, $\star = u, s$, such that

$$\lim_{|z|\to\infty} \phi^{0,\star}(z,\tau) = 0, \ \forall (z,\tau) \in D^{\star,\mathrm{in}}_{\theta,\kappa} \times \mathbb{T}.$$

In order to make explicit the dependence of these solutions on Δ, we shall denote $\phi^{0,\star}$ by $\phi^{0,\star}_{\Delta}$.

We also recall, that in Theorem 3.3, we have proved that for

$$z \in \mathcal{R}^{\mathrm{in},+}_{\theta,\kappa} = D^{u,\mathrm{in}}_{\theta,\kappa} \cap D^{s,\mathrm{in}}_{\theta,\kappa} \cap \{z;\ z \in i\mathbb{R} \text{ and } \mathrm{Im}(z) < 0\},$$

we have

$$\phi^{0,u}_{\Delta}(z,\tau) - \phi^{0,s}_{\Delta}(z,\tau) = e^{-i\mu_3 z}\left(\mathcal{C}_{\mathrm{in}}(\Delta)\sin(3\tau) + \chi(z,\tau;\Delta)\right), \quad \mu_3 = 2\sqrt{2}, \tag{11.4}$$

where we have denoted $\mathcal{C}_{\mathrm{in}}(\Delta) = C_{\mathrm{in}}(f)$, the Stokes constant, and $\chi_\Delta(z,\tau) = \chi(z,\tau;\Delta)$ are analytic functions in the variables z satisfying

$$\|\chi_\Delta\|_{\ell_1}(z), \|\partial_\tau \chi_\Delta\|_{\ell_1}(z) \le \frac{M_2}{|z|} \quad \textit{and} \quad \|\partial_z \chi_\Delta\|_{\ell_1}(z) \le \frac{M_2}{|z|^2},\ \forall z \in \mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}.$$

For $\Delta \equiv 0$, which corresponds to the sine-Gordon equation, from the explicit formula (1.1) and the asymptotics (3.11) of $\phi^{0,\star}$, a direct computation allows us to verify that the inner equation (11.2) admits the solutions

$$\phi^{0,u}(z,\tau) = \phi^{0,s}(z,\tau) = \phi^0_{\mathrm{b}}(z,\tau) = \frac{4}{\sqrt{2}}\arctan\left(-\frac{i\sin(\tau)}{z}\right), \tag{11.5}$$

which implies that $\mathcal{C}_{\mathrm{in}}(0) = C_{\mathrm{in}}(f^{\mathrm{sg}}) = 0$, where $f^{\mathrm{sg}}(\phi_0) = -\frac{1}{\sqrt{2}}\sin(\sqrt{2}\phi_0) + \phi_0 - \frac{1}{3}(\phi_0)^3$.

To prove Theorem 1.4, we also consider the *parameterized inner equation:*

$$\partial^2_\tau \phi_0 - \partial^2_z \phi_0 + \frac{1}{\sqrt{2}}\sin(\sqrt{2}\phi_0) + \mu\Delta(\phi_0) = 0. \tag{11.6}$$

where $\mu \in \mathbb{R}$ is a parameter.

Observe that equation (11.6) corresponds to taking in equation (11.1), the μ-dependent function:

$$f(\phi_0,\mu) = -\frac{1}{\sqrt{2}}\sin(\sqrt{2}\phi_0) + \phi_0 - \frac{\phi_0^3}{3} - \mu\Delta(\phi_0).$$

As equation (11.1), with $f(\phi_0,\mu)$, depends analytically (in fact linearly) on μ, by Theorem 3.3 so do the solutions $\phi^{u,s}$ and the Stokes constant $C_{\mathrm{in}}(f(\cdot,\mu)) = \mathcal{C}_{\mathrm{in}}(\mu\Delta)$.

For a given function $\Delta \in \mathcal{F}_r$, let us denote by

$$c^\Delta_{\mathrm{in}}(\mu) = \mathcal{C}_{\mathrm{in}}(\mu\Delta)$$

which is an analytic function of μ. Consider the directional derivative

$$\begin{aligned} c^d_{\mathrm{in}} &: \mathcal{F}_r \to \mathbb{C} \\ \Delta &\mapsto c^d_{\mathrm{in}}(\Delta) = \frac{dc^\Delta_{\mathrm{in}}}{d\mu}(0). \end{aligned} \tag{11.7}$$

By the analyticity of C_{in}, $c^d_{\mathrm{in}} : \mathcal{F}_r \to \mathbb{C}$ is a bounded linear operator. We first state the following propositions.

Proposition 11.1. *For any* $\Delta \notin \ker(c^d_{\mathrm{in}})$*, the set*

$$\{\mu \in \mathbb{R} : \mathcal{C}_{\mathrm{in}}(\mu\Delta) = 0\}$$

is a discrete subset of $\mathbb{R}$*.*

This proposition follows directly from 1.) the analyticity of $\mathcal{C}_{\mathrm{in}}(\mu\Delta)$ in μ as given in Theorem 3.3(3) and 2.) $\mathcal{C}_{\mathrm{in}}(\mu\Delta)$ does not vanish identically due to the assumption $\Delta \notin \ker(c^d_{\mathrm{in}})$.

Proposition 11.2. *The operator* c^d_{in} *satisfies* $c^d_{\mathrm{in}} \neq 0$*.*

We shall prove this proposition after we show that

$$\mathcal{V} = \{\Delta \in \mathcal{F}_r,\ \mathcal{C}_{\mathrm{in}}(\Delta) \neq 0\}$$

is open and dense, which completes the proof of Theorem 1.4. In fact

- $\mathcal{V} \subset \mathcal{F}_r$ is open due to the continuity of $\mathcal{C}_{\rm in}(\Delta)$ in Δ as given in Theorem 3.3.
- $c_{\rm in}^d \neq 0$ implies that there exists $\Delta_1 \in \mathcal{F}_r \setminus \ker(c_{\rm in}^d)$. Therefore for any $\Delta \in \mathcal{F}_r$, there exists $(\mu_1, \mu_2) \in \mathbb{R}^2$ arbitrarily close to $(1,0)$ such that $\Delta + \mu_2\Delta_1 \notin \ker(c_{\rm in}^d)$ and $\mathcal{C}_{\rm in}(\mu_1(\Delta + \mu_2\Delta_1)) \neq 0$, which implies the density of $\mathcal{V} \subset \mathcal{F}_r$.
- In fact, since the real dimension of $\ker(c_{\rm in}^d)$ is at most 2, the implication of the above propositions is much stronger than that $\mathcal{V}$ is open and dense.

The rest of the section is devoted to prove Proposition 11.2. We shall first derive a formula for $c_{\rm in}^d(\Delta)$ defined in (11.7) for $\Delta \in \mathcal{F}_r$. In order to do this, we consider the parameterized inner equation (11.6) and we shall compute $c_{\rm in}^d(\Delta)$ through a *Melnikov-like analysis*. Thus, we write the solutions of equation (11.6) as

$$\phi_{\mu\Delta}^{0,\star}(z,\tau) = \phi_b^0(z,\tau) + \mu\psi_1^\star(z,\tau) + \mu^2 R^\star(z,\tau;\Delta,\mu), \quad \psi_1^\star = \left(\frac{d}{d\mu}\phi_{\mu\Delta}^{0,\star}\right)\Big|_{\mu=0}, \qquad \star = u, s, \tag{11.8}$$

where ϕ_b^0 is given in (11.5) and $R^\star(z,\tau;\Delta,\mu)$ is analytic in the variable z and also in the parameter μ. Note that the functions $\psi_1^\star$ satisfy the same estimates as $\psi^\star$ in (3.12) in Theorem 3.3. A direct computation shows that $\psi_1^\star$ satisfies the non-homogeneous linear equation

$$\partial_\tau^2\psi_1^\star - \partial_z^2\psi_1^\star + \cos(\sqrt{2}\phi_b^0)\psi_1^\star - \Delta(\phi_b^0) = 0. \tag{11.9}$$

As in the standard Melnikov analysis, each solution to the corresponding homogeneous equation – the variational equation of (11.6) around ϕ_b^0 at $\mu = 0$ – can be used to measure the splitting between ψ_1^u and ψ_1^s in a certain direction, which yields the splitting of $\phi_{\mu\Delta}^{0,u}(z,\tau)$ and $\phi_{\mu\Delta}^{0,s}(z,\tau)$ in the leading order of μ for $|\mu| \ll 1$. Next lemma gives the solutions of the variational equations around ϕ_b^0.

Lemma 11.3. *The homogeneous linear partial differential equation*

$$\partial_\tau^2\xi - \partial_z^2\xi + \cos(\sqrt{2}\phi_b^0)\xi = 0 \tag{11.10}$$

has a family of solutions given by

$$\xi_n^\pm(z,\tau) = \frac{2}{\mu_n^2}\left(\chi_n^\pm(z,\tau) - \chi_{-n}^\pm(z,\tau)\right), \tag{11.11}$$

where $n \in \mathbb{N}$, $n \geq 2$, $\mu_n = \sqrt{n^2-1}$ and, for each $l \in \mathbb{Z}$, $\chi_l^\pm$ are the functions given by

$$\begin{aligned} \chi_l^\pm(z,\tau) = \quad & e^{\pm i\mu_l z + il\tau}\left(1 - \frac{\sin^2(\tau)}{z^2}\right)^{-1} \\ & \times\left\{\pm\frac{\mu_l}{2z} - \frac{l}{2}\frac{\cos(\tau)\sin(\tau)}{z^2} - \frac{i}{4}\mu_l^2 + i\frac{(l^2+1)}{4}\frac{\sin^2(\tau)}{z^2}\right\}. \end{aligned} \tag{11.12}$$

The proof of this lemma is obtained through a direct verification. In fact, the result is a consequence of a particular case of Lemma 4 in [20].

Next proposition gives a Melnikov integral type expression of the desired function $c_{\rm in}^d(\Delta)$:

Proposition 11.4. *For any function $\Delta \in \mathcal{F}_r$, $c_{\rm in}^d(\Delta) \in \mathbb{C}$ satisfies*

$$c_{\rm in}^d(\Delta) = \frac{d}{d\mu}\left(\mathcal{C}_{\rm in}(\mu\Delta)\right)\Big|_{\mu=0} = \frac{1}{2\pi i\mu_3}\int_{-\infty}^{\infty}\int_0^{2\pi}\Delta(\phi_b^0(z+s,\tau))\xi_3^+(z+s,\tau)d\tau ds, \tag{11.13}$$

which is independent of $z \in \mathcal{R}_{\theta,\kappa}^{{\rm in},+}$, where ξ_3^+ is given in (11.11).

Proof. Consider ξ_3^+ given in (11.11). Since ψ_1^u satisfies (11.9), multiplying it by ξ_3^+, we obtain

$$\xi_3^+\left(\partial_\tau^2\psi_1^u - \partial_z^2\psi_1^u + \cos(\sqrt{2}\phi_b^0)\psi_1^u\right) = \xi_3^+\Delta(\phi_b^0). \tag{11.14}$$

Thus, for $z \in D_{\theta,\kappa}^{u,{\rm in}}$, we have

$$\int_{-\infty}^0\int_0^{2\pi}\xi_3^+\left(\partial_\tau^2\psi_1^u - \partial_z^2\psi_1^u + \cos(\sqrt{2}\phi_b^0)\psi_1^u\right)(s+z,\tau)d\tau ds = \int_{-\infty}^0\int_0^{2\pi}\xi_3^+\Delta(\phi_b^0)(s+z,\tau)d\tau ds. \tag{11.15}$$

This integrals are well defined since ψ_1^u satisfies the estimates (3.12) in Theorem 3.3. Integrating by parts with respect to τ twice and using that the functions are 2π-periodic in τ we have that

$$\int_{-\infty}^{0}\int_0^{2\pi}\xi_3^+(s+z,\tau)\partial_\tau^2\psi_1^u(s+z,\tau)d\tau ds=\int_{-\infty}^{0}\int_0^{2\pi}\psi_1^u(s+z,\tau)\partial_\tau^2\xi_3^+(s+z,\tau)d\tau ds. \tag{11.16}$$

Now, integrating by parts with respect to s twice and using the expression of ξ_3^+, we have that

$$\begin{aligned}\int_{-\infty}^{0}\int_0^{2\pi}\xi_3^+(s+z,\tau)\partial_z^2\psi_1^u(s+z,\tau)d\tau ds&=\int_{-\infty}^{0}\int_0^{2\pi}\psi_1^u(s+z,\tau)\partial_z^2\xi_3^+(s+z,\tau)d\tau ds\\&\quad-\int_0^{2\pi}\left(\psi_1^u(z,\tau)\partial_z\xi_3^+(z,\tau)-\partial_z\psi_1^u(z,\tau)\xi_3^+(z,\tau)\right)d\tau.\end{aligned} \tag{11.17}$$

Replacing (11.16),(11.17) in (11.15) and using that ξ_3^+ satisfies (11.10), we have

$$\int_{-\infty}^{0}\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\xi_3^+(s+z,\tau)d\tau ds=\int_0^{2\pi}\left[\psi_1^u(z,\tau)\partial_z\xi_3^+(z,\tau)-\partial_z\psi_1^u(z,\tau)\xi_3^+(z,\tau)\right]d\tau. \tag{11.18}$$

Analogously, if $z\in D^{s,\mathrm{in}}_{\theta,\kappa}$, we obtain

$$\int_{+\infty}^{0}\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\xi_3^+(s+z,\tau)d\tau ds=\int_0^{2\pi}\left[\psi_1^s(z,\tau)\partial_z\xi_3^+(z,\tau)-\partial_z\psi_1^s(z,\tau)\xi_3^+(z,\tau)\right]d\tau. \tag{11.19}$$

Hence, subtracting (11.18), (11.19) we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\xi_3^+(s+z,\tau)d\tau ds&=\int_0^{2\pi}(\psi_1^u-\psi_1^s)(z,\tau)\partial_z\xi_3^+(z,\tau)d\tau\\&\quad-\int_0^{2\pi}\partial_z(\psi_1^u-\psi_1^s)(z,\tau)\xi_3^+(z,\tau)d\tau.\end{aligned} \tag{11.20}$$

Recall that, if $\mu=0$, then $\mathcal{C}_{\mathrm{in}}(0)=0$ and $\chi(z,\tau;0)\equiv 0$ in (11.4). Now, using (11.8) and (11.4), expanding $\mathcal{C}_{\mathrm{in}}(\mu\Delta)$ and χ around $\mu=0$ and taking $\mu\to 0$, it follows that

$$\begin{aligned}\psi_1^u(z,\tau)-\psi_1^s(z,\tau)&=e^{-i\mu_3 z}\left(\frac{d}{d\mu}\mathcal{C}_{\mathrm{in}}(\mu\Delta)\Big|_{\mu=0}\sin(3\tau)+\partial_\mu\chi(z,\tau;0)\right)\\&=e^{-i\mu_3 z}\left(c^d_{\mathrm{in}}(\Delta)\sin(3\tau)+\mathcal{O}_{\ell_1}(z^{-1})\right).\end{aligned} \tag{11.21}$$

Since (11.11) and (11.12) yield

$$\xi_3^+(z,\tau)=e^{i\mu_3 z}\left(\sin(3\tau)+\mathcal{O}_{\ell_1}(z^{-1})\right),\quad \partial_z\xi_3^+(z,\tau)=i\mu_3e^{i\mu_3 z}\left(\sin(3\tau)+\mathcal{O}_{\ell_1}(z^{-1})\right),\quad \text{as } |z|\to\infty,$$

a straightforward computation of the right-hand side of (11.20) shows that, for each $z\in\mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}$,

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\xi_3^+(s+z,\tau)d\tau ds=2\pi i\mu_3c^d_{\mathrm{in}}(\Delta)+Q(z,\tau), \tag{11.22}$$

where $Q(z,\tau)=\mathcal{O}_{\ell_1}(z^{-1})$. Since the left-hand side of (11.22) does not depend on z (one can just make the change of variables $\sigma=s+z$), the decay of Q implies that $Q\equiv 0$, and thus (11.13) holds. □

With a formula for $c^d_{\mathrm{in}}(\Delta)$, $\Delta\in\mathcal{F}_r$, the following lemma finishes the proof of Proposition 11.2.

Lemma 11.5. *If* $\Delta_0(u)=\left(-i\tan\left(\frac{\sqrt{2}}{4}u\right)\right)^5\left(1+\tan^2\left(\frac{\sqrt{2}}{4}u\right)\right)$, *then*

$$c^d_{\mathrm{in}}(\Delta_0)=\int_{-\infty}^{\infty}\int_0^{2\pi}\Delta_0(\phi_b^0(z+s,\tau))\xi_3^+(z+s,\tau)d\tau ds=\frac{26\pi^2}{15}i.$$

Proof. First, notice that

$$\Delta_0(\phi_b^0(s+z,\tau))=-\frac{\sin^5(\tau)}{(s+z)^5}\left(1-\frac{\sin^2(\tau)}{(s+z)^2}\right),$$

and

$$\xi_3^+(s+z,\tau) = \frac{e^{i2\sqrt{2}(s+z)}}{8(s+z)^2\left(1-\dfrac{\sin^2(\tau)}{(s+z)^2}\right)}\left(4\sin(\tau)+(8(s+z)^2+i4\sqrt{2}(s+z)-5)\sin(3\tau)+\sin(5\tau)\right).$$

Therefore,

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\Delta_0(\phi_b^0(z+s,\tau))\xi_3^+(z+s,\tau)d\tau ds = \int_{-\infty}^{\infty}F(z,s)ds,$$

where

$$F(z,s) = \frac{\pi e^{i2\sqrt{2}(s+z)}}{64(s+z)^7}\left(-33+i10\sqrt{2}(s+z)+20(s+z)^2\right).$$

Recall that this integral is independent of $z \in \mathcal{R}_{\theta,\kappa}^{\mathrm{in},+}$. Since $z=-i\kappa$, for some $\kappa>0$ sufficiently big, we have that $F(-i\kappa,s)$ has a pole at $s=i\kappa$.

For $R>\kappa$ sufficiently big, consider $C_R=\{z; |z|=R,\ \Im(z)\geq 0\}$ and L_R be the line segment between the points $-R$ and R. Let $\gamma_R(\theta)=Re^{i\theta}$, $0\leq\theta\leq\pi$, be a parameterization of C_R and notice that

$$F(-i\kappa,\gamma_R(\theta)) = \frac{\pi e^{i2\sqrt{2}(Re^{i\theta}-i\kappa)}}{64(Re^{i\theta}-i\kappa)^7}\left(-33+i10\sqrt{2}(Re^{i\theta}-i\kappa)+20(Re^{i\theta}-i\kappa)^2\right),$$

which means, for some $M>0$ independent of R,

$$|F(-i\kappa,\gamma_R(\theta))| \leq M\frac{e^{-2\sqrt{2}R\sin(\theta)}}{|Re^{i\theta}-i\kappa|^5}.$$

Since $0\leq\theta\leq\pi$, we have

$$\lim_{R\to\infty}\int_{C_R}F(-i\kappa,s)ds = 0.$$

From Residue Theorem, it follows that

$$\oint_{C_R\cup L_R}F(-i\kappa,s)ds = 2\pi i\mathrm{Res}(F(-i\kappa,s),i\kappa) = \frac{26\pi^2}{15}i,$$

which implies

$$\int_{-\infty}^{\infty}F(-i\kappa,s)ds = \frac{26\pi^2}{15}i.$$

□

11.2. **A conjecture on the Stokes constant.** As stated in Theorem 1.4, generically C_{in} does not vanish. We heuristically explain this fact from points of views different compared to those given in Sections 5 and 11.1. Consider first a toy model of (11.1) near the breather (11.5) decomposed into Fourier modes in the form of (2.12). A "simplified" equation for the third mode can be taken the form

$$\gamma_3''(z)+\mu_3^2\gamma_3(z) = g(z) = \sum_{l=3}^{\infty}\frac{a_l}{z^l},$$

where we may start with the assumption that the power series on the righthand side is convergent outside a disk centered at the origin. The same proof as in Section 5 yields two solutions γ^s, γ^u such that

$$\gamma^\star(z) = \mathcal{O}(z^{-3}),\quad z\in D_{\theta,\kappa}^{\star,\mathrm{in}},\quad \star=u,s,$$

where $D_{\theta,\kappa}^{\star,\mathrm{in}}$ are the sectorial complex domains with vertex at $z=\pm\infty$ defined in (3.10). However, in general, $\gamma^{u,s}$ cannot be extended to analytic functions defined in a neighborhood of ∞. In fact, $\gamma^{u,s}$ have the same formal asymptotic expansion $\tilde{\gamma}$, as $|z|\to\infty$,

$$\gamma^{u,s}(z)\sim\tilde{\gamma}(z) = \sum_{l=3}^{\infty}\frac{\gamma_l}{z^l},\quad \gamma_l = \sum_{j=0}^{[\frac{l-3}{2}]}(-1)^j\mu_3^{-2(j+1)}\frac{(l-1)!}{(l-2j-1)!}a_{l-2j},\qquad l\geq 3,$$

where $[b]$ denotes the greatest integer no greater than b. We observe that $\tilde{\gamma}$ is generally a divergent series, but in the Gevrey-1 class, namely

$$\sup_l \left(\frac{|\gamma_l|}{l!}\right)^{\frac{1}{l}} < \infty.$$

Hence, one may expect $\gamma^u \neq \gamma^s$ in general. There are several ways to see that and to provide an algorithm to compute their difference.

Borel resummation. One possibility is using the Borel resummation method as well as the Resurgence theory of Écalle. The main idea is to consider the formal Borel transform $\hat{\gamma}$ (the inverse of the Laplace transform) term by term of the series $\tilde{\gamma}$

$$\hat{\gamma}(\xi) = \sum_{l=3}^{\infty} \frac{\gamma_l}{l!}\xi^{l-1}, \qquad l \geq 3.$$

If $\tilde{\gamma}$ is of Gevrey-1 class, then $\hat{\gamma}$ is a convergent series in a disk around $\xi = 0$ which gives an analytic function that we also denote $\hat{\gamma}$. As a first step, one can study the analytic extension of this function $\hat{\gamma}$ to the complex plane and its singularities.

In our toy model, one can easily compute the equation satisfied by $\hat{\gamma}$,

$$(\xi^2 + \mu_3^2)\hat{\gamma}(\xi) = \hat{g}(\xi)$$

where $\hat{g}(\xi) = \sum_{l=3}^{\infty} \frac{a_l}{l!}\xi^{l-1}$ is an entire function. Clearly, in this model, the only singularities of $\hat{\gamma}$ are simple poles located at $\xi = \pm\mu_3 i$.

The second step to recover the original functions is to compute the Laplace transform of the function $\hat{\gamma}$ along "rays" to infinity. The existence of singularities of $\hat{\gamma}$ makes the Laplace transforms to be different when choosing different paths according to their decay requirements, obtaining two different functions $\gamma^{s,u}$, whose difference can be computed by means of the residuum theorem. Again, using our toy model and assuming growth conditions on $\hat{\gamma}$ of the form $|\hat{\gamma}(z)| \lesssim e^{C|\xi|}$, one can recover $\gamma^{s,u}$ by choosing Laplace transforms of $\hat{\gamma}$ along positive or negative real axis

$$\gamma^s(z) = \int_0^{\infty} e^{-z\xi}\hat{\gamma}(\xi)d\xi, \text{ for } \quad \operatorname{Re} z > C,$$
$$\gamma^u(z) = \int_0^{-\infty} e^{-z\xi}\hat{\gamma}(\xi)d\xi, \text{ for } \quad \operatorname{Re} z < -C.$$

One can easily study the analytic continuation of these functions by changing the paths of integration. In this way we extend the functions to sectorial domains similar to $D^{\star,\mathrm{in}}_{\theta,\kappa}$ and study its difference. For instance, for $z = -i\kappa$, $\kappa > 0$, for γ^s, we change the path to $\Gamma^+ = \{\arg(\xi) = \theta\}$, $\theta \in (0, \pi/2)$. For γ^u, we change the path to $\Gamma^- = \{\arg(\xi) = \pi - \theta\}$. Hence, we have

$$\begin{aligned}\gamma^u(-i\kappa) - \gamma^s(-i\kappa) &= \int_{\Gamma^-} e^{i\kappa\xi}\hat{\gamma}(\xi)d\xi - \int_{\Gamma^+} e^{i\kappa\xi}\hat{\gamma}(\xi)d\xi \\ &= -2\pi i \operatorname{Res}(e^{i\kappa\xi}\hat{\gamma}(\xi), \xi = \mu_3 i) = -\pi e^{-\mu_3\kappa}\frac{\hat{g}(\mu_3 i)}{\mu_3},\end{aligned} \tag{11.23}$$

as we have assumed that $\hat{\gamma}$ has moderate growth at infinity and, since $\hat{\gamma}(\xi) = \frac{\hat{g}(\xi)}{\xi^2+\mu_3^2}$, it has a simple pole at $\xi = \mu_3 i$ with residuum $\frac{\hat{g}(\mu_3 i)}{2\mu_3 i}$.

Resurgence theory gives rigor to this argument when one construct the solutions $\phi^{0,\star}$, $\star = u, s$, for the full nonlinear equation (11.1). Roughly speaking, the constant C_{in} appears in the computation of residue of the extension of such $\hat{\gamma}$ in the singularity closest to the origin. Using these ideas, one can develop an algorithm to compute C_{in}.

A more direct approach through Perron integrals. Let us end this section by proposing another approach to illustrate $\gamma^s \neq \gamma^u$ and also giving an algorithm to compute C_{in}. We can write an integral representation of the functions γ^s, γ^u using their decay at infinity:

$$\gamma^u(z) = \frac{1}{2i\mu_3}\int_{-\infty}^{z} e^{-i\mu_3(s-z)}g(s)ds - \frac{1}{2i\mu_3}\int_{-\infty}^{z} e^{i\mu_3(s-z)}g(s)ds,$$
$$\gamma^s(z) = \frac{1}{2i\mu_3}\int_{+\infty}^{z} e^{-i\mu_3(s-z)}g(s)ds - \frac{1}{2i\mu_3}\int_{+\infty}^{z} e^{i\mu_3(s-z)}g(s)ds.$$

For $\kappa > 0$, let $B_\kappa \subset \mathbb{C}$ be the disk centered at 0 with radius κ and S be the path going from $-\infty$ to $-\kappa$ along the negative real axis, then to κ along the lower half of ∂B_κ, then to $+\infty$ along the real axis. By the Cauchy integral theorem we obtain

$$\begin{aligned}\gamma^u(-i\kappa) - \gamma^s(-i\kappa) =& \frac{1}{2i\mu_3}\int_S e^{-i\mu_3(s+i\kappa)}g(s)ds - \frac{1}{2i\mu_3}\int_S e^{i\mu_3(s+i\kappa)}g(s)ds\\ =& -\frac{e^{-\mu_3\kappa}}{2i\mu_3}\oint_{\partial B_\kappa} e^{i\mu_3 s}g(s)ds = -\pi e^{-\mu_3\kappa}\sum_{l=3}^{\infty}\frac{i^{l-1}\mu_3^{l-2}}{(l-1)!}a_l.\end{aligned}$$

The above right side are related to $\hat{g}$, the Borel transform of g, evaluated at $i\mu_3$ and gives the difference between γ^u and γ^s. In fact, this formula is exactly the same as (11.23). This means that in the derivation of the stable/unstable solutions $\phi^{0,\star}$, $\star = u, s$, through the Lyapunov-Perron approach, a nonzero splitting appears even after the first iteration.

An algorithm to compute C_{in}. As mentioned above, by the Borel-Laplace summation theory, $\phi^{0,u}$ and $\phi^{0,s}$, while analytic on their own domains and non-equal in the intersection of the domains, share the same formal series as $z \to \infty$ in suitable sectors

$$\phi^{0,\star} \sim \sum_{j=3}^{\infty}\frac{b_j}{z^j},$$

which is generally divergent, but belongs to the Gevrey-1 class, Moreover, the right hand sides F_n of (2.12) are also associated to formal series

$$F_n\left(\phi^{u,s}\right)(z) \sim \sum_{j=3}^{\infty}\frac{\beta_{n,j}}{z^j},$$

in the Gevrey-1 class. The above considerations motivate us to make the following conjecture.

Conjecture. *The constant C_{in} introduced in Theorem 1.3 can be expressed as*

$$C_{\text{in}} = -\pi\sum_{l=3}^{\infty}\frac{i^{l-1}\mu_3^{l-2}}{(l-1)!}\beta_{3,l}.$$

Even though the formula of the splitting constant C_{in} in this conjecture is still very complicated, if proved, it would give an algorithm to compute C_{in} which may be implemented by numerical computations. The proof of this conjecture is beyond this paper.

Appendix A. Proof of Proposition 4.3

Unless stated otherwise, M denotes any constant independent of κ and ε. The proof of items (1) and (2) are straightforward using that (3.1) acts on the Fourier coefficients of ξ. To prove item (3), we consider $h \in \mathcal{E}_{m,\alpha}$ and we estimate $\mathcal{G}_1(h_1)$ and $\mathcal{G}_n(h_n)$ (see (4.2) and (4.3)). For $\mathcal{G}_n$, using Lemma 5.5 in [33], one can see that

$$\|\mathcal{G}_n(h)\|_{m,\alpha} \leq \frac{M\varepsilon^2}{\lambda_n^2}\|h\|_{m,\alpha},\ n \geq 2. \tag{A.1}$$

Now we estimate $\mathcal{G}_1$ given by

$$\mathcal{G}_1(h)(y) = -\zeta_1(y)\int_0^y \zeta_2(s)h(s)ds + \zeta_2(y)\int_{-\infty}^{y}\zeta_1(s)h(s)ds.$$

First, we bound $\mathcal{G}_1(h)(y)$ for values of y in $D_\kappa^{\text{out},u} \cap \{\text{Re}(y) \leq -1\}$. Notice that the functions $\zeta_1(y), \zeta_2(y)$ given in (4.4) satisfy

$$|\zeta_1(y)| \leq \frac{M}{|\cosh(y)|} \quad \text{and} \quad |\zeta_2(y)| \leq M|\cosh(y)|, \tag{A.2}$$

for every

$$y \in D_\kappa^{\mathrm{out},u} \cap \{y \in \mathbb{C};\ |\operatorname{Im}(y)| \leq -K \operatorname{Re}(y)\} \qquad \text{where} \qquad K = \left(\tan(\beta) + \frac{\pi}{2} - \kappa\varepsilon\right).$$

The second integral in $\mathcal{G}_1$ satisfies that, for every $y \in D_\kappa^{\mathrm{out},u} \cap \{\operatorname{Re}(y) \leq -1\}$,

$$\left|\int_{-\infty}^{y} \zeta_1(s)h(s)ds\right| \leq M\|h\|_{m,\alpha} \int_{-\infty}^{0} \frac{1}{|\cosh^{m+1}(s+y)|}ds \leq M\frac{\|h\|_{m,\alpha}}{|\cosh^{m+1}(y)|}.$$

Therefore

$$\left|\zeta_2(y)\int_{-\infty}^{y} \zeta_1(s)h(s)ds\right| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh^{m}(y)|}. \tag{A.3}$$

Now, to estimate the first integral in $\mathcal{G}_1$, let y^* be the unique point in the segment of line between 0 and y such that $\operatorname{Re}(y^*) = -1$. Hence, it follows from (A.2) that,

(1) If s is in the line between 0 and y^*, then

$$|\zeta_2(s)h(s)| \leq \frac{M\|h\|_{m,\alpha}|\cosh(s)|}{|s^2+\pi^2/4|^\alpha} \leq M\|h\|_{m,\alpha}.$$

(2) If s is in the line between y^* and y, then

$$|\zeta_2(s)h(s)| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh^{m-1}(s)|}.$$

Thus since $m > 1$, using the previous estimates, we have that

$$\left|\int_{0}^{y} \zeta_2(s)h(s)ds\right| \leq \left|\int_{y^*}^{0} \zeta_2(s)h(s)ds\right| + \left|\int_{y}^{y^*} \zeta_2(s)h(s)ds\right| \leq M\|h\|_{m,\alpha},$$

and consequently

$$\left|\zeta_1(y)\int_{0}^{y} \zeta_2(s)h(s)ds\right| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh(y)|}. \tag{A.4}$$

Now, from (4.2), (A.3) and (A.4), we obtain that

$$\sup_{y \in D_\kappa^{\mathrm{out},u} \cap \{\operatorname{Re}(y) \leq -1\}} |\cosh(y)\mathcal{G}_1(h)(y)| \leq M\|h\|_{m,\alpha}. \tag{A.5}$$

For the region $D_\kappa^{\mathrm{out},u} \cap \{\operatorname{Re}(y) \geq -1\}$, we consider a new set of fundamental solutions $\{\zeta_+, \zeta_-\}$ of $\mathcal{L}_1(\zeta) = 0$ which has good properties at $\pm i\pi/2$. We rewrite the solutions $\zeta_1(y)$ and $\zeta_2(y)$ as linear combinations of $\zeta_+(y)$ and $\zeta_-(y)$ and use them to obtain a new expression of the operator $\mathcal{G}_1$. We emphasize that the operator $\mathcal{G}_1$ is already defined. We only express it in a different way.

Lemma A.1. *The functions*

$$\zeta_\pm(y) = \zeta_1(y)\int_{\pm i\frac{\pi}{2}}^{y} \frac{1}{\zeta_1^2(s)}ds = -\frac{\sqrt{2}}{4}\frac{1}{\cosh^2(y)}\left(\frac{3y\sinh(y)}{2} - \cosh(y) + \frac{1}{4}\sinh(y)\sinh(2y) \mp i\frac{3\pi}{4}\sinh(y)\right)$$

are solutions of equation $\mathcal{L}_1(\zeta) = 0$ and have the following properties.

- *The Wronksian satisfies*

$$W(\zeta_+, \zeta_-) = \zeta_+\dot{\zeta}_- - \zeta_-\dot{\zeta}_+ = -i\frac{3\pi}{16}.$$

  and therefore $\zeta_\pm$ are linearly independent.
- *They can be written as*

$$\zeta_\pm(y) = \frac{(y \mp i\pi/2)^3}{(y \pm i\pi/2)^2}\eta_\pm(y), \tag{A.6}$$

  where $\eta_\pm$ are analytic functions in $D_\kappa^{\mathrm{out},u} \cap \{\operatorname{Re}(y) \geq -1\}$ uniformly bounded (with respect to ε and κ).

- *The operator $\mathcal{G}_1$ given by (4.2) can be rewritten as*

$$\mathcal{G}_1(h) = i\frac{16}{3\pi}\left(-\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-(y)\int_0^y \zeta_+(s)h(s)ds\right) + \zeta_2(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds,$$

where ζ_1, ζ_2 are given in (4.4).

The proof of this lemma is a straightforward computation using the relation between $\zeta_\pm$ and ζ_1, ζ_2.

Using this lemma, we bound $\mathcal{G}_1(h)$ for $y \in D_\kappa^{\mathrm{out},u}$ satisfying $\mathrm{Re}(y) \geq -1$. First, notice that we can use (A.2) to see that

$$\begin{aligned}\left|\int_{-\infty}^0 \zeta_1(s)h(s)ds\right| &\leq M\|h\|_{m,\alpha}\left(\int_{-\infty}^{-1}\frac{1}{|\cosh^{m+1}(s)|}ds + \int_{-1}^0 \frac{1}{|\cosh(s)(s^2+\pi^2/4)^\alpha|}ds\right)\\ &\leq M\|h\|_{m,\alpha}.\end{aligned}$$

From the expression of $\zeta_2(y)$ in (4.4), we have that $\zeta_2(y)$ has poles of order 2 at $\pm i\pi/2 + i2k\pi$. Since $\alpha \geq 5$,

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_2(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds\right| \leq M\|h\|_{m,\alpha}.$$

Now, we use that $\alpha \geq 5$ and equation (A.6) to see that

$$\begin{aligned}\left|\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds\right| &\leq M\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\int_0^y \frac{|s+i\pi/2|^3}{|s-i\pi/2|^2}|h(s)|ds\\ &\leq M\|h\|_{m,\alpha}\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\int_0^y \frac{1}{|s+i\pi/2|^{\alpha-3}|s-i\pi/2|^{\alpha+2}}ds\\ &\leq \frac{M\|h\|_{m,\alpha}}{|y^2+\pi^2/4|^{\alpha-2}}.\end{aligned}$$

We conclude that

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds\right| \leq M\|h\|_{m,\alpha}.$$

In a similar way, we can prove that

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_-(y)\int_0^y \zeta_+(s)h(s)ds\right| \leq M\|h\|_{m,\alpha}.$$

Therefore

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\mathcal{G}_1(h)(y)\right| \leq M\|h\|_{m,\alpha}. \tag{A.7}$$

Hence, using (4.1), (A.1), (A.5) and (A.7) , one obtains item (3) of Proposition 4.3.

To prove the estimates on $\partial_\tau\mathcal{G}(h)$ and $\partial_\tau^2\mathcal{G}(h)$ it is sufficient to use (A.1) and

$$\Pi_n[\partial_\tau^2\mathcal{G}(h)] = -n^2\Pi_n[\mathcal{G}(h)].$$

Finally, for item (5), notice that

$$\partial_y \circ \mathcal{G}_n(h) = \frac{1}{2}e^{i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{-i\frac{\lambda_n}{\varepsilon}s}h(s)ds + \frac{1}{2}e^{-i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{i\frac{\lambda_n}{\varepsilon}s}h(s)ds,\ n\geq 2,$$

and thus, one can easily obtain

$$\|\partial_y \circ \mathcal{G}_n(h)\|_{m,\alpha} \leq \frac{M\varepsilon}{\lambda_n}\|h\|_{m,\alpha},\ n\geq 2.$$

The decay of $\partial_y \circ \mathcal{G}_n(h)$ for $n \geq 2$ also implies that $\partial_y \circ \mathcal{G}_n(h) = \mathcal{G}(\partial_y h)$ and thus we also have

$$\|\partial_y \circ \mathcal{G}_n(h)\|_{m,\alpha} \leq \frac{M\varepsilon^2}{\lambda_n^2}\|\partial_y h\|_{m,\alpha},\ n\geq 2.$$

For the first mode, since

$$\begin{aligned}\partial_y \circ \mathcal{G}_1(h) =& i\frac{16}{3\pi}\left(-\zeta_+'(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-'(y)\int_0^y \zeta_+(s)h(s)ds\right) + \zeta_2'(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds \\ =& -\zeta_1'(y)\int_0^y \zeta_2(s)h(s)ds + \zeta_2'(y)\int_{-\infty}^y \zeta_1(s)h(s)ds,\end{aligned}$$

one has $\|\partial_y \circ \mathcal{G}_1(h)\|_{1,\alpha-1} \leq M\|h\|_{m,\alpha}$.

Acknowledgements

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 757802). O.M.L.G. has been partially supported by the Brazilian FAPESP grants 2015/22762-5, 2016/23716-0 and 2019/01682-4, and by the Brazilian CNPq grant 438975/2018-9. T. M. S. has also been partly supported by the Spanish MINECO-FEDER Grant PGC2018-098676-B-100 (AEI/FEDER/UE) and the Catalan grant 2017SGR1049. M. G. and T. M. S. are supported by the Catalan Institution for Research and Advanced Studies via an ICREA Academia Prize 2019. C. Z. has been partially supported by the US NSF grant DMS 19000083 and DMS-2350115. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2018 semester.

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