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ADIABATIC INVARIANT OF THE HARMONIC OSCILLATOR, COMPLEX MATCHING AND RESURGENCE *

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Abstract. The linear oscillator equation with a frequency slowly dependent on time is used to test a method to compute exponentially small quantities. This work presents the matching method in the complex plane as a tool to obtain rigorously the asymptotic variation of the action of the associated Hamiltonian *beyond all orders*.

The solution in the complex plane is approximated by a series in which all terms present a singularity at the same point. Following matching techniques near this singularity one is led to an equation which does not depend on any parameter, the so-called inner equation, of a Riccati-type. This equation is studied by resurgence methods.

Key words. Adiabatic invariants, exponentially small, matching theory, resurgence theory

AMS subject classifications. 30B, 34E, 40C, 58F

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1. Introduction. We consider one degree of freedom Hamiltonian system depending on a parameter that changes slowly with time modelled by a Hamiltonian of the form (see [1])

$$H(I,\varphi,\varepsilon t) = H_0(I,\lambda(\varepsilon t)) + \varepsilon\lambda'(\varepsilon t)H_1(I,\varphi,\varepsilon t)\,,$$

where $\lambda(\varepsilon t)$ is a function with definite limits at $\pm\infty$ and such that $\lambda^{k)}(\tilde t) \to 0$ when $\tilde t \to \pm\infty$, for all $k \in \mathbf{N}$. The equations of the motion are given by

$$\text{(1.1)}\qquad \left\{\begin{array}{rcl} \dot I &=& -\varepsilon\lambda'\dfrac{\partial H_1}{\partial\varphi},\\ \dot\varphi &=& \dfrac{\partial H_0}{\partial I} + \varepsilon\lambda'\dfrac{\partial H_1}{\partial I}\,. \end{array}\right.$$

This is a quasi-integrable system in the sense that we can apply the classical averaging procedure looking for a change of variables, close to the identity in powers of ε,

$$\text{(1.2)}\qquad \left\{\begin{array}{rcl} I &=& J + \varepsilon u_1(J,\psi,t) + \varepsilon^2 u_2(J,\psi,t) + \cdots,\\ \varphi &=& \psi + \varepsilon v_1(J,\psi,t) + \varepsilon^2 v_2(J,\psi,t) + \cdots, \end{array}\right.$$

in order to eliminate the angle variables of the Hamiltonian.

If we truncate the formal series (1.2) at order n, the system obtained is of the form

$$\text{(1.3)}\qquad \left\{\begin{array}{rcl} \dot J &=& \varepsilon^n\lambda'(\varepsilon t)\ldots,\\ \dot\psi &=& \dfrac{\partial\mathcal{H}}{\partial I}(I,\varepsilon) + \varepsilon^n\ldots. \end{array}\right.$$

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Poincaré proved that even though the series (1.2) are divergent they are asymptotic. In our case that means that the actions of systems (1.1) and (1.3) satisfy

$$|I(t) - J(t) - \varepsilon u_1(J(t), \psi(t), t) - \cdots - \varepsilon^{n-1} u_{n-1}(J(t), \psi(t), t)| \leq K\varepsilon^n,$$

for all $t \in \mathbf{R}$. As a consequence, $I(t)$ is an adiabatic invariant for system (1.1), in the sense that its variation is small for a long time interval. Moreover, due to the asymptotic properties of λ it happens that $u_n(J, \psi, t) \to 0$ and $v_n(J, \psi, t) \to 0$, as $t \to \pm\infty$. Then, one can see that I and J have limits at $\pm\infty$ verifying $I(\pm\infty, \varepsilon) = J(\pm\infty, \varepsilon) + O(\varepsilon^n)$, for all $n \in \mathbf{N}$. Moreover, from (1.3) and taking into account that $\lambda(\varepsilon t)$ is bounded, one has that $J(+\infty, \varepsilon) = J(-\infty, \varepsilon) + O(\varepsilon^n)$, $\forall n \in \mathbf{N}$. Hence, it follows that

$$\Delta I(\varepsilon) := I(+\infty, \varepsilon) - I(-\infty, \varepsilon) = O(\varepsilon^n), \quad \forall n \in \mathbf{N}. \tag{1.4}$$

That is, $I(t, \varepsilon)$ is a perpetual adiabatic invariant at all orders.

Nevertheless, this discrepancy is nonzero (otherwise, system (1.1) would be integrable) but it cannot be viewed directly from the asymptotic series (1.2). The goal of this paper is to present a method to compute the asymptotic expansion of the adiabatic invariant "beyond all orders."

1.1. Matching and resurgence. In fact, we have an asymptotic development of I uniformly valid for all $t \in \mathbf{R}$ and the problem is to catch the part of I invisible in the series (1.2). Matching theory principle says that in order to see the *hidden* properties of a function defined by an asymptotic series we must go to the regions, called *boundary layers*, where these series are no longer asymptotic. Boundary layers can be found by two fundamental methods: the first one is an a priori knowledge of its location provided by heuristic arguments and the second one is to look for the singularities of the series terms.

If we follow the second method for (1.2), we see that the terms of these series do not have singularities in $\mathbf{R}$ (due to the asymptotic properties of λ) and therefore, we are led to look for the boundary layers in $\mathbf{C}$. This is the principal reason why these problems are formulated using complex numbers and why the equation requires analyticity properties. Furthermore, working with analytic functions and complex asymptotic theory gives us more chances to obtain refined results.

Among others, we use as a basic tool in this paper resurgence theory for understanding the nature of the divergence of the series. But instead of analyzing the outer expansion (1.2), we apply resurgence theory to the *inner expansion* (the series near the boundary layer) to compute $\Delta I(\varepsilon)$ given in (1.4). These techniques have been used by V. Hakim and K. Mallick in [7] to compute formally the separatrix splitting of the standard map.

In the present paper we use their aproach to compute the behavior of the adiabatic invariant for a simple oscillator

$$\ddot{x} + \phi^2(\varepsilon\tau)x = 0, \tag{1.5}$$

obtaining rigorously an asymptotic expression for the adiabatic invariant $\Delta I(\varepsilon)$ *beyond all orders.* This problem is quite well understood [3] but we think useful and clarifying to treat it joining matching techniques and the resurgence theory. We have followed closely Wasow [18] and Meyer [12] formulation reducing (1.5) to a Riccati equation.

1.2. Wasow formulation and reduction to a Riccati equation. Following [18] and taking $t = \varepsilon\tau$ in (1.5), let us consider the equation

$$\varepsilon^2 \ddot{u} + \phi^2(t) u = 0, \quad t \in \mathbf{R} \tag{1.6}$$

where $\phi(t)$ satisfies
H1 $\phi(t) > 0$, $\forall t \in \mathbf{R}$,
H2 $\lim_{t\to\pm\infty} \phi(t) = \phi_\pm > 0$,
H3 $\phi \in \mathcal{C}^\infty(\mathbf{R})$ and $\phi^{k)} \in L_1((-\infty, +\infty))$, $k \in \mathbf{N}$, (i.e., $\dot{\phi}$ is a gentle function).
Now, given any solution $u(t,\varepsilon)$ of (1.6), let us denote by $I(t,\varepsilon)$ the function

$$I^2(t,\varepsilon) := \phi(t) u^2(t,\varepsilon) + \varepsilon^2 \frac{\dot{u}^2(t,\varepsilon)}{\phi(t)}$$

(when ϕ is a constant, $I(t,\varepsilon)$ is the action variable of the integrable Hamiltonian system associated to (1.6)). Littlewood proved in [9] that for any solution $u(t,\varepsilon)$ the limits $I(\pm\infty,\varepsilon)$ exist, and

$$\Delta I^2(\varepsilon) = I^2(+\infty,\varepsilon) - I^2(-\infty,\varepsilon) = O(\varepsilon^n), \quad \forall n \in \mathbf{N}.$$

Moreover, Wasow proved that ΔI^2 satisfies

$$\Delta I^2(\varepsilon) = 2\varepsilon \mathrm{Re}\left[\left(\sqrt{\phi(0)}u_0 + i\frac{\varepsilon}{\sqrt{\phi(0)}}\dot{u}_0\right)^2 \hat{p}(+\infty,\varepsilon)\right](1 + O(\varepsilon)), \tag{1.7}$$

where $(u_0, \dot{u}_0)$ are the initial conditions of $u(t,\varepsilon)$, and $\hat{p}(t,\varepsilon) = e^{-(2i/\varepsilon)\int_0^t \phi(s)ds} p(t,\varepsilon)$, with $p(t,\varepsilon)$ being the solution of the Riccati equation

$$\begin{cases} \varepsilon\dot{p} = 2i\phi(t)p + \dfrac{\dot{\phi}(t)}{2\phi(t)}(1 - \varepsilon p^2), \\ p(-\infty,\varepsilon) = 0 \end{cases} \tag{1.8}$$

for all $\varepsilon > 0$. Looking for the solution as a power series in ε, one can prove Littlewood's results, but in order to obtain more acurate estimates for $\Delta I^2(\varepsilon)$ we will need to extend our problem to the complex domain for the variable t.

By the change of variable $x = \int_0^t \phi(s)ds$ (1.8) becomes

$$\varepsilon w' = 2iw + f(x)(1 - \varepsilon^2 w^2), \tag{1.9}$$

where $f(x) = \frac{\dot{\phi}(t)}{2\phi^2(t)}$. Now, due to hypotheses **H1**, **H2**, **H3**, on ϕ, it is clear that $f(x)$ is a real function with gentle properties. But in order to study the problem on $\mathbf{C}$, let us make the following extra hypotheses on f:

H4 f is real analytic in $\bar{\Gamma} - \{x_0\}$, where $x_0 \in \mathbf{C}$, such that Im $(x_0) < 0$ and $\Gamma = \{x \in \mathbf{C} : \mathrm{Im}\,(x_0) < \mathrm{Im}\,(x) \le 0\}$, and for $|x - x_0| \le 1$ one has

$$f(x) = \frac{1}{6(x - x_0)}\left[1 + \tilde{f}((x - x_0)^{2/3})\right]$$

with $\tilde{f}(u)$ being an holomorphic function such that $\tilde{f}(0) = 0$.

H5 f is $\mathbf{C}$-gentle in the sense that for all $x \in \bar{\Gamma} - \{x_0\}$ one has

$$\lim_{\mathrm{Re}\, x\to\pm\infty} \int_{C_\pm(x)} \left|f^{k)}(s)\right| ds = 0, \quad k \in \mathbf{N},$$

uniformly on x, where

$$C_+(x) = \{t \in \mathbf{C} : \mathrm{Im}\ (t) = \mathrm{Im}\ (x), \quad \mathrm{Re}\ (t) \geq \mathrm{Re}\ (x)\ \}$$

and

$$C_-(x) = \{t \in \mathbf{C} : \mathrm{Im}\ (t) = \mathrm{Im}\ (x), \quad \mathrm{Re}\ (t) \leq \mathrm{Re}\ (x)\ \}\,.$$

Although our hypotheses **H4** and **H5** of f can seem capricious, they are deduced from the more natural hypotheses on ϕ made by Wasow in [19], namely, ϕ^2 has an analytic continuation to the complex domain and has a simple zero in $\mathbf{C}$ noted t_0, with Im $(t_0) < 0$, such that $x_0 = \int_0^{t_0} \phi(s)ds$ (the case Im $(t_0) > 0$ can be studied in an analogous way).

The aim of this paper is to compute $w(+\infty, \varepsilon)$, where $w(x, \varepsilon)$ is the solution of the Riccati equation (1.9) such that $w(-\infty, \varepsilon) = 0$. The rest of this paper is structured as follows: First of all, in section 2 we seek for $w(x, \varepsilon)$ as a power series in ε, for complex values of x. We will study its asymptotic validity until some neighborhood of the singularity x_0 which is called the *inner region*. As is usual in matching methods, in the inner region a change of variables will be needed in order to enlarge the validity of the solution. This is done in section 3, obtaining as a first approximation in this region the solution of the so-called *inner equation*. This inner equation is studied by the help of resurgence theory in a self-contained way in section 5. In the inner region we can catch some terms of our solution hidden in the power series, and in section 4 we prove that they are going to be exponentially small on ε (but not zero!) at $+\infty$. Finally, in section 6 we make some remarks for more general nonlinear inner equations. We defer for another paper the general study of (1.1) in a Hamiltonian form (see [16]). Recently, Ramis and Schäfke [14] have obtained upper bounds for $\Delta I(\varepsilon)$ showing the Gevrey-1 character (see footnote 1 in section 5) of the series (1.2) in the general case.

All of this is summarized in the following theorem.

THEOREM 1.1 (main theorem). *Let $w(x, \varepsilon)$ be the solution of the Riccati equation* (1.9) *such that* $\lim w(x, \varepsilon) = 0$ *when $x \to -\infty$. Then, if hypotheses* **H1**,...,**H5** *are satisfied one has*

$$\lim_{x \to +\infty} \hat{w}(x, \varepsilon) = -\frac{i}{\varepsilon} e^{-2ix_0/\varepsilon}(1 + O(\varepsilon^{2\gamma/3}))$$

where $\hat{w}(x, \varepsilon) := e^{-2ix/\varepsilon} w(x, \varepsilon)$ and γ is any number verifying $0 < \gamma < 1/2$ *. Moreover, the variation of the action of the Hamiltonian system associated to* (1.6) *is given by*

$$\Delta I^2(\varepsilon) = -2\phi(0) u_0^2 e^{\frac{2\mathrm{Im}\ (x_0)}{\varepsilon}} \sin\left(\frac{2\mathrm{Re}\ (x_0)}{\varepsilon}\right)(1 + O(\varepsilon^{2\gamma/3}));$$

therefore, it is a quantity exponentially small in ε.

2. The solution in the outer left domain. In this section we prove the existence of the solution $w(x, \varepsilon)$ of the Riccati equation (1.9),

$$\varepsilon w' = 2iw + f(x)(1 - \varepsilon^2 w^2)\,,$$

such that $\lim w(x, \varepsilon) = 0$, for Re $(x) \to -\infty$ and $x \in \Gamma$ (where Γ is defined in hypothesis **H4**), and we give an asymptotic expression of the solution in a suitable subdomain of Γ.

First of all, we look for a formal solution of (1.9) in the following proposition.

PROPOSITION 2.1. *There exists a series* $\sum_{n\geq 0} \varepsilon^n w_n(x)$ *that formally satisfies the Riccati equation* (1.9). *The functions* $w_n(x)$,

i. *verify the recurrence*

$$\left\{ \begin{array}{rcl} w_0(x) & = & \dfrac{-f(x)}{2i} \\ w_1(x) & = & \dfrac{w_0'(x)}{2i} \\ w_n(x) & = & \dfrac{w_{n-1}'(x)}{2i} + \dfrac{f(x)}{2i} \sum_{i+j=n-2} w_i(x) w_j(x), \quad n > 1; \end{array} \right. \tag{2.1}$$

ii. *are* **C**-*gentle functions (see hypothesis* **H5***);*

iii. *are analytic functions in* $\bar{\Gamma} - \{x_0\}$ *with a singularity at* $x = x_0$ *such that*

$$|w_n^{k)}(x)| \leq C_{n,k} \; |x - x_0|^{-(n+k+1)} \;, \quad \text{if } |x - x_0| \leq 1,\ k \in \mathbf{N}. \tag{2.2}$$

Remark. Due to the fact that $w_n(x)$ are **C**-gentle functions uniformly bounded for $|x - x_0| \geq 1$, we can choose the constants $C_{n,k}$ such that

$$|w_n^{k)}(x)| \leq C_{n,k} \;, \quad \text{if } |x - x_0| \geq 1,\ k \in \mathbf{N}\,. \tag{2.3}$$

Proof. The recurrence is obtained directly by the substitution of the series into (1.9) and the properties of $w_n(x)$ follow from hypotheses **H4** and **H5** on $f(x)$. □

Now we will prove that if we are not close to the singularity x_0, the formal series of Proposition 2.1 is asymptotic to a **C**-gentle function $\hat{w}(x,\varepsilon)$. Unfortunately, $\hat{w}(x,\varepsilon)$ will not be a solution of (1.9) but, nevertheless, it will help us to prove the existence of the solution of (1.9) and its asymptoticity to the formal series.

Let $\Gamma_{\varepsilon^\gamma}$ be the following subdomain of Γ for a suitable $\gamma > 0$.

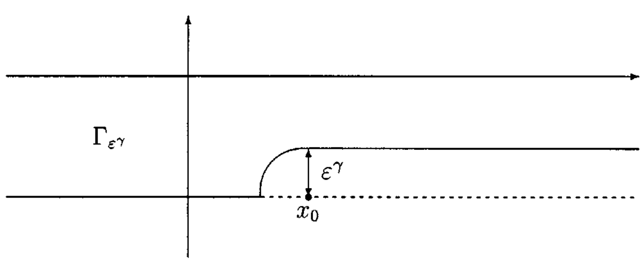

FIG. 1.

PROPOSITION 2.2. *Let* $w_n(x)$, $n \geq 0$, *be functions defined in* Γ *verifying* ii) *and* iii) *of Proposition* 2.1. *Then, for* $0 \leq \gamma < 1$ *and* $\varepsilon > 0$ *sufficiently small, there exists an analytic function* $\hat{w}(x,\varepsilon)$ *defined for* $x \in \Gamma_{\varepsilon^\gamma}$, *such that*

i. *for any* $\delta > 0$ *and* $k \in \mathbf{N}$,

$$|\hat{w}^{k)}(x,\varepsilon) - \sum_{n=0}^{N} \varepsilon^n w_n^{k)}(x)| \leq \hat{C}_{N,k} \varepsilon^{-(\gamma+\delta)} \varepsilon^{(N+1)(1-\gamma)-\gamma k}\,,$$

for all $x \in \Gamma_{\varepsilon^\gamma}$, *where* $\hat{C}_{N,k}$ *are constants independent of* ε *and* δ,

ii. $\hat{w}(x,\varepsilon)$ *is a* $\mathbf{C}$*-gentle function for* $\mathrm{Im}\,(x) \geq \mathrm{Im}\,(x_0) - \varepsilon^\gamma$.

Remark. Let us note that if $\gamma = 0$ (this means that we are far away from the singularity) this proposition says that the series $\sum_{n\geq 0} \varepsilon^n w_n(x)$ is asymptotic to the function $\hat{w}(x,\varepsilon)$ on Γ_1. In this sense we can look at i) as a weak form of asymptotic expansion near the singularity.

Proof. First of all, let us define

$$K_n(\varepsilon) := \max_{0\leq k\leq n} \left\{ \sup_{\Gamma_{\varepsilon^\gamma}} |w_n^{k)}(x)|, \sup_{\Gamma_{\varepsilon^\gamma}} \int_{C_\pm(x)} |w_n^{k)}(s)ds| \right\}.$$

From bounds (2.2) and (2.3), as $x \in \Gamma_{\varepsilon^\gamma}$ it follows that

$$K_n(\varepsilon) \leq C_n \varepsilon^{-\gamma(2n+1)}\,,$$

where $C_n := \max\{C_{n,k} \,:\, 0 \leq k \leq n\}$ are independent of ε. Secondly, let us define, for any $\delta > 0$,

$$\alpha_n(\varepsilon) := 1 - e^{-1/(\varepsilon^\delta C_n)},$$

and note that $\alpha_n(\varepsilon) < \frac{1}{\varepsilon^\delta C_n}$.

Then, let us consider

$$\hat{\omega}^{k)}(x,\varepsilon) = \sum_{n\geq 0} \alpha_n(\varepsilon) w_n^{k)}(x)\varepsilon^n,$$

and let us define

$$L_k := \max\left\{1; \frac{C_{0,k}}{C_0}; \ldots; \frac{C_{k-1,k}}{C_{k-1}}; C_0; \ldots; C_k; C_{0,k}; \ldots; C_{k,k}\right\}.$$

Using bounds (2.2) and (2.3) it follows that

$$|\alpha_n(\varepsilon) w_n^{k)}(x)| \leq \frac{C_{n,k}}{C_n} \varepsilon^{-(\delta+\gamma)} \varepsilon^{-\gamma(n+k)}.$$

Thus, for any $k \geq 0$ and $n \geq 0$, we have that

$$|\alpha_n(\varepsilon) w_n^{k)}(x)| \leq L_k \varepsilon^{-(\gamma+\delta)} \varepsilon^{-\gamma(n+k)}. \tag{2.4}$$

So, from (2.4) and taking ε small enough, we obtain that

$$|\hat{w}^{k)}(x,\varepsilon)| \leq \sum_{n=0}^{\infty} |\alpha_n(\varepsilon) w_n^{k)}(x)\varepsilon^n| \leq 2L_k \varepsilon^{-\gamma-\delta} \varepsilon^{-k\gamma}\,, \tag{2.5}$$

and thus, that $\hat{w}^{k)}(x,\varepsilon)$ converges uniformly in $\Gamma_{\varepsilon^\gamma}$, for $0 \leq \gamma < 1$, $k \in \mathbf{N}$, and ε sufficiently small. Furthermore, if we define $\hat{w}(x,\varepsilon) := \hat{w}^{0)}(x,\varepsilon)$, we have that $\hat{w}^{k)}(x,\varepsilon)$ are the k-derivatives of $\hat{w}(x,\varepsilon)$.

Now, in order to see i) let us take $N > 0$ and let us again use the bounds (2.2), (2.3), and (2.4). It follows

$$|\hat{w}^{k)}(x,\varepsilon) - \sum_{n=0}^{N} w_n^{k)}(x)\varepsilon^n| = |\hat{w}^{k)}(x,\varepsilon) - \sum_{n=0}^{N} \alpha_n(\varepsilon) w_n^{k)}(x)\varepsilon^n + \sum_{n=0}^{N} (\alpha_n(\varepsilon) - 1) w_n^{k)}(x)\varepsilon^n|$$

$$\leq \sum_{n=N+1}^{\infty} |\alpha_n(\varepsilon) w_n^{k)}(x)\varepsilon^n| + \sum_{n=0}^{N} |(\alpha_n(\varepsilon)-1) w_n^{k)}(x)\varepsilon^n|$$
$$\leq L_k \sum_{n=N+1}^{\infty} \varepsilon^{n-\gamma(n+k+1)-\delta} + e^{-1/(L_N\varepsilon^\delta)} L_N \sum_{n=0}^{N} \varepsilon^{n-\gamma(n+k+1)}$$
$$\leq \hat{C}_{N,k}\varepsilon^{-(\delta+\gamma)}\varepsilon^{(N+1)(1-\gamma)-\gamma k}.$$

(we have used that $e^{-1/(L_N\varepsilon^\delta)}$ is exponentially small in ε).

By an analogous argument, using the integrals of $\hat{w}(x,\varepsilon)$ on $C_\pm(x)$, we can prove ii). $\square$

Finally, the following theorem proves the existence of the solution $w(x,\varepsilon)$ and give us estimates on its domain of definition. In this domain we will also prove that the series of Proposition 2.1 is weakly asymptotic to $w(x,\varepsilon)$.

THEOREM 2.3. *Let us take* $0<\delta<1$ *and* $0\leq\gamma<1-\delta$. *Then, if* $\varepsilon>0$ *is small enough, the Riccati equation* (1.9) *defined for* $x\in\Gamma_{\varepsilon^\gamma}$, *has a unique solution* $w(x,\varepsilon)$ *such that* $\lim w(x,\varepsilon)=0$, *when* $\mathrm{Re}\,(x)\to-\infty$. *Furthermore, the solution* $w(x,\varepsilon)$ *satisfies that*

$$|w^{k)}(x,\varepsilon) - \sum_{n=0}^{N}\varepsilon^n w_n^{k)}(x)| \leq K_{N,k}\ \varepsilon^{-(\gamma+\delta)}\varepsilon^{(N+1)(1-\gamma)-\gamma k}, \tag{2.6}$$

for all $x\in\Gamma_{\varepsilon^\gamma}$ *and* $k,N\in\mathbf{N}$, *where the* $K_{N,k}$ *are constants independent of* ε *and* δ.

Proof. Let us take $w(x,\varepsilon)=\hat{w}(x,\varepsilon)+Q(x,\varepsilon)$, where $\hat{w}(x,\varepsilon)$ is the gentle function obtained in Proposition 2.2. Then, $w(x,\varepsilon)$ will be the solution of (1.9) if $Q(x,\varepsilon)$ is the solution of the equation

$$\varepsilon Q' = 2iQ - f(x)\varepsilon^2(\hat{w}Q - Q^2) + q(x,\varepsilon)\,, \tag{2.7}$$

where $q(x,\varepsilon):=2i\hat{w}+f(x)(1-\varepsilon^2\hat{w}^2)-\varepsilon\hat{w}'$ is an analytic $\mathbf{C}$-gentle function in $\Gamma_{\varepsilon^\gamma}$ such that $q(x,\varepsilon)=O(\varepsilon^n)$, for all $n\in\mathbf{N}$ and for all $x\in\Gamma_{\varepsilon^\gamma}$. Let us note that this implies that q verifies that, for any $n\in\mathbf{N}$

$$\int_{C_-(x)} \left|\frac{q(s,\varepsilon)}{\varepsilon}\right| \leq K_n\varepsilon^n \tag{2.8}$$

for some constant K_n.

Let us now consider the operator

$$T(Q) = \int_{C_-(x)} e^{2i(x-s)/\varepsilon}\left(\frac{q(s,\varepsilon)}{\varepsilon} - f(s)\varepsilon\left[\hat{w}(s,\varepsilon)Q(s,\varepsilon) - Q^2(s,\varepsilon)\right]\right)ds$$

defined in the Banach space of continuous bounded functions, with the supremum norm. Then, using bounds (2.5) and (2.8), and hypothesis **H4** we have, for ε small enough, that

1) if $||Q||\leq 1$,

$$||T(Q)|| \leq K_n\varepsilon^n + \varepsilon\ln\varepsilon^\gamma(2L_0\varepsilon^{-(\gamma+\delta)}+1) \leq 1,$$

2) if $||Q_i|| \leq 1$, for $i = 1, 2$,

$$||T(Q_1) - T(Q_2)|| \leq ||Q_1 - Q_2||\varepsilon(||\hat{w}|| + 2) \int_{C_-(x)} |f(s)| ds$$
$$\leq ||Q_1 - Q_2||\varepsilon(2L_0\varepsilon^{-(\gamma+\delta)} + 2)|\ln \varepsilon^\gamma| \leq \frac{1}{2}||Q_1 - Q_2||.$$

So, by the fixed-point theorem the integral equation $T(Q) = Q$, and thus the differential equation (2.7) have a unique solution Q. Moreover, using again (2.8), one has, for any $n \in \mathbf{N}$

$$||Q|| = ||T(Q)|| \leq 2||T(0)|| \leq 2 \int_{C_-(x)} \left| \frac{q(s,\varepsilon)}{\varepsilon} \right| ds \leq 2K_n\varepsilon^n .$$

Finally, using that $Q(x,\varepsilon) = w(x,\varepsilon) - \hat{w}(x,\varepsilon)$ and the bound of $\hat{w}(x,\varepsilon)$ given in Proposition 2.2 we obtain the desired result. To obtain the bounds for the derivatives $w^{k)}(x,\varepsilon)$ we only have to use (2.7) to see that all the derivatives of Q are asymptotic to zero. □

Unfortunately, with Theorem 2.3 we have proved that $\lim w(x,\varepsilon) = O(\varepsilon^n)$ when Re $(x) \to +\infty$, for all $n \in \mathbf{N}$, but we cannot obtain a more refined description of it at infinity. So, if we want to obtain an asymptotic expression for this limit, we will need to study the solution near the singularity $x = x_0$ of $w_n(x)$. In order to simplify the exposition, we will assume from now on $0 < \gamma < 1/2$.

3. The solution in the inner domain. The goal of this section is to obtain an asymptotic representation of $w(x,\varepsilon)$ near the singularity $x = x_0$ of $w_n(x)$. Of course, we cannot obtain it at $x = x_0$ but as we will see in section 4 it will be sufficient to work at a distance of order ε of this singularity. So, we will extend $w(x,\varepsilon)$ of Theorem 2.3 from a point x^* such that $|x - x^*| = \varepsilon^\gamma$, Im $(x^*) \geq$ Im $(x_0) + \varepsilon$, and Re $(x^*) \leq$ Re (x_0) (i.e., x^* belongs to the boundary of the left domain) up to the point $\tilde{x}^*$ symmetric of x^* with respect to the line $\{\text{Re }(x) = \text{Re }(x_0)\}$. From $\tilde{x}^*$ we will continue the solution in the next section.

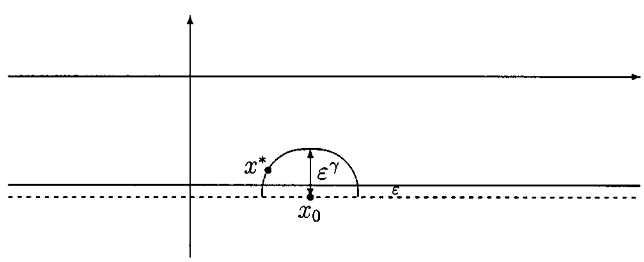

FIG. 2.

Note that, taking into account the bound (2.6), for $N = 0$, the asymptotic expression of f given by hypothesis **H4** and that $0 < \gamma < 1/2$, the initial condition of $w(x,\varepsilon)$ in the inner domain verifies

$$|w(x^*,\varepsilon) - \frac{1}{12i(x^* - x_0)}| \leq \bar{K}\varepsilon^{-\gamma}(\varepsilon^{2/3\gamma} + \varepsilon^{1-\gamma-\delta}) \leq K^*\varepsilon^{-\gamma/3},$$

where $\bar{K}$ and K^* are constants independent of ε.

Then, if we consider the change of variable and function

$$\tau = \frac{x - x_0}{\varepsilon} \qquad p(\tau, \varepsilon) = \varepsilon w(x_0 + \varepsilon\tau, \varepsilon)$$

(1.9) is transformed into

$$p' = 2ip + \varepsilon f(x_0 + \varepsilon\tau)(1 - p^2) \tag{3.1}$$

and defining $\tau^* = \frac{x^* - x_0}{\varepsilon}$, the initial condition for $p(\tau, \varepsilon)$ in the inner domain must verify

$$\left| p(\tau^*, \varepsilon) - \frac{1}{12i\tau^*} \right| \leq K^* \varepsilon^{1-\gamma/3}. \tag{3.2}$$

So, we have to study the solution of (3.1) verifying (3.2) for $\tau \in \mathbf{C}$ such that Im $\tau \geq 1$ and Re $(\tau^*) <$ Re $(\tau) <$ Re $(\tilde{\tau}^*)$, where $\tilde{\tau}^* = \frac{\tilde{x}^* - x_0}{\varepsilon}$ (we will establish in Theorem 3.2 the unicity of such a solution). In order to do this, we will compare $p(\tau, \varepsilon)$ with the solution of

$$p_0' = 2ip_0 + \frac{1}{6\tau}(1 - p_0^2) \tag{3.3}$$

such that $\lim p_0(\tau) = 0$ when Re $(\tau) \to -\infty$. This equation is obtained from (3.1) when ε tends to 0 where the initial condition is obtained "matching" the inner solution with the outer solution at τ^*.

It is easy to see, as in Proposition 2.1, that there exists a formal solution $\sum_{n\geq 0} a_n \tau^{-n-1}$ of (3.3). Moreover, looking at (2.1) and the behavior of f assumed in hypothesis **H4** one can see that the a_n are the principal parts of the terms of the outer series near the singularity, that is

$$w_n(x) = \frac{a_n}{(x - x_0)^{n+1}}(1 + O((x - x_0)^{2/3})).$$

In the next theorem existence, analytic properties, and asymptoticity of $p_0(\tau)$ are described. The proof, done with resurgence theory methods, is given in section 5.

THEOREM 3.1. i. *Equation* (3.3) *admits a unique solution* $p_0(\tau)$ *analytic in a sectorial neighborhood of* $-\infty$ *such that*

$$\lim_{\text{Re } \tau \to -\infty} p_0(\tau) = 0.$$

Moreover, this function is analytic in $\mathbf{C} - \mathbf{R}^+$*, and is asymptotic to the formal solution of* (3.3) *in every proper subsector of this set.*

ii. *Equation* (3.3) *admits a unique solution* $\tilde{p}_0(\tau)$ *analytic in a sectorial neighborhood of* $+\infty$ *such that*

$$\lim_{\text{Re } \tau \to +\infty} \tilde{p}_0(\tau) = 0.$$

Moreover, this function is analytic in $\mathbf{C} - \mathbf{R}^-$*, and is asymptotic to the formal solution of* (3.3) *in every proper subsector of this set.*

iii. *If* Re $\tau > 0$ *and* Im $\tau > 0$,

$$p_0(\tau) - \tilde{p}_0(\tau) = -ie^{2i\tau}(1 + O(\tau^{-1})). \tag{3.4}$$

Now, if we compare $p(\tau,\varepsilon)$ with $p_0(\tau)$ we have the following theorem.

THEOREM 3.2. *The problem* (3.1), (3.2) *has a unique solution* $p(\tau,\varepsilon)$ *defined for* $D_{\tau^*} = \{\tau \in \mathbf{C} : \mathrm{Re}\,(\tau^*) \le \mathrm{Re}\,(\tau) \le \mathrm{Re}\,(\tilde{\tau}^*)\,, \mathrm{Im}\,\tau \ge 1\}$. *Moreover,* $p(\tau,\varepsilon)$ *satisfies that*

$$|p(\tau,\varepsilon) - p_0(\tau)| \le L\varepsilon^{(2/3)\gamma}$$

for all $\tau \in D_{\tau^*}$, *where* L *is independent of* ε.

For the proof of this theorem we will need the following lemma.

LEMMA 3.3. *There exists a constant* B, *independent of* ε, *such that for* τ, τ_1, *and* $\tau_2 \in D_{\tau^*}$:

i. $|p_0(\tau)| \le B$,

ii. $|\int_{\tau_1}^{\tau_2} \frac{p_0(s)}{s} ds| \le B$,

iii. $\int_{\tau_1}^{\tau_2} |\frac{\tilde{f}((\varepsilon s)^{2/3})}{s}| ds \le B\varepsilon^{(2/3)\gamma}$, *where* $\tilde{f}$ *is defined in hypothesis* **H4**.

Proof. Let us take $p_0(\tau)$ the unique solution of Theorem 3.1, and $0 < \alpha < \pi/2$ some fixed angle. Then there exists some constant C_α such that for $\tau \in \mathbf{C}$,

a. if $|\arg(\tau)| > \alpha$,

$$\left| p_0(\tau) + \frac{1}{12i\tau} \right| \le C_\alpha \frac{1}{\tau^2} \tag{3.5}$$

(use that $p_0(\tau)$ is asymptotic to the series $\sum_{n\ge 0} a_n \tau^{-n-1}$, where $a_0 = -\frac{1}{12i}$);

b. if $-\pi + \alpha \le \arg(\tau) \le \pi - \alpha$,

$$\left| \tilde{p}_0(\tau) + \frac{1}{12i\tau} \right| \le C_\alpha \frac{1}{\tau^2}$$

(use the same argument as before for $\tilde{p}_0(\tau)$);

c. if $|\arg(\tau)| < \alpha$,

$$|p_0(\tau) + ie^{2i\tau} - \tilde{p}_0(\tau)| \le C_\alpha \frac{1}{\tau}$$

(use (3.4)).

From these inequalities i) follows inmediately. In order to prove ii) we only need to integrate by parts and show that $\int_{\tau_1}^{\tau_2} \frac{e^{2is} ds}{s}$ is bounded for any τ_1, τ_2 in D_{τ^*}. Finally, iii) follows from hypothesis **H4** taking into account that $|\tau_i| \le \varepsilon^{\gamma - 1}$. □

Proof (of Theorem 3.2). If we consider $v := p - p_0$, the problem that we have to study is

$$\left\{ \begin{array}{rcl} v' &=& \left[2i - \dfrac{p_0(\tau)}{3\tau}(1 + \tilde{f}((\varepsilon\tau)^{2/3})) \right] v - \dfrac{1}{6\tau}[1 + \tilde{f}((\varepsilon\tau)^{2/3})] v^2 \\ && -\dfrac{1}{6\tau} \tilde{f}((\varepsilon\tau)^{2/3})(1 - p_0^2), \\ v(\tau^*,\varepsilon) &=& p(\tau^*,\varepsilon) - p_0(\tau^*). \end{array} \right. \tag{3.6}$$

Taking into account (3.2) and (3.5) (note that $|\arg \tau^*| > \alpha$) we have that

$$|v(\tau^*,\varepsilon)| \le K^* \varepsilon^{1-\gamma/3} + C_\alpha \varepsilon^{2(1-\gamma)} \le 2K^* \varepsilon^{1-\gamma/3}\,. \tag{3.7}$$

Now, let us consider the operator

$$
\begin{aligned}
T(v) = {} & e^{2i(\tau-\tau^*)}e^{-\int_{\tau^*}^{\tau}(p_0(r)/3r)(1+\tilde{f})dr}v(\tau^*,\varepsilon)\\
& -\int_{\tau^*}^{\tau}e^{2i(\tau-s)}e^{-\int_{s}^{\tau}(p_0(r)/3r)(1+\tilde{f})dr}\frac{1}{6s}v^2(s,\varepsilon)ds\\
& +\int_{\tau^*}^{\tau}e^{2i(\tau-s)}e^{-\int_{s}^{\tau}(p_0(r)/3r)(1+\tilde{f})dr}\frac{1}{6s}\tilde{f}(1-p_0^2(s))ds,
\end{aligned}
$$

defined in the Banach space of continuous functions on D_{τ^*} with the supremum norm, such that $\|v\| \le L\varepsilon^{2/3\gamma}$ with $L = \frac{1}{2}e^{2B/3}B(1+B^2)$. Taking into account the bounds (3.7), (3.3), (3.3), and (3.3) of Lemma 3.3, for ε small enough, we have that

1) if $\|v\| \le L\varepsilon^{2/3\gamma}$,

$$
\|T(v)\| \le \frac{1}{6}e^{2B/3}\left(12K^*\varepsilon^{1-\gamma}+2L^2\varepsilon^{(2/3)\gamma}\ln\varepsilon^{\gamma-1}+B(1+B^2)\right)\varepsilon^{(2/3)\gamma} \le L\varepsilon^{(2/3)\gamma};
$$

2) if $\|v_i\| \le L\varepsilon^{(2/3)\gamma}$, for $i = 1, 2$,

$$
\|T(v_1)-T(v_2)\| \le 2e^{2B/3}L\varepsilon^{(1/3)\gamma}\ln\varepsilon^{\gamma-1}\|v_1-v_2\| \le \frac{1}{2}\|v_1-v_2\|.
$$

So, by the fixed-point theorem, the integral equation $T(v) = v$ and thus the differential equation (3.6) has a unique solution. Moreover, this solution can be bounded by

$$
|v(\tau,\varepsilon)| \le L\varepsilon^{(2/3)\gamma},
$$

for $\tau \in D_{\tau^*}$.

Finally, using that $v = p - p_0$ we finish the proof of the theorem. ∎

As we have seen, Theorem 3.2 gives us a bound of the function $w(x,\varepsilon)$ on the right side of the inner domain. In fact, at the point $\tilde{x}^*$ symmetric of x^*, we have

$$
\left|w(\tilde{x}^*,\varepsilon)-\frac{1}{\varepsilon}p_0\left(\frac{\tilde{x}^*-x_0}{\varepsilon}\right)\right| \le L\varepsilon^{(2/3)\gamma-1}, \tag{3.8}
$$

which will be used in the next section.

4. The solution in the outer right domain. In this section, we will extend the solution $w(x,\varepsilon)$ from the end point $\tilde{x}^*$ of the inner domain up to $+\infty$. We will do this comparing $w(x,\varepsilon)$ with the solution $\tilde{w}(x,\varepsilon)$ of (1.9) such that $\lim\tilde{w}(x,\varepsilon) = 0$, for $\mathrm{Re}\,(x) \to +\infty$. The existence and the properties of $\tilde{w}(x,\varepsilon)$ are analogous to $w(x,\varepsilon)$ now considering x belonging to the outer right domain $\tilde{\Gamma}_{\varepsilon^\gamma}$.

All of this is summarized in the following theorem.

THEOREM 4.1. *Let us take $\delta > 0$. The Riccati equation* (1.9) *defined for $x \in \tilde{\Gamma}_{\varepsilon^\gamma}$, $0 < \gamma < 1-\delta$, and $\varepsilon > 0$ sufficiently small has a unique solution $\tilde{w}(x,\varepsilon)$ such that $\lim\tilde{w}(x,\varepsilon) = 0$ when* $\mathrm{Re}\,(x) \to +\infty$. *Furthermore, the solution $\tilde{w}(x,\varepsilon)$ satisfies that*

$$
\left|\tilde{w}^{k)}(x,\varepsilon)-\sum_{n=0}^{N}\varepsilon^n w_n^{k)}(x)\right| \le K_{N,k}\ \varepsilon^{-(\gamma+\delta)}\varepsilon^{(N+1)(1-\gamma)-\gamma k}, \quad k \in \mathbf{N},
$$

for all $x \in \tilde{\Gamma}_{\varepsilon^\gamma}$, where $K_{N,k}$ are constants independent of ε and of δ, and $w_n(x)$ are the functions given in Proposition 2.1.

Proof. It is analogous to the proof of Theorem 2.3. ∎

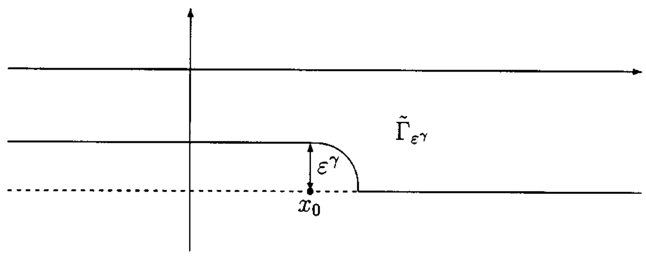

FIG. 3.

Remark. As in the previous section, in order to simplify the exposition, we will take from now on $0 < \gamma < 1/2$.

Now, let us take again $\tilde{x}^* = x_0 + \varepsilon\tilde{\tau}^*$. We want to have an estimate of $w(\tilde{x}^*, \varepsilon)$ in order to consider it as an initial condition to extend $w(x, \varepsilon)$. As we have seen in (3.8),

$$\left| w(\tilde{x}^*, \varepsilon) - \frac{1}{\varepsilon} p_0 \left(\frac{\tilde{x}^* - x_0}{\varepsilon} \right) \right| \le L\varepsilon^{(2/3)\gamma - 1},$$

and from (3.4) with $\tau = \tilde{\tau}^*$ it follows that

$$\left| p_0 \left(\frac{\tilde{x}^* - x_0}{\varepsilon} \right) + i e^{2i(\tilde{x}^* - x_0)/\varepsilon} - \tilde{p}_0 \left(\frac{\tilde{x}^* - x_0}{\varepsilon} \right) \right| \le C_\alpha \varepsilon^{1-\gamma}.$$

On the other hand, using Theorem 3.1 and 4.1, the asymptotic expression of f, and that $0 < \gamma < 1/2$ we obtain an analogous formula to (3.7), i.e.,

$$|\tilde{w}(\tilde{x}^*, \varepsilon) - \frac{1}{\varepsilon} \tilde{p}_0 \left(\frac{\tilde{x}^* - x_0}{\varepsilon} \right) | \le 2\tilde{K}^* \varepsilon^{-\gamma/3}.$$

Considering this altogether, one can obtain that

$$\left| w(\tilde{x}^*, \varepsilon) - \tilde{w}(\tilde{x}^*, \varepsilon) + \frac{i}{\varepsilon} e^{2i(\tilde{x}^* - x_0)/\varepsilon} \right| \le 3L\varepsilon^{(2/3)\gamma - 1}. \tag{4.1}$$

Now we are willing to prove the following theorem.

THEOREM 4.2. *The solution $w(x, \varepsilon)$ of (1.9) exists for $x \in \mathbf{C}$ such that* Re $(\tilde{x}^*) \le$ Re (x) *and* Im $(x_0) + \varepsilon \le$ Im $(x) \le 0$ *and verifies*

$$|w(x, \varepsilon) - \tilde{w}(x, \varepsilon) + \frac{i}{\varepsilon} e^{(2i/\varepsilon)(x - x_0)}| \le C e^{-2/\varepsilon \mathrm{Im}\ (x - x_0)} \varepsilon^{(2/3)\gamma - 1}. \tag{4.2}$$

In order to prove this theorem we need the following lemma.

LEMMA 4.3. *For $x \in \mathbf{C}$ such that* Re $(x) \ge$ Re $(\tilde{x}^*)$ *and* Im $(x) \ge$ Im $(x_0 + \varepsilon)$ *the following bounds hold:*

i. $| \int_{\tilde{x}^*}^{x} \varepsilon f(t) \tilde{w}(t, \varepsilon) dt | \le C\varepsilon^{1-\gamma}$,

ii. $\left| \int_{\tilde{x}^*}^{x} e^{-(2i/\varepsilon)(\tilde{x}^* - s) + \int_s^{\tilde{x}^*} 2\varepsilon f(t) \tilde{w}(t, \varepsilon) dt} \varepsilon f(s) ds \right| \le \varepsilon^{2-\gamma}$,

where C is a constant independent of ε.

Proof. The first bound follows inmediately from hypothesis **H4** and the asymptoticity of $\tilde{w}$ given in Theorem 4.1. For the second one it is sufficient to integrate by parts. ∎

Proof (of Theorem 4.2). Let us define $z(x,\varepsilon) := w(x,\varepsilon) - \tilde{w}(x,\varepsilon)$. From (1.9) and (4.1), $z(x,\varepsilon)$ verifies

$$\varepsilon z'(x,\varepsilon) = (2i - 2\varepsilon^2 f(x)\tilde{w}(x,\varepsilon))z(x,\varepsilon) - \varepsilon^2 f(x) z^2$$

and

$$\left| z(\tilde{x}^*,\varepsilon) + \frac{i}{\varepsilon} e^{2i(\tilde{x}^* - x_0)/\varepsilon} \right| \leq 3L\varepsilon^{2/3\gamma - 1}.$$

Thus, noting that z is the solution of a Bernoulli equation one can obtain the following integral expression for z:

$$z(x,\varepsilon) = \frac{e^{(2i/\varepsilon)(x-\tilde{x}^*) - \int_{\tilde{x}^*}^{x} 2\varepsilon f(t)\tilde{w}(t,\varepsilon)dt} z(\tilde{x}^*,\varepsilon)}{1 + z(\tilde{x}^*) \displaystyle\int_{\tilde{x}^*}^{x} e^{-(2i/\varepsilon)(\tilde{x}^* - s) + \int_{s}^{\tilde{x}^*} 2\varepsilon f(t)\tilde{w}(t,\varepsilon)dt} \varepsilon f(s) ds}. \tag{4.3}$$

Now, using the bounds given by Lemma 4.3 and (4.3), there exist some constants $\tilde{C}_i$, $i = 1,2,3$, independent of ε such that

$$|z(x,\varepsilon) - e^{(2i/\varepsilon)(x-\tilde{x}^*)} z(\tilde{x}^*,\varepsilon)| \leq \tilde{C}_1 \varepsilon^{-\gamma} |e^{(2i/\varepsilon)(x-\tilde{x}^*)}|$$

and then

$$\left| z(x,\varepsilon) + \frac{i}{\varepsilon} e^{(2i/\varepsilon)(x-x_0)} \right| \leq \tilde{C}_2 \varepsilon^{2/3\gamma - 1} |e^{(2i/\varepsilon)(x-\tilde{x}^*)}|.$$

Now, using that $\mathrm{Im}\ (\tilde{x}^*) = \mathrm{Im}\ (x_0) + \varepsilon$, we obtain

$$\left| z(x,\varepsilon) + \frac{i}{\varepsilon} e^{(2i/\varepsilon)(x-x_0)} \right| \leq \tilde{C}_3 \varepsilon^{2/3\gamma - 1} |e^{(2i/\varepsilon)(x-x_0)}| = \tilde{C}_3 \varepsilon^{2/3\gamma - 1} e^{-2/\varepsilon \mathrm{Im}\ (x-x_0)}.$$

Finally, taking into account that $z(x,\varepsilon) = w(x,\varepsilon) - \tilde{w}(x,\varepsilon)$ we obtain the theorem. ∎

Now we are in a position to prove the main Theorem 1.1 by taking the limit as $\mathrm{Re}\ (x) \to +\infty$ in inequality (4.2):

$$\left| \lim_{\mathrm{Re}\ (x) \to +\infty} e^{-(2i/\varepsilon)x} w(x,\varepsilon) + \frac{i}{\varepsilon} e^{-(2i/\varepsilon)x_0} \right| \leq e^{2\mathrm{Im}\ (x_0)/\varepsilon} \varepsilon^{(2/3)\gamma - 1}.$$

5. Resurgence of the solutions of the inner equation – Proof of Theorem 3.1.

5.1. Introduction. In this part of the article we provide a self-contained introduction to Écalle's theory of resurgent functions, and we show how our inner problem (3.3) fits within this framework.

We already know a formal solution to it:

$$\sum_{n \geq 0} a_n \tau^{-n-1},$$

and it is easy to see that there is no other formal solution. Our goal is to prove Theorem 3.1.

After the change of variable

$$z = 2i\tau,$$

(3.3) may be viewed as a particular case of singular Riccati equation of the type

$$\frac{dY}{dz} = Y + H^-(z) + H^+(z)Y^2 \tag{5.1}$$

where $H^\pm \in z^{-1}\mathbf{C}\{z^{-1}\}$ (analytic germs at infinity, vanishing at infinity); in our case, $H^-(z) = -H^+(z) = 1/6z$.

Now, resurgence is a good tool for analyzing all the equations of this kind; in fact, Écalle's theory allows the analytic classification of local equations in far more general contexts [4, 5, 2] (Of course, resurgence is not the only possible approach; see [10, 11] for another method of classifying singular local equations.), but the study of (5.1) provides a nice elementary introduction to some aspects of Écalle's work, even if many simplifications arise in the case of Riccati equations.

5.2. Singular Riccati equations and resurgence.

5.2.1. Resurgence of the formal solution. In the sequel, H^+ and H^- are two fixed analytic germs, vanishing at infinity. As (3.3), (5.1) admits a unique solution among formal expansions in negative powers of the variable; let's denote $Y_- \in z^{-1}\mathbf{C}[[z^{-1}]]$ this unique formal solution. We shall show that it is generically divergent using formal Borel transform $\mathcal{B}$.

The linear mapping $\mathcal{B}$ is defined by

$$\begin{cases} z^{-1}\mathbf{C}[[z^{-1}]] & \to & \mathbf{C}[[\zeta]], \\ z^{-n-1} & \mapsto & \zeta^n/n! \end{cases}$$

and it induces an isomorphism between the multiplicative algebra of Gevrey-1[1] formal series (without constant term) and the convolutive algebra of analytic germs at the origin $\mathbf{C}\{\zeta\}$, that is,

$$\begin{array}{rcl} \varphi_1(z) & \mapsto & \hat\varphi_1(\zeta), \\ \varphi_2(z) & \mapsto & \hat\varphi_2(\zeta), \\ \varphi_1(z)\varphi_2(z) & \mapsto & \hat\varphi_1 * \hat\varphi_2(\zeta) = \displaystyle\int_0^\zeta \varphi_1(\zeta_1)\varphi_2(\zeta-\zeta_1)d\zeta_1. \end{array}$$

Moreover, a formal series $\varphi(z)$ converges for $|z| > \rho$ if and only if its Borel transform is an entire function of exponential type at most ρ : $|\hat\varphi(\zeta)| \le \text{const } e^{\rho|\zeta|}$. Hence, finite radius of convergence for $\hat\varphi$ (i.e., existence of singularities in ζ-plane) means divergence for φ.

We shall call *resurgent function* a Gevrey-1 formal series φ whose Borel transform has the following property: on any broken line issuing from the origin, there is a finite set of points such that $\hat\varphi$ may be continued analytically along any path that closely follows the broken line in the forward direction, while circumventing (to the right or to the left) those singular points. A nontrivial fact is the stability under convolution of this property. Indeed, resurgent functions form an algebra which can be considered either as a subalgebra of $\mathbf{C}[[z^{-1}]]$ (*formal model*) or, via $\mathcal{B}$, as a subalgebra of $\mathbf{C}\{\zeta\}$

[1] Let us recall that a formal power series $\sum_{n\ge0} a_n\tau^{-n-1}$ is said to be Gevrey-1 if there exist two positive constants M, K such that $|a_n| \le Mn!K^n$.

(*convolutive model*). The Borel transform of a given resurgent function is often called its *minor*.

PROPOSITION 5.1. *The formal solution of* (5.1) *is a resurgent function, with singularities in the convolutive model over the negative integers only.*

Proof. We start by performing Borel transform on (5.1) itself; differentiation with respect to z yields multiplication by $-\zeta$ and we obtain an equation for $\hat{Y}_-$:

$$-(\zeta+1)\hat{Y}(\zeta) = \hat{H}^- + \hat{H}^+ * \hat{Y}^{*2}, \tag{5.2}$$

where $\hat{H}^+$ and $\hat{H}^-$ are some entire series with infinite radius of convergence.

Let's define inductively a sequence of $\mathbf{C}[[\zeta]]$ by

- $\hat{Y}_0(\zeta) = -\hat{H}^-(\zeta)/(\zeta+1)$,
- $\forall n \geq 1$,

$$\hat{Y}_n(\zeta) = \frac{-1}{\zeta+1}\left(\hat{H}^+ * \sum_{n_1+n_2=n-1} \hat{Y}_{n_1} * \hat{Y}_{n_2}\right).$$

The valuation of $\hat{Y}_n$ being at least $2n$, the series $\sum_{n\geq 0}\hat{Y}_n$ converges formally in $\mathbf{C}[[\zeta]]$ towards the unique solution $\hat{Y}_-$ of (5.2). Now we observe that $\hat{H}^+$ and $\hat{H}^-$ define entire functions of at most exponential growth in any direction; $\hat{Y}_0$ defines thus a meromorphic function with a simple pole at -1, and, by successive convolutions, we only get for the $\hat{Y}_n$'s other simple poles at the negative integers together with ramification (logarithmic singularities).

In particular, for each integer n, $\hat{Y}_n$ is analytic in the universal covering of $\mathbf{C}\setminus(-\mathbf{N}^*)$; with some technical but easy work, one can prove that the series of holomorphic functions $\sum \hat{Y}_n$ is uniformly convergent in every compact subset of this universal covering. Therefore, $\hat{Y}_-$ is convergent at the origin and satisfies the required property of Borel transforms of resurgent functions. □

Remark 1. The definition of a general resurgent function doesn't impose anything on the nature of singularities one may encounter in following the analytic continuation of its minor and visiting the various leaves of its Riemann surface. We shall call *simple resurgent function* a resurgent function $\varphi(z)$ whose minor $\hat{\varphi}(\zeta)$ has only singularities of the form

$$\hat{\varphi}(\omega+\zeta) = \frac{c}{2\pi i\zeta} + \hat{\psi}(\zeta)\frac{\log\zeta}{2\pi i} + \hat{R}(\zeta)$$

with $c \in \mathbf{C}$ and $\hat{\psi}, \hat{R} \in \mathbf{C}\{\zeta\}$. Simple resurgent functions form a subalgebra, which contains Y_- and all the other resurgent functions that appear in the sequel.

Remark 2. When writing in details the proof of Proposition 5.1, one obtains, in fact, exponential bounds in any sector $S_\alpha^+ = \{\zeta \in \mathbf{C}^*/-\pi+\alpha \leq \arg\zeta \leq \pi-\alpha\}$ (α being a small positive angle):

$$\forall\zeta \in S_\alpha^+,\ |\hat{Y}_-(\zeta)| \leq \text{const } e^{\rho|\zeta|},$$

where the positive number ρ depends on the radii of convergence of H^+ and H^- and on α. This allows us to apply *Laplace transform* in any direction different from the direction of $\mathbf{R}^-$.

Laplace transform in a direction θ is defined by

$$\mathcal{L}^\theta :\ \hat{\varphi}(\zeta) \mapsto \varphi^\theta(z) = \int_0^{e^{i\theta}\infty} \hat{\varphi}(\zeta)e^{-z\zeta}d\zeta.$$

When applied to an analytic function of exponential type at most ρ in direction θ, it yields a function φ^θ analytic in a half-plane bisected by the conjugate direction: Re $(ze^{i\theta}) > \rho$. If $\hat\varphi$ has at most exponential growth and no singularity in a sector of aperture α (in ζ-plane), by moving the direction of integration and using Cauchy theorem, we get a function analytic in a sectorial neighborhood of infinity of aperture $\pi+\alpha$ (in z-plane)[2]; moreover, in this neighborhood, $\varphi^\theta(z)$ is asymptotic in Gevrey-1 sense[3] to the formal series $\varphi = \mathcal{B}^{-1}\hat\varphi$ (a series which is the result of termwise application of Laplace transform to the Taylor series of $\hat\varphi$: $\mathcal{B}^{-1}$ is, in fact, the *formal Laplace transform*).

So, by choosing different values for θ, it is possible to associate to the formal series $\varphi(z)$ a family of sectorial germs $\{\varphi^\theta(z)\}$. When the series φ is convergent, the different φ^θ's yield the same analytic germ at infinity: the sum of φ. In general, the passage from φ to φ^θ through $\mathcal{L}^\theta \circ \mathcal{B}$ may be considered as a *resummation* process, since multiplication of formal series is taken to multiplication of sectorial germs, and differentiation w.r.t. z is respected too. We sum up this Borel–Laplace process in the following diagram.

$$\begin{tikzcd}
\text{Sectorial germ } \varphi^\theta & \\
& \hat\varphi \text{ (convolutive model)} \arrow[ul, "\mathcal{L}^\theta"'] \\
\text{Formal series } \varphi \arrow[uu, dashed] \arrow[ur, "\mathcal{B}"] &
\end{tikzcd}$$

FIG. 4.

Applying Laplace transform $\mathcal{L}^\theta$ to $\hat Y_-$ with $\theta \in]-\pi,\pi[$, we get an analytic function defined in a sectorial neighborhood of infinity of aperture 3π in z-plane, which is a solution of (5.1). In particular, we have two possible summations of the formal solution Y_- in the half-plane $\{\text{Re } z < 0\}$ near infinity: Y_-^θ with θ close to π, and $Y_-^{\theta'}$ with θ' close to $-\pi$. These functions correspond respectively to the solutions $p_0(\tau)$ and $\tilde p_0(\tau)$ Theorem 3.1 refers to.

The question now is to compute the difference $Y_-^{\theta'} - Y_-^\theta$; we shall do it by analyzing the singularities of the minor $\hat Y_-$.

5.2.2. Formal integral. Before that, we will study a formal object, more general than the formal solution Y_-, which solves (5.1) too: the *formal integral.* We shall see that Y_- is the first term of a sequence $(\phi_n(z))$ of simple resurgent functions such that

$$Y(z,u) = \sum_{n\geq 0} u^n e^{nz}\phi_n(z) \in \mathbf{C}[[z^{-1}, ue^z]]$$

[2]This is a subset of $\mathbf{C}$ which contains, for all $\delta \in]0,\pi+\alpha[$, a sector $\{z \in \mathbf{C}/|\arg(ze^{i\theta})| < \delta/2\,,\ |z| > \rho\}$ for some positive ρ.

[3] If $\varphi(z) = \sum \varphi_n z^{-n-1}$, this means that in every closed subsector $\bar S$ of the domain, there exist $C, K > 0$ such that:

$$\forall N \geq 1, \forall z \in \bar S,\ |z|^{N+1}|\varphi^\theta(z) - \sum_{n=0}^{N-1} \varphi_n z^{-n-1}| \leq CK^N N!.$$

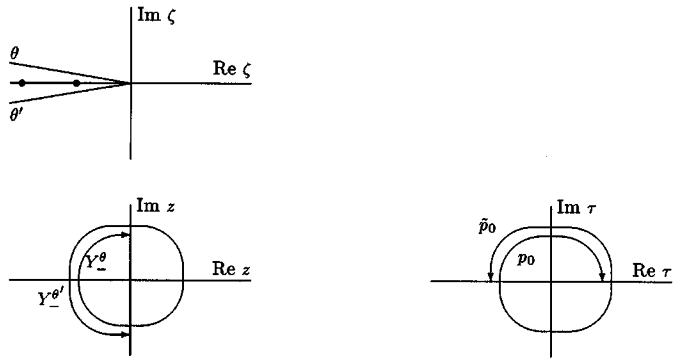

FIG. 5.

formally satisfies the equation. This means that (5.1) is formally conjugated to

$$\frac{dX}{dz} = X \tag{5.3}$$

through the formal diffeomorphism

$$Y = \Phi(z, X) = \sum_{n\geq 0} X^n \phi_n(z) \in \mathbf{C}[[z^{-1}, X]]\,.$$

Due to the fact that we deal with a Riccati equation, the formal integral admits a simple expression.

PROPOSITION 5.2. *There are formal series* $Y_+ \in z^{-1}\mathbf{C}[[z^{-1}]]$ *and* $Y_0(z) \in 1 + z^{-1}\mathbf{C}[[z^{-1}]]$ *such that*

$$Y(z, u) = \frac{ue^z Y_0(z) + Y_-(z)}{ue^z Y_0(z) Y_+(z) + 1}$$

formally solves (5.1). *Like* Y_-, *these formal series are simple resurgent functions;* [4] *their Borel transforms have singularities over* $\mathbf{Z}$ *only, and at most exponential growth at infinity.*

Proof. First we observe that our equation is equivalent to

$$-\frac{d}{dz}(1/Y) = 1/Y + H^+(z) + H^-(z)(1/Y)^2\,. \tag{5.4}$$

Thus, using the same arguments that we used for proving Proposition 5.1, we see that there is a unique formal series $Y_+ \in z^{-1}\mathbf{C}[[z^{-1}]]$ whose inverse solves (5.1), and that it is a simple resurgent function whose Borel transform has singularities over the positive integers only and at most exponential growth.

Expecting a linear fractional dependence on the free parameter, we perform the change of unknown function

$$Y = \frac{a + Y_-(z)}{aY_+(z) + 1}.$$

[4] The definition of resurgent functions can be extended to allow them to have a constant term. Being the unity of the convolution, the Borel transform of 1 may be considered as the Dirac distribution δ at $\zeta = 0$. If $\varphi = c + \psi$ is a resurgent function of constant term c, its Borel transform is $\mathcal{B}\varphi = c\delta + \mathcal{B}\psi$, but we still call minor the germ $\hat{\varphi} = \mathcal{B}\psi$. See section 5.3 for one further generalization.

It yields the equation $da/dz = a(1 + H^+Y_- + H^-Y_+)$: the general solution is $a = ue^{z+\alpha(z)}$, where α is the unique formal series without constant term of derivative $H^+Y_- + H^-Y_+$. The series α is a simple resurgent function; its Borel transform

$$\hat{\alpha}(\zeta) = -\frac{1}{\zeta}(\hat{H}^+ * \hat{Y}_- + \hat{H}^- * \hat{Y}_+)$$

has singularities over $\mathbf{Z}^*$ only and at most exponential growth. Its exponential $Y_0 = e^\alpha$ inherits this property, by general properties of exponentiation of resurgent functions ([4, 2] : Y_0 has constant term 1 and its minor $\hat{Y}_0(\zeta) = \sum_{n\geq 1} \hat{\alpha}^{*n}/n!$ is analytic in the universal covering of $\mathbf{C} \setminus \mathbf{Z}$, with no singularity at the origin on the main sheet). $\square$

So, we have $Y(z,u) = \sum_{n\geq 0} u^n e^{nz}\phi_n(z)$ with $\phi_0 = Y_-$, and for positive n,

$$\phi_n = (-1)^{n-1} Y_0^n Y_+^{n-1}(1 - Y_-Y_+)\,.$$

If we apply Laplace transform in a nonsingular direction θ, we obtain a one-parameter family of analytic solutions of (5.1):

$$Y^\theta(z,u) = \sum_{n\geq 0} u^n e^{nz}\mathcal{L}^\theta\hat{\phi}_n = \frac{ue^z Y_0^\theta(z) + Y_-^\theta(z)}{ue^z Y_0^\theta(z) Y_+^\theta(z) + 1},$$

defined for $\operatorname{Re}\,(ze^{i\theta}) - \rho > \text{const.}|ue^z|$ (a condition meant to ensure that the Laplace transforms of $\hat{Y}_0$, $\hat{Y}_-$, $\hat{Y}_+$ are defined and that the denominator in $Y^\theta(z,u)$ does not vanish).

In the convolutive model, we can apply Cauchy theorem and move the direction of integration in the upper or lower half-plane (depending on the value of θ). This provides an analytic continuation of $Y^\theta(z,u)$ allowing z to vary in a sectorial neighborhood of infinity of aperture 2π, that we call $Y^+(z,u)$ or $Y^-(z,u)$ as illustrated in the following diagram.

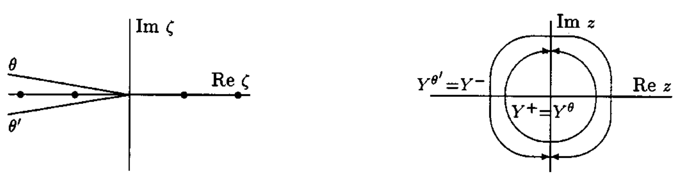

FIG. 6.

Thus, we essentially have two one-parameter families of analytic solutions of (5.1), characterized by their asymptotic behavior in the above-mentioned domains in z-plane. There must be some connection between them: a member $Y^+(.\,,u)$ of the first family has to coincide with some member $Y^-(.\,,u')$ of the other family for values of z with negative real part, and with some solution $Y^-(.\,,u'')$ for values of z with positive real part. These connection formulae will be computed in the following sections.

We are especially concerned with the functions $Y^+(z,0)$ and $Y^-(z,0)$ which correspond respectively to $p_0(\tau)$ and $\tilde{p}_0(\tau)$. At this stage, the first two statements of Theorem 3.1 are proved; the unicity assertion for the second one, for instance, is a consequence of the following easy lemma.

LEMMA 5.3. *If* $u \in \mathbf{C}^*$, $Y^-(z,u)$ *is defined for* $\operatorname{Re} z \leq 0$, $\operatorname{Im} z \geq 0$, *and* $|z|$ *big enough, and*

$$Y^-(z,u) - Y^-(z,0) = ue^z(1 + O(z^{-1})).$$

Indeed any solution of (5.1) analytic in a neighborhood of $i.\infty$ on the imaginary axis has to coincide with some function $Y^-(z,u)$; only one tends to 0 as Im z tends to infinity, and it corresponds to $u=0$.

And now we see that in order to prove the last statement of the theorem, we simply need to compute the value u_0 of the parameter such that $Y^+(z,0)=Y^-(z,u_0)$ for Re $z<0$ and to apply the lemma.

5.2.3. Alien calculus. It is essential to be able to analyze the singularities that appear in the convolutive model, for they are responsible for the divergence in the formal model. This is done by means of *alien calculus*, one of the main features of Écalle's theory, which relies on a new family of derivations: the *alien derivations*. Let's introduce them in the case of simple resurgent functions.

Let ω be in $\mathbf{C}^*$. We define an operator Δ_ω in the following way: given a simple resurgent function $\varphi(z)$, let's try to follow the analytic continuation of its minor $\hat{\varphi}(\zeta)$ along the half-line issuing from the origin and passing by ω (the minor is defined by the Borel transform of φ without taking into account the constant term if there is any); on this line, there is an ordered sequence $(\omega_1,\omega_2,\ldots)$ of singular points to be circumvented. If $r\geq 1$, we obtain in this way 2^{r-1} determinations of the minor in the segment $]\omega_{r-1},\omega_r[$ (with the convention $\omega_0=0$ if $r=1$ — in this case, there is only one determination), and we denote them by

$$\hat{\varphi}^{\varepsilon_1,\ldots,\varepsilon_{r-1}}_{\omega_1,\ldots,\omega_{r-1}},$$

each ε_i being a plus or minus sign indicating whether ω_i is circumvented to the right or to the left:

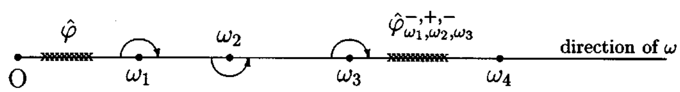

FIG. 7.

- If $\omega\notin\{\omega_1,\omega_2,\ldots\}$, we set

$$\Delta_\omega\varphi=0.$$

- If $\omega=\omega_r$ for $r\geq 1$, each of the above-mentioned determinations may have a singularity at ω:

$$\hat{\varphi}^{\varepsilon_1,\ldots,\varepsilon_{r-1}}_{\omega_1,\ldots,\omega_{r-1}}(\omega+\zeta)=\frac{c^{\varepsilon_1,\ldots,\varepsilon_{r-1}}_{\omega_1,\ldots,\omega_{r-1}}}{2\pi i\zeta}+\hat{\psi}^{\varepsilon_1,\ldots,\varepsilon_{r-1}}_{\omega_1,\ldots,\omega_{r-1}}(\zeta)\frac{\log\zeta}{2\pi i}+\text{ regular function,}$$

  and we set

$$\Delta_{\omega_r}\varphi=\sum_{\varepsilon_1,\ldots,\varepsilon_{r-1}}\frac{p(\varepsilon)!q(\varepsilon)!}{r!}\,(c^{\varepsilon_1,\ldots,\varepsilon_{r-1}}_{\omega_1,\ldots,\omega_{r-1}}+\mathcal{B}^{-1}\hat{\psi}^{\varepsilon_1,\ldots,\varepsilon_{r-1}}_{\omega_1,\ldots,\omega_{r-1}})\,,\tag{5.5}$$

  where the integers p and $q=r-1-p$ are the numbers of plus signs and of minus signs in the sequence $(\varepsilon_1,\ldots,\varepsilon_{r-1})$.

It is easy to check the consistency of this definition. In some sense, $\Delta_\omega\varphi$ is a well-balanced average of the singularities of the determinations of the minor over ω; adding or removing false singularities in the list $(\omega_1,\omega_2,\ldots)$ would not affect the result, which is a simple resurgent function (the definitions of section 5.2.1 were formulated exactly for this purpose).

The definition of operators Δ_ω for more general algebras of resurgence is given in [4, 5, 6]. In fact, these operators encode the whole singular behavior of the minor; given a sequence of points $(\omega_1, \dots, \omega_n)$ in $\mathbf{C}^*$, not necessarily on the same line, the composed operator $\Delta_{\omega_n} \cdots \Delta_{\omega_1}$ measures singularities over the point $\omega_1 + \cdots + \omega_n$.

The main property that makes these operators very useful in practice is the following: *the Δ_ω are derivations of the algebra of resurgent functions, i.e.,*

$$\forall \omega \in \mathbf{C}^*, \forall \varphi_1, \varphi_2 \text{ resurgent functions}, \ \Delta_\omega(\varphi_1\varphi_2) = (\Delta_\omega\varphi_1)\varphi_2 + \varphi_1(\Delta_\omega\varphi_2)\,.$$

By contrast with the natural derivation $\frac{d}{dz}$, they are called *alien derivations.*

Alien derivations interact with natural derivations according to the rule

$$\frac{d}{dz}\Delta_\omega\varphi = \Delta_\omega\frac{d\varphi}{dz} + \omega\Delta_\omega\varphi,$$

which reads

$$\frac{d}{dz}\dot{\Delta}_\omega\varphi = \dot{\Delta}_\omega\frac{d\varphi}{dz} \tag{5.6}$$

when one introduces the symbol $\dot{\Delta}_\omega = e^{-\omega z}\Delta_\omega$ (*pointed alien derivation*). But the $(\Delta_\omega)_{\omega\in\mathbf{C}^*}$ generate a free Lie algebra.

Finally, let's state the other property that we shall use: suppose that all the singularities of the minor of φ in a sector $\{\theta < \arg\zeta < \theta'\}$ form an ordered sequence $(\omega_1, \omega_2, \dots)$ on a half-line inside the sector

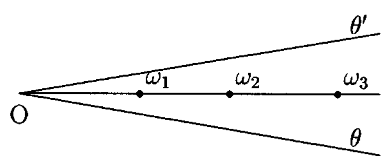

FIG. 8.

and that we can apply Borel–Laplace summation process, then

(5.7)

$$\varphi^\theta = \varphi^{\theta'} + \sum_{r\geq 1;\, i_1,\dots,i_r\geq 1} \frac{1}{r!}e^{-(\omega_{i_1}+\dots+\omega_{i_r})z}(\Delta_{\omega_{i_1}}\cdots\Delta_{\omega_{i_r}}\varphi)^{\theta'} = \left[\left(\exp\sum_{i\geq 1}\dot{\Delta}_{\omega_i}\right).\varphi\right]^{\theta'}$$

if we systematically use the notation ψ^θ for $\mathcal{L}^\theta\mathcal{B}\psi$, and $(e^{-\omega z}\psi)^\theta$ for $e^{-\omega z}\psi^\theta$.

5.2.4. Bridge equation. Coming back to our Riccati equation (5.1), let's try to compute the alien derivatives of the various simple resurgent functions that appear in the formal integral of Proposition 5.2. We shall use the generating series

$$Y(z,u) = \sum_{n\geq 0} u^n e^{nz}\phi_n(z)\,,\ \Delta_\omega Y = \sum_{n\geq 0} u^n e^{nz}\Delta_\omega\phi_n,$$

and we shall assume $\omega \in \mathbf{Z}^*$, since we already know that $\Delta_\omega Y$ vanishes if it is not the case.

Of course, it is equivalent to look for $\dot{\Delta}_\omega Y$, and it turns out that one can easily derive from (5.1) a deep relation, simply by applying $\dot{\Delta}_\omega$ to the equation itself. One obtains a linear equation for $\dot{\Delta}_\omega Y$

$$\frac{d}{dz}(\dot{\Delta}_\omega Y) = (1 + 2H^+ Y)\dot{\Delta}_\omega Y$$

(because pointed alien derivations commute with natural derivation, vanish on convergent series like $H^\pm$, and satisfy Leibniz rule), which admits $\partial Y/\partial u$ as a nontrivial solution, so that there must be some proportionality relationship

$$\dot{\Delta}_\omega Y = A_\omega(u)\frac{\partial Y}{\partial u}.$$

Simple arguments show that the coefficient $A_\omega(u)$ must be zero if $\omega \leq -2$ (because $\phi_1 \neq 0$, so the valuation of $\partial Y/\partial u$ w.r.t. e^z is exactly 1), that it is of the form $A_\omega(u) = A_\omega u^{\omega+1}$ (for homogeneity reasons), and finally that it is zero if $\omega \geq 2$ (Because one can repeat everything with (5.4); it's only here that we use the fact that (5.1) is a Riccati equation and not a more general nonlinear equation). We end up with the following proposition.

PROPOSITION 5.4. *There exist $A^-, A^+ \in \mathbf{C}$ such that*

$$\begin{aligned} \dot{\Delta}_{-1} Y &= A^- \partial\frac{Y}{\partial u}, \\ \dot{\Delta}_{+1} Y &= -A^+ u^2 \partial\frac{Y}{\partial u}, \\ \dot{\Delta}_\omega Y &= 0 \ \textit{if}\ \omega \notin \{-1, +1\}. \end{aligned}$$

So, the action of alien derivations on the formal integral is equivalent to the action of some differential operator: this important and very general result was called *bridge equation* by Écalle, since it throws a bridge between alien and ordinary calculus. When interpreted in the convolutive model, it expresses a strong link between an analytic germ at the origin and its singularities: in some ways, the germ reproduces itself at singular points, and this was the reason for naming "resurgent" such an object. Of course, with our definitions, not all resurgent functions have this property, but Écalle observed that it holds for all resurgent functions that arise "naturally" (as solutions of some analytic problem).

For instance, bridge equation holds for more general nonlinear equations, but in contrast with Riccati case, there can then be an infinity of numbers A_ω, $\omega \in \{-1, 1, 2, \dots\}$; they are called *analytic invariants* of the equation, because it can be proven that two such equations are analytically (not only formally) conjugated if and only if they have the same set of A_ω's.

Thus, in our problem, inside the class of formally conjugated equations (5.1) (they are all conjugated to (5.3)), analytic classes are parametrized by a pair of two numbers.

Alien derivatives can be computed explicitly in terms of the two analytic invariants A^- and A^+:

$$\begin{aligned} \Delta_{-1} Y_- &= A^- Y_0 (1 - Y_- Y_+), & \Delta_{+1} Y_- &= 0, \\ \Delta_{-1} Y_+ &= 0, & \Delta_{+1} Y_+ &= A^+ Y_0^{-1}(1 - Y_- Y_+), \\ \Delta_{-1} Y_0 &= -A^- Y_0^2 Y_+, & \Delta_{+1} Y_0 &= A^+ Y_-. \end{aligned}$$

In particular, $\Delta_{\pm 1} Y_\pm = A^\pm + O(z^{-1})$, which means that $A^\pm$ is the residuum of $\hat{Y}_\pm(\zeta)$ at the point ± 1. These two numbers are transcendent functions of the convergent germs H^+ and H^-; we shall see later how to compute them in special cases. The vanishing of alien derivatives at integer points other than ± 1 does not mean that there is no singularity at those points: these other singularities can be detected by iterating bridge equation.

Bridge equation may be used for other purposes than analytic classification: formula (5.7) can be justified with $\varphi = Y(\,.\,, u)$, and we are now in a position to compare $Y^-(z,u)$ and $Y^+(z,u)$, the two families of solutions of (5.1) we obtained by resummation at the end of section 5.2.2. In the sequel, we shall take various values of z with big enough modulus and appropriated values of u in order to have $Y^\pm(z,u)$ defined.

If Re $z < 0$, applying formula (5.7) with $\theta < \pi < \theta'$ (and these angles both close to π) yields

$$Y^\theta(z,u) = [\exp \dot{\Delta}_{-1}]^{\theta'}(z,u) = \left[\exp\left(A^- \frac{\partial}{\partial u}\right)\right]^{\theta'}(z,u) = Y^{\theta'}(z, u + A^-)\,,$$

that is

$$Y^+(z,u) = Y^-(z, u + A^-)\,; \tag{5.8}$$

and similarly, if Re $z > 0$, choosing $\theta < 0 < \theta'$,

$$Y^-(z,u) = Y^+\left(z, \frac{u}{1 + A^+ u}\right). \tag{5.9}$$

Let's take $u = 0$: we already knew that $Y^+(z,0)$ and $Y^-(z,0)$ coincide for Re $z > 0$ (as was noticed at the end of section 5.2.1), but now, formula (5.8) and Lemma 5.3 show that, for Re $z < 0$ and Im $z > 0$,

$$Y^+(z,0) - Y^-(z,0) = A^- e^z (1 + O(z^{-1}))\,.$$

Finally, when the original variable $\tau = -iz/2$ has positive real and imaginary parts,

$$p_0(\tau) - \tilde{p}_0(\tau) = A^- e^{2i\tau}(1 + O(\tau^{-1}))\,. \tag{5.10}$$

5.3. Computation of analytic invariants. To complete the proof of Theorem 3.1, we only need to compute the coefficient A^- associated with an equation

$$\frac{dY}{dz} = Y + \frac{1}{6z}(1 - Y^2) \tag{5.11}$$

deduced from (3.3) by the change of variable $z = 2i\tau$, and to put it inside formula (5.10).

We shall, in fact, compute the pair of analytic invariants for all equations

$$\frac{dY}{dz} = Y - \frac{1}{2\pi i z}(B^- + B^+ Y^2), \tag{5.12}$$

where $B^\pm \in \mathbf{C}$. The result is proved in the second volume of [4], but we present here an alternative method.

Proposition 5.5. *The analytic invariants of* (5.12) *are given by*

$$A^- = B^- \sigma(B^- B^+)\,,\; A^+ = -B^+ \sigma(B^- B^+),$$

where $\sigma(b) = \frac{2}{b^{1/2}} \sin \frac{b^{1/2}}{2}$. Note that this implies $A^- = -i$ in the case of (5.11), as required for ending the proof of the theorem.

Proof. Let's begin with a few simple remarks. The coefficient $A^\pm$ is the residuum at ± 1 of the Borel transform of $Y_\pm$, where $Y_-(z)$ is the unique formal solution of (5.12) and $Y_+(z)$ the unique formal solution of the equation corresponding to (5.4). It is easy to see that

$$Y_-(z) = B^- y(z)\,,\; Y_+(z) = -B^+ y(-z)\,,$$

where y is the unique formal solution of an equation depending only on $b = B^- B^+$:

$$\frac{dy}{dz} = y - \frac{1}{2\pi i z}(1 + by^2).$$

If we solve the Borel transform of this equation like we did in the proof of Proposition 5.1 (by expanding everything in powers of b), we find for the residuum of $\hat{y}(\zeta)$ at -1 an analytic function $\sigma(b)$ such that $\sigma(0) = 1$, and we obtain $A^- = B^- \sigma(b)$ and $A^+ = -B^+ \sigma(b)$.

Rather than studying the power expansion of $\sigma(b)$ (this is more or less what is done in [4, Vol. 2, pp. 476–480], but in a very efficient manner through the theory of *moulds*), we prefer to perform the change of unknown function

$$y(z) = \frac{2\pi i}{b} \cdot \frac{z q'(z)}{q(z)}\,,$$

which leads us to a second-order linear equation

$$z^2 q'' + (z - z^2) q' - \beta^2 q = 0$$

where $\beta^2 = \frac{b}{4\pi^2}$. We assume in the the sequel that $\operatorname{Re} \beta > 0$ (excluding real nonpositive values of b is innocuous since the function σ is analytic).

We exploit the peculiar form of this new equation, and write its unique formal solution with constant term 1 as the product of a monomial and expansion in fractional powers of z:

$$q(z) = z^\beta r(z) = 1 + O(z^{-1})\,,\quad r(z) \in z^{-\beta}\mathbf{C}[[z^{-1}]]. \tag{5.13}$$

The series $r(z)$ may be called resurgent if we extend the definition of Borel transform by

$$z^{-\nu} \mapsto \zeta^{\nu-1}/\Gamma(\nu)\,,\ \text{if}\ \ \nu \in \mathbf{C}\ \ \text{and}\ \ \operatorname{Re} \nu > 0,$$

and admit among resurgent functions all formal series (with possibly fractional powers) whose Borel transform, which may now be ramified at the origin and has endless analytic continuation. The convolution of minors is defined as before and we are still dealing with an algebra.

The point is that alien derivatives of r are easy to compute, for the equation it satisfies

$$r'' + (-1 + (2\beta + 1) z^{-1}) r' - \beta z^{-1} r = 0 \tag{5.14}$$

can be solved explicitly in the convolutive model.

LEMMA 5.6. *The Borel transform $\hat{r} = \mathcal{B}r$ is given by*

$$\hat{r}(\zeta) = \frac{\zeta^{\beta-1}}{\Gamma(\beta)}(1+\zeta)^{\beta}.$$

Proof. The Borel transform of (5.14)

$$(\zeta^2+\zeta)\hat{r} - (2\beta+1)1 * (\zeta\hat{r}) - \beta(1 * \hat{r}) = 0$$

is equivalent to a first-order linear equation obtained by differentiation with respect to ζ (this was the only purpose of the change of unknown function (5.13)),

$$\zeta(\zeta+1)\frac{d\hat{r}}{d\zeta} = [(2\beta-1)\zeta + \beta - 1]\hat{r}. \qquad \square$$

Alien calculus applies in this slightly generalized context. We only need to be careful about the determination of ζ^{β} we use (let's say it is the principal one) and about the sheet of the Riemann surface of the logarithm we look at. In particular, alien derivations are now indexed by points in this Riemann surface rather than by points in $\mathbf{C}^*$, and in order to compute $\Delta_{e^{i\pi}}\, r$ we perform a translation,

$$\hat{r}(e^{i\pi}+\zeta) = e^{i\pi(\beta-1)}\frac{\zeta^{\beta}}{\Gamma(\beta)}(1-\zeta)^{\beta-1},$$

and take the variation of the resulting singular germ (just as we were asked to retain the coefficient of $\log\zeta/2\pi i$ in the case of pure logarithmic singularities, according to formula (5.5)),

$$\mathcal{B}\,\Delta_{e^{i\pi}}\, r = -e^{i\pi\beta}(1-e^{2i\pi\beta})\frac{\zeta^{\beta}}{\Gamma(\beta)}(1-\zeta)^{\beta-1}.$$

We deduce that

$$\Delta_{e^{i\pi}}\, r = -e^{i\pi\beta}(1-e^{2i\pi\beta})\beta z^{-\beta-1}(1+O(z^{-1}))$$

and

$$\Delta_{-1}\, q = z^{\beta}\Delta_{e^{i\pi}}\, r = -2i\beta\sin(\pi\beta)z^{-1}(1+O(z^{-1})).$$

Finally we use Leibniz rule and formula (5.6):

$$\Delta_{-1}\, y = \frac{2\pi i}{b}\, z\,\Delta_{-1}(q'/q) = \frac{2}{b^{1/2}}\sin\frac{b^{1/2}}{2} + O(z^{-1}).$$

The constant term of the alien derivative is the residuum $\sigma(b)$. $\square$

6. Remarks on general nonlinear inner equations: The Kruskal–Segur strategy. Formula $p_0(\tau) - \tilde{p}_0(\tau) = -ie^{2i\tau}(1+O(\tau^{-1}))$ obtained in Theorem 3.1 for the inner equation (3.3) has been crucial for determining $w(+\infty,\varepsilon)$ and thus $\Delta I^2(\varepsilon)$, and this was the aim of the previous section. There (3.3) is studied by the resurgence theory obtaining the formula (5.10): $p_0(\tau) - \tilde{p}_0(\tau) = A^- e^{2i\tau}(1+O(\tau^{-1}))$. In order to compute exactly the coefficient $A^-(=-i)$ in the subsection 5.3, it is essential to be assured that (3.3) can be transformed into a second-order linear equation. However, in most applications (for example, for standard maps, see [7]) the inner equation is not a Riccati equation and the method outlined in section 5 does not give quantitative results. Nevertheless, in the general case we can join the resurgence method with the Kruskal and Segur strategy [8]. This strategy adapted to our case would have the following form.

The main idea is that A^- can be computed looking at the coefficients of the formal solution of (3.3). Following the Kruskal–Segur strategy one can see that the growth of a_n is controlled comparing them with the coefficients b_n of the associated linear problem.

In this sense let $\sum_{n\geq 0} a_n\tau^{-n-1}$ be the formal solution of (3.3), which vanish at $-\infty$ (and, in fact, at $+\infty$), and let $\sum_{n\geq 0} b_n\tau^{-n-1}$ be the associated formal solution of the linear part of (3.3) $q_0' = 2iq_0 + (1)/6\tau$. We obtain that $b_n = (-1)^{n+1}(1)/12i(n!)/(2i)^n$ and as a first step in this method we would have to prove the following.

Proposition 6.1. *$a_n = k_n b_n$, where $k_n = 3/\pi + O(\frac{1}{n})$, as $n \to \infty$.*

In our case we have numerical evidences of this result, and probably using the special form of our equation it could be proved analytically (see [17]). In this paper, we have not adopted this strategy because of the special form of our equation where this result is a consequence of the computation of $\Delta_{-1}Y_-$ in section 5.

Proposition 6.1 would be essential to control the behavior of the Borel transform of our solution. Let $\varphi(\xi) := \sum_{n\geq 0}(a_n\xi^n)/n!$ be the Borel transformation of $\sum_{n\geq 0} a_n\tau^{-n-1}$ and $\varphi_0(\xi) := 3/\pi\sum_{n\geq 0}(b_n\xi^n)/n! = -(1/2\pi)(1/2i+\xi)$ the Borel transformation of $3/\pi\sum_{n\geq 0} b_n\tau^{-n-1}$, and let us call $\varphi_1 := \varphi - \varphi_0$. Finally, let $f(\tau)$ be the Borel resummation of $\varphi(\xi)$, that is, its Laplace transform in some direction of the upper plane $Im\xi > 0$. As we know exactly $\varphi_0(\xi)$ in all the complex plane, this method allows us to compute its contribution to the resummation $f(\tau)$. Nevertheless, it would remain to prove that $\varphi_1(\xi)$ contributes to $f(\tau)$ only with higher-order terms. In order to prove this, it would be necessary to know the behavior of $\varphi_1(\xi)$, and, for our case, this is done in Proposition 6.2.

Proposition 6.2.

i. *φ_0 has a unique singularity at $-2i$, which is a pole with residue $-\frac{1}{2\pi}$,*
ii. *φ_1 has logarithmic singularities at $-2i, -4i, -6i, \ldots,$*
iii. *Moreover, $f(\tau)$ is Gevrey-1 asymptotic to the formal solution $\sum_{n\geq 0} a_n\tau^{-n-1}$ in the sector $-3\pi/2+\alpha \leq \arg\tau \leq \pi/2-\alpha$, when $|\tau| \to \infty \ldots$*

An analogous proposition is studied in section 5 with the help of resurgent theory for our equation, and it seems that this can be generalized to other equations, as in [15]. Resurgent theory gives us the location of the singularities of φ_1, as well as their type and, consequently, their contribution to the "resummation" $f(\tau)$.

Finally, as a last step of this method, putting together Propositions 6.1 and 6.2 we would have the following.

Proposition 6.3.

i. $f(\tau) = p_0(\tau)$ *if* $\pi/2 < \arg\tau < 2\pi$,
ii. $f(\tau) = \tilde{p}_0(\tau)$ *if* $-\pi < \arg\tau < \pi/2$.

Then, for τ such that $0 \leq \arg\tau < \pi/2$, one can use the analytic continuation of $f(\tau)$, and using these propositions and the Cauchy theorem, one can see that in our equation:

$$p_0(\tau) - \tilde{p}_0(\tau) = \int_{-\infty}^{+\infty} e^{-\tau s}\varphi(s)ds$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} e^{-\tau s}\varphi_0(s)ds + \int_{-\infty}^{+\infty} e^{-\tau s}\varphi_1(s)ds \\
&= e^{2i\tau}(-i + O(\tau^{-1})).
\end{aligned}
$$

Observe that following the Kruskal–Segur strategy we can compute the coefficient $A^-/(2\pi i)$ as the residue of the Borel transform φ_0 at its pole $-2i$. We are convinced that the link between the resurgence approach and Kruskal–Segur strategy rests on the fact that, in general, all the successive approximations of the corresponding (5.2) have a pole at $-2i$. Then φ_0 would be the summation of all the polar parts at $-2i$ of those approximations, and its residue the sum of their residues (which corresponds to the coefficient A^- computed in Proposition 5.5).

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