# Regularization of non-smooth systems using analytic functions.

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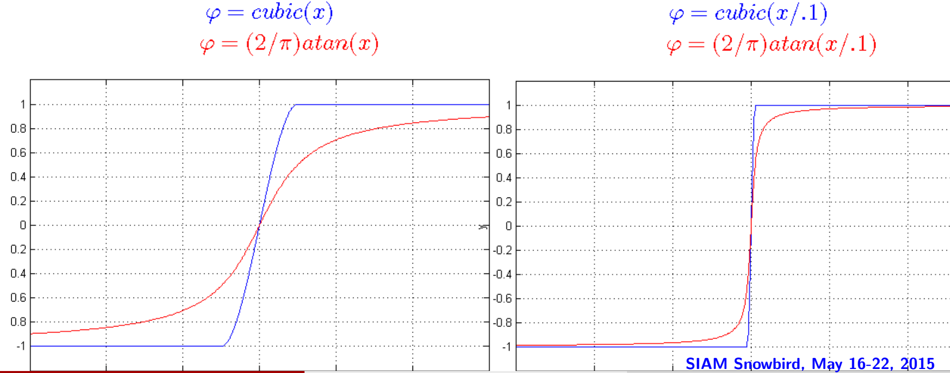
When with Tere M-Seara were working in the paper *Regularization of sliding global bifurcations derived from the local fold singularity of Filippov systems*, we used a Sotomayor-Teixeira regularization with a $C^k$-function $\varphi$ at $x = \pm 1$. Nevertheless in simulations is common to use analytic functions flat at infinity, instead. And the results seem to work out. Then we proposed us to study the deviation of the Fenichel manifold for these cases.



For the sake of simplicity, in this talk, we consider the Filippov System:

$$Z(x,y) = \begin{cases} X^+(x,y) = (2,4x),\ y > 0 \\ X^-(x,y) = (0,1),\ \ y < 0. \end{cases}$$

Then we regularize this system with $\varphi = \frac{2}{\pi} atan(\frac{y}{\epsilon})$. The regularized system can be written, with the change of variable $y = \epsilon v$ as:

$$\begin{aligned} \dot{x} &= 1 + \varphi(v) \\ \epsilon\dot{v} &= 1 + 2x + \varphi(v)(2x-1) \end{aligned}$$

We use the same methods (Fenichel theory and asymptotic methods), but now the problem is slightly different. This system doesn't have a natural "regularization zone" such that outside of it the fields are the original ones. Now the regularized field is different in all the phase space.
In spite of this, we can follow the Fenichel manifold and prove that the deviation will be of order $\mathcal{O}(\epsilon^{\frac{2}{3}})$ with respect to the parabolic solution of $X(x,y) = (2,4x)$, $y = x^2$, at $y = y_0 = x_0^2, x_0 > 0$. Also we see that in this case the deviation is $\sqrt{y_0} - \frac{\bar{\Omega}_0}{2\sqrt{y_0}}\epsilon^{2/3}$, with $\bar{\Omega}_0$ is related to $\Omega_0 = 2.338107...$, the standard constant appearing in the Riccati equation.

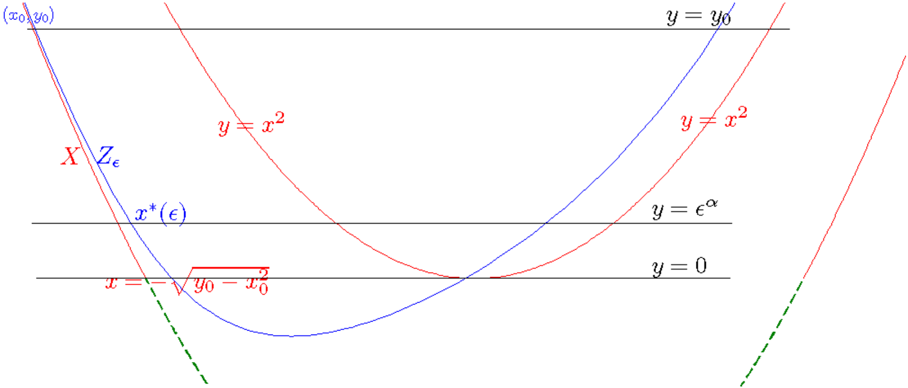
## The first step

First we take the regularized system $Z_\epsilon$:

$$\begin{aligned}\dot{x} &= 1+\varphi(\tfrac{y}{\epsilon})\\ \dot{y} &= 1+2x+\varphi(\tfrac{y}{\epsilon})(2x-1)\end{aligned}$$

with $\varphi = \frac{2}{\pi} atan(\frac{y}{\epsilon})$
We will take a point $(x_0, y_0)$, with $x_0 < 0$ and $y_0 > x_0^2$ and we will compare the orbits of this regularized system with those of $X$ beginning at $(x_0, y_0)$

# The first step




We see that we cannot arrive at $y = \epsilon$, cause the difference between $Z_\epsilon$ and $X$ is large there. So we arrive only till $y = \epsilon^\alpha$, with $\frac{1}{2} < \alpha < 1$. Later we will see why.

As

$$\varphi(z) = 1 - \frac{2}{\pi z} + \mathcal{O}(\frac{1}{z^3}),\ z \to \infty.$$

in the region $y \geq \epsilon^\alpha$, $0 \leq \alpha < 1$, we will have $Z_\epsilon = X + \mathcal{O}(\epsilon^{1-\alpha})$.
We denote as $x^*(\epsilon)$ the intersection with $y = \epsilon^\alpha$ of the orbit of $Z_\epsilon$ issuing of $(x_0, y_0)$, and we will have

$$x^*(\epsilon) = -\sqrt{y_0 - x_0^2} + \mathcal{O}(\epsilon^{1-\alpha}),\ \frac{1}{2} < \alpha < 1.$$

where $x^*(0) = -\sqrt{y_0 - x_0^2}$ is the intersection of the orbit of $X$ with $y = 0$.
From now on we track the orbit of $Z_\epsilon$ issuing from the point $(x^*(\epsilon), \epsilon^\alpha)$.
The method, as usual, will be to arrive to exponentially small neighbourhoods of the Fenichel manifold(s) and then we will focus in tracking this one.

# The second step

In this step we will see the atraction to the Fenichel manifold. Now we depart from system:

$$\begin{array}{rcl} \dot{x} & = & 1+\varphi(v) \\ \epsilon\dot{v} & = & 1+2x+\varphi(v)(2x-1) \end{array}$$

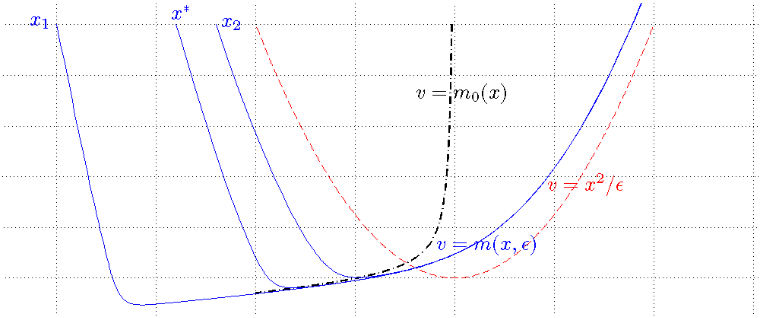
with $(x^*(\epsilon),\epsilon^{\alpha-1})$ as initial conditions. The slow manifold in this case is

$$atan(v)=\frac{\pi}{2}\frac{1+2x}{1-2x},\ \text{or}\ x=\frac{1}{2}\frac{2atan(v)-\pi}{2atan(v)+\pi}$$

If we denote by $v=m_0(x)$ and $v=m(x,\epsilon)$ the equations of slow manifold and the Fenichel manifold for $x_1<x^*(\epsilon)<x_2<0$ we know that

$$m(x,\epsilon)=m_0(x)+\mathcal{O}(\epsilon)$$

uniformly for $x\in[x_1,x_2],\epsilon<\epsilon_0$. And if $x_1$ is small enough we can suppose that $m_0(x), and, m(x,\epsilon)$ are increasing functions. All this is summarized in the next figure




with the change $w = v - m(x,\epsilon)$ , taking account of the properties of $m_0(x), m(x,\epsilon)$, and a Gronwall argument, the solution $w(t)$ of the tresformed system fullfils :

$$0 \leq w(t,\epsilon) \leq (\epsilon^{\alpha-1} - m(x^*(\epsilon),\epsilon))e^{-L\epsilon^{1-2\alpha}t},\ 0 \leq t \leq \frac{x_2 - x^*(\epsilon)}{2}$$

and as $x^*(0) = -\sqrt{y_0 - x_0^2}, \alpha > \frac{1}{2}$, if $x_2$ is small enough, we have the result.

# Third step

From now on, we only must track ( the halo of) the Fenichel manifold. As we can consider $x_2$ small enough, system $Z_\epsilon$ is equivalent to the equation

$$\frac{dx}{dv} = \epsilon \frac{1+\varphi(v)}{1+2x+\varphi(v)(2x-1)}$$

That is, from now on, we will look for the Fenichel manifold and also for all nearby orbits as graphs over the $v$ variable, because it is already inside the region $1+2x+\varphi(v)(2x-1) > 0$ and cannot scape there, for the monotony properties.
Then we try to track the Fenichel manifold in a region $M \leq v \leq \epsilon^{\alpha-1}$, where $M$ can be taken as big as we need.

We seek the Fenichel manifold of the form:

$$x = n(v;\epsilon) = n_0(v) + \epsilon n_1(v) + \epsilon^2 n_2(v) + \cdots + \mathcal{O}(\epsilon^k)$$

where $n_0(v) = \frac{1}{2}\frac{\varphi(v)-1}{\varphi(v)+1}$, obtaining:

$$\begin{aligned} n_1(v) &= \frac{1}{2}\frac{1}{n_0'(v)}, \\ n_2(v) &= -2n_1'(v)n_1^2(v) = \frac{1}{2}\frac{n_0''(v)}{(n_0'(v))^2}, \end{aligned}$$

These functions are singular at $+\infty$ and behave as:

$$\begin{aligned} n_0(v) &= -\frac{1}{2\pi v} + \mathcal{O}(\frac{1}{v^2}), \\ n_1(v) &= \pi v^2 + \mathcal{O}(v), \\ n_2(v) &= \mathcal{O}(v^5), \end{aligned}$$

One can see that $n_k(v) = \mathcal{O}(v^{3k-1})$, as $v \to \infty$.

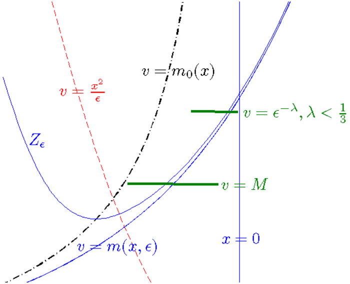
This suggest that the validity of this expansion is:

$$M \leq v \leq \epsilon^{-\lambda}, \quad 0 < \lambda < \frac{1}{3}$$

One can be proved that the region:

$$B_1 = \{(x, v),\ M \leq v \leq \epsilon^{-\lambda},\ n_0(v) \leq x \leq n_0(v) + \epsilon K_1 v^2\}$$

is positively invariant and contains the Fenichel manifold.




Note, however, that $n_0(\epsilon^{-\lambda}) < 0$!

# fourth step

To continue the Fenichel manifold to the region $x > 0$, we perform the change to the inner variables:

$$\eta = \epsilon^{-\frac{1}{3}} x, \quad u = \epsilon^{\frac{1}{3}} v, \quad \mu = \epsilon^{\frac{1}{3}}$$

The equation in these variables reads:

$$\frac{d\eta}{du} = \frac{\mu(1 + \varphi(\frac{u}{\mu}))}{\Big(1 + 2\mu\eta(u) + \varphi(\frac{u}{\mu})(2\mu\eta(u) - 1)\Big)}$$

We will study this system for $\epsilon^{\frac{1}{3}-\lambda} \leq u < \infty$.
Looking for the solution as:

$$\eta = \eta(u, \mu) = \eta_0(u) + \mu\eta_1(u) + \dots$$

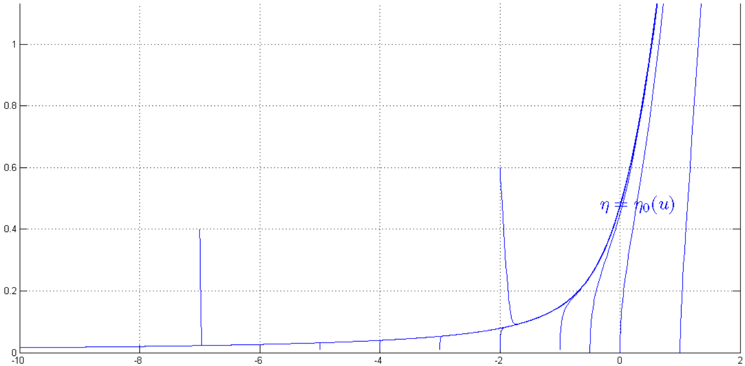
The equation for $\eta_0$ can be transformed with the change $w = u - \eta_0^2$ and some scalings into the well known Ricatti equation:

$$\frac{d\eta}{dw} = w + \eta^2$$

The solutions of the equation for $\eta_0$ in $u > 0$:

$$\frac{d\eta_0}{du} = \frac{1}{2\eta_0 + \frac{1}{\pi u}}$$

are shown in the next figure. Here $\eta = \eta_0(u)$ denotes the only solution that $\eta \to -\infty$ as $u \to o^+$



# fourth step

Using the results of Mischenko-Rozov one obtains for this unique solution

$$\begin{aligned} \eta_0(u) &= -\frac{1}{2\pi u} + \pi^2 + \mathcal{O}(u^5), \quad u \to 0^+ \\ \eta_0(u) &= \sqrt{u} - \frac{\bar{\Omega}_0}{2\sqrt{u}} + \mathcal{O}(\tfrac{1}{u}), \quad u \to +\infty, \quad \bar{\Omega}_0 = \frac{\Omega_0}{\pi^{\frac{2}{3}}} \end{aligned}$$

where $\Omega_0$ is the constant appearing in the standard Riccati equation. The next term in the expansion $\eta_1(u)$ can be computed and one can see that:

$$\begin{aligned} \eta_1(u) &\simeq -\frac{1}{2(\pi u)^2}, \quad u \to 0^+ \\ \eta_1(u) &\simeq \frac{1}{\sqrt{u}}, \quad u \to +\infty, \end{aligned}$$

With this information we can build the blocks which contain the Fenichel manifold.

# The three blocks

We take $u_0$ small enough and $K_2$ big enough but independent of $\epsilon$ to build our first block:

$$B_2 = \{(\eta, u),\ \epsilon^{\frac{1}{3}-\lambda} \le u \le u_0,\ |\eta - \eta_0(u)| \le K_2 \frac{\epsilon^{1/3}}{u^2}\}$$

The next block is:

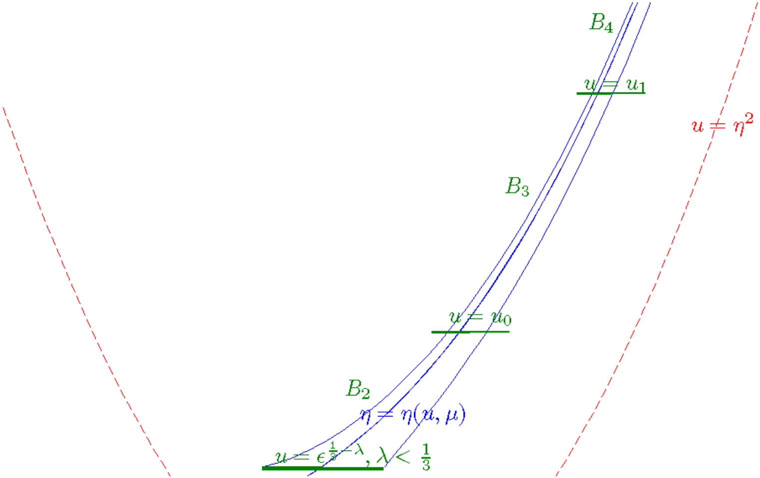
$$B_3 = \{(\eta, u),\ u_0 \le u \le u_1,\ |\eta - \eta_0(u)| \le K_3 \epsilon^{1/3}\}$$

where $u_1$ and $K_3$ are big enough but independent of $\epsilon$. Finally, the last block is:

$$B_4 = \{(\eta, u),\ u_1 \le u < \infty,\ |\eta - \eta_0(u)| \le K_4 \frac{\epsilon^{1/3}}{\sqrt{u}}\}$$

where $K_4$ is big enough but independent of $\epsilon$

# The three blocks



We prove that the Fenichel manifold enters the three blocks and stays in $B_4$ and there is given by

$$\eta(u) = \eta_0(u) + \mathcal{O}(\frac{\epsilon^{1/3}}{\sqrt{u}})$$

Going back to the original variables, the Fenichel manifold arrives to the section $y = y_0$ in a point $(x^{**}(\epsilon), y_0)$ given by:

$$x^{**} = \epsilon^{1/3}\eta_0(\frac{y_0}{\epsilon^{2/3}}) + \mathcal{O}(\frac{\epsilon}{\sqrt{y_0}}) = \sqrt{y_0} - \frac{\bar{\Omega}_0}{2\sqrt{y_0}}\epsilon^{2/3} + \mathcal{O}(\epsilon)$$