

SPARSE GRADIENT BOUNDS FOR DIVERGENCE FORM ELLIPTIC EQUATIONS

OLLI SAARI, HUA-YANG WANG, AND YUANHONG WEI

ABSTRACT. We provide sparse estimates for gradients of solutions to divergence form elliptic partial differential equations in terms of the source data. We give a general result of Meyers (or Gehring) type, a result for linear equations with VMO coefficients and a result for linear equations with Dini continuous coefficients. In addition, we provide an abstract theorem conditional on PDE estimates available. The linear results have the full range of weighted estimates with Muckenhoupt weights as a consequence.

1. INTRODUCTION

Let $\theta \in (0, 1)$. A family of cubes \mathcal{Q} in \mathbb{R}^n , $n \geq 1$, is said to be θ -sparse if for each $Q \in \mathcal{Q}$ there exists $E_Q \subset Q$ such that $|E_Q| \geq \theta|Q|$ and for all $x \in \mathbb{R}^n$

$$\sum_{Q \in \mathcal{Q}} 1_{E_Q}(x) \leq 1.$$

An operator T acting on some test function class is said to satisfy a θ -sparse (s, r) -bound if for each pair of test functions (f, g) there exists a θ -sparse family \mathcal{Q} such that

$$\begin{aligned} & \left| \int T f(x) g(x) dx \right| \\ & \leq C \sum_{Q \in \mathcal{Q}} |Q| \left(\frac{1}{|3Q|} \int_{3Q} |f(x)|^s dx \right)^{1/s} \left(\frac{1}{|3Q|} \int_{3Q} |g(x)|^r dx \right)^{1/r}. \end{aligned} \quad (1.1)$$

Despite its repelling appearance, a sparse bound is a useful estimate. Its validity implies immediately L^p bounds for $s < p < r'$. Another immediate consequence is the quantitative weighted estimate, which we detail in Proposition 6.1, quoted from [5].

Sparse bounds were originally introduced in the context of quantitative weighted estimates for Calderón–Zygmund singular integrals by Lerner [29]. Their use led to astonishingly simple proofs of the A_2 theorem [27, 30, 32], whose original proofs in the cases of Beurling–Ahlfors transform [36], the Hilbert transform [35] and also in the general case [21] are somewhat challenging. Lerner’s original work involved so-called sparse domination in Banach function spaces norms, which was later

improved to pointwise bounds [11, 31]. The bounds in terms of duality pairings as in (1.1) were first used in [5], where their applicability outside of the theory of singular integral operators was demonstrated. Although weaker than pointwise bounds, they are extremely important as they can be used to model L^p bounds on a restricted range unlike the pointwise bounds. This opened the door for using the sparse approach to many operators previously out of reach. Since the publication [35], sparse bounds have been proved for the spherical maximal function [28], pseudodifferential operators [2], bilinear Hilbert transforms and generalizations [12, 3] and many other operators. The literature has become very extensive, and it is impossible to give an exhaustive list.

The purpose of the present paper is to study sparse bounds thematically close but historically far from their origins. We consider solutions to Dirichlet problems to elliptic partial differential equations of the form

$$\operatorname{div} a(x, \nabla u(x)) = \operatorname{div} F(x)$$

where $F \in L^2_{loc}(\mathbb{R}^n)$ is a source term and a a coefficient function as in Definition 2.1. Estimating $|\nabla u|$ in terms of $|F|$ is the topic of the other branch of Calderón–Zygmund theory. The literature on various L^p and weighted L^p bounds for various coefficient functions a is again too vast to be listed. In addition to the linear theory, there has been a number of works studying related questions for nonlinear equations [22] and nonlinear potential theory [17, 26], and there have been works addressing more and more irregular coefficients such as [9]. To the best of our knowledge, however, not too many sparse bounds are known in the context of differential equations. Results in [5] do not deal with the exact setup we are interested in and the results in [2] only deal with pseudodifferential operators whose symbols are too smooth to really be applied here. Hence we try to give an overview on phenomena in Calderón–Zygmund theory of linear and nonlinear equations from the sparse point of view.

Our first main result is a sparse analogue of Meyers’ estimate, which deals with source data almost in the natural energy space. Here the setup is completely nonlinear. We do not assume any spatial smoothness on the coefficients and consequently this is a result that does not have an obvious analogue in the point of view based on kernels. See Section 2 for precise definitions.

Theorem 1.1. *Let $\theta \in (0, 1)$, $0 < \lambda \leq \Lambda < \infty$, Q_0 be a cube, and a (λ, Λ) -elliptic coefficient a in Q_0 be given. Let $\Omega \subset Q_0$ be a Lipschitz domain. There exists $q = q(\lambda, \Lambda, n, \Omega) > 2$ such that the following holds.*

Let $p \in (2, q]$, $F \in L^p(3Q_0; \mathbb{R}^n)$ and $f \in L^{p_}(3Q_0)$ where $p_* = np/(n + p)$. Let $u \in W_0^{1,2}(\Omega)$ be a solution to*

$$\operatorname{div} a(x, \nabla u(x)) = \operatorname{div} F(x) + f(x), \quad x \in \Omega$$

and let $g \in L^\infty(3Q_0)$ be non-negative.

Then there exists a $(1 - \theta)$ -sparse family \mathcal{P} of cubes P such that for all $P \in \mathcal{P}$ it holds $3P \subset 3Q_0$ and

$$\begin{aligned} \int_{\Omega} |\nabla u(x)|g(x) dx &\leq C \sum_{P \in \mathcal{P}} |P| \langle |F| \rangle_{3P,2} \langle g \rangle_{P,q'} \\ &\quad + C \sum_{P \in \mathcal{P}} |P| 3\ell(P) \langle |f| \rangle_{3P,2^*} \langle g \rangle_{P,q'} \end{aligned}$$

where $C = C(\lambda, \Lambda, n, q, \Omega, \theta)$.

Note that by Hölder's inequality and a maximal function argument, the sparse estimate above implies

$$\|\nabla u\|_{L^p(\Omega)} \leq C \|F\|_{L^p(\Omega)} + C \|f\|_{L^{p^*}(\Omega)}.$$

Estimates of this latter type were first proved by Bojarski [7], Meyers [33], Elcrat–Meyers [34] and Gehring [18]. The L^p theory for general nonlinear equations with less restricted range of p is quite well developed, and we refer to [9] as an example of a more general result with BMO coefficients. When it comes to sparse estimates, however, a peculiarity of the linearization argument we use impedes us from advancing beyond the Meyers type result in the nonlinear setting. No matter how smooth the coefficient function is, we end up with a linearization that has no a priori smoothness beyond measurability.

In the setting of linear equations, a duality argument allows us to make the range of admissible (s, r) for a sparse bound a symmetric rectangle centred at $(1/2, 1/2)$. We postpone a more quantitative statement to the bulk of the paper, but here we give an example of what we can say about relatively good equations. We start with the case where all L^p -bounds are known to hold. Recall that a coefficient matrix $A = [A_{ij}]_{n \times n}$ is said to be of vanishing mean oscillation (see [37]) if for each A_{ij} with $1 \leq i, j \leq n$ it holds

$$\lim_{s \rightarrow \infty} \mathcal{V}_s(A_{ij}) < \infty, \quad \lim_{s \rightarrow 0} \mathcal{V}_s(A_{ij}) = 0$$

where

$$\mathcal{V}_s(A_{ij}) = \sup_{x \in \mathbb{R}^n, r \leq s} \inf_{c \in \mathbb{R}} \frac{1}{r^n} \int_{Q_r(x)} |A_{ij}(y) - c| dy.$$

Here we denote

$$Q_r(x) = \prod_{i=1}^n [x_i - 2^{-1}r, x_i + 2^{-1}r].$$

Theorem 1.2. *Let $\theta \in (0, 1)$, $0 < \lambda \leq \Lambda < \infty$, Q_0 be a cube, and a linear (λ, Λ) -elliptic coefficient A in Q_0 be given. Let $\Omega \subset Q_0$ be a C^2 -domain. Assume that A is of vanishing mean oscillation. Let $q \in (2, \infty)$ be given. Let $p \in (2, q]$ and $F \in L^p(3Q_0; \mathbb{R}^n)$. Let $u \in W_0^{1,2}(\Omega)$ be a solution to*

$$\operatorname{div} A(x) \nabla u(x) = \operatorname{div} F(x), \quad x \in \Omega$$

and let $g \in L^\infty(3Q_0)$ be non-negative.

Then there exists a $(1-\theta)$ -sparse family \mathcal{P} of cubes $P \subset Q_0$ such that for all $P \in \mathcal{P}$ it holds $3P \subset 3Q_0$ and

$$\int_{\Omega} |\nabla u(x)|g(x) dx \leq C \sum_{P \in \mathcal{P}} |P| \langle |F| \rangle_{3P, q'} \langle g \rangle_{P, q'}$$

where $C = C(A, n, q, \Omega, \theta)$.

This is a sparse variant of the results of Di Fazio [13], Iwaniec–Sbordone [23] and Kinnunen–Zhou [24, 25] (linear case only). We have omitted the function f on the source data for simplicity. It could be included, but additional restrictions on the exponents would appear and make reading more difficult.

The sparse bound as above has a corollary easy to state in terms of Muckenhoupt weights. Recall that a locally integrable function $w \geq 0$ is said to be an A_p weight with $1 < p < \infty$ if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all axis parallel cubes in \mathbb{R}^n . A version of Corollary 1.3 with continuous coefficient matrix can be found as Theorem 2.5 in [8] and with VMO regular coefficient matrix as Theorem 2.1 in [1]. These smoothness conditions are not optimal, and further generalizations can be found for instance in [15], where coefficients with small BMO norm are treated.

Corollary 1.3. *Let A be a (λ, Λ) -elliptic coefficient matrix in Q_0 . Let $\Omega \subset Q_0$ be a C^2 -domain. Assume that A is of vanishing mean oscillation. If $p \in (1, \infty)$, $w \in A_p$, $F \in L^p(\Omega, w dx)$ and $u \in W_0^{1,2}(\Omega)$ is a solution to*

$$\operatorname{div} A(x) \nabla u(x) = \operatorname{div} F(x), \quad x \in \Omega,$$

then

$$\|\nabla u\|_{L^p(\Omega, w dx)} \leq C \|F\|_{L^p(\Omega, w dx)}$$

where $C = C(A, p, [w]_{A_p}, n, \Omega)$.

We do not have good control on how the bound depends on the A_p constant. We refer to Proposition 6.1 for some estimates in this setting.

Next we turn to the setting of Dini continuous coefficients, where we get exactly as good results as in the singular integral theory of Dini continuous Calderón–Zygmund kernels. Recall that $\omega : [0, 1) \rightarrow [0, \infty)$ is said to be a Dini function if there are constants $c_1, c_2 > 0$ such that whenever $0 < s < t < 2s < 1$, it holds

$$c_1 \omega(s) \leq \omega(t) \leq c_2 \omega(2s)$$

and for all $t \in (0, 1)$

$$\int_0^t \frac{\omega(s)}{s} ds < \infty.$$

A function $g : 3Q_0 \rightarrow \mathbb{R}$ is said to be of ω -Dini mean oscillation if

$$\omega_g(r) = \sup_{x \in \Omega} \int_{Q_r(x) \cap \Omega} |g(y) - \langle g \rangle_{Q_r(x) \cap \Omega}| dy$$

is a Dini function.

Theorem 1.4. *Let $\theta \in (0, 1)$, $0 < \lambda \leq \Lambda < \infty$ and let Q_0 be a cube. Let $\Omega \subset Q_0$ be a C^2 -domain. Let a linear (λ, Λ) -elliptic coefficient A with Dini mean oscillation in Ω be given. Assume $\omega_A(r) \leq c|\log r|^{-2}$ for some c and all $r \in (0, 1/2)$. Let $F \in L^p(3Q_0; \mathbb{R}^n)$ for some $p \in (1, \infty)$. Let $u \in W_0^{1,2}(\Omega)$ be a solution to*

$$\operatorname{div} A(x) \nabla u(x) = \operatorname{div} F(x), \quad x \in \Omega$$

and let $g \in L^\infty(3Q_0)$ be non-negative.

Then there exists a $(1 - \theta)$ -sparse family \mathcal{P} of cubes P such that $3P \subset 3Q_0$ and

$$\int_{\Omega} |\nabla u(x)| g(x) dx \leq C \sum_{P \in \mathcal{P}} |P| \langle |F| \rangle_{3P,1} \langle g \rangle_{P,1}$$

where C depends on $(\lambda, \Lambda, n, \Omega, \theta)$ and the Dini data of A .

Corollary 1.5. *Let A be a (λ, Λ) -elliptic coefficient matrix in $3Q_0$. Let $\Omega \subset Q_0$ be a smooth domain. Assume $\omega_A(r) \leq c|\log r|^{-2}$ for some c and all $r \in (0, 1/2)$. If $p \in (1, \infty)$, $w \in A_p$, $F \in L^p(Q_0, w dx)$ and $u \in W_0^{1,2}(\Omega)$ is a solution to*

$$\operatorname{div} A(x) \nabla u(x) = \operatorname{div} F(x), \quad x \in \Omega$$

then

$$\|\nabla u\|_{L^p(\Omega, w dx)} \leq C [w]_{A_p}^{\max(\frac{1}{p-1}, 1)} \|F\|_{L^p(3Q_0, w dx)}$$

where C depends on $(n, p, \lambda, \Lambda, \Omega)$ and the Dini data of A .

We conclude the introduction with a few words about our proofs. We were mostly inspired by the arguments of Lerner [30] and Lerner–Ombrosi [32]. After putting the PDE and singular integral theory quantities in a correspondence and finding all the necessary estimates, our proof is very similar to theirs. However, our formulations go through boundary value problems rather than Poisson problems as would be closer to the spirit of their writing. Although it might be possible to reduce the Dini theorem to a situation where their result can be applied as a black box, we do not believe there is much gain of insight or brevity in attempting to do that. Hence we stick to the way of Dirichlet problems. The idea of using reverse Hölder inequalities as a workhorse comes from two sources. In PDE, our original plan and motivation was to follow the work of Caffarelli and Peral [10]. In the sparse context, the reverse Hölder inequalities were exploited very efficiently by Lacey in [28]. However, we note that several arguments become invalid when the coefficient a appearing in the equation becomes too rough. This is

why we present two arguments, one for the general setting (which carries us until the VMO theorem) and another one which takes advantage of full regularity theory for equations with Dini continuous coefficients, in particular the estimates of Dong, Escauriaza and Kim [16, 14].

Acknowledgement. O. Saari is supported by Generalitat de Catalunya (2021 SGR 00087), Ministerio de Ciencia e Innovación and the European Union – Next Generation EU (RYC2021-032950-I), (PID2021-123903NB-I00) and the Spanish State Research Agency through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). Y. Wei is supported by the National Natural Science Foundation of China (Grant No. 11871242, 11971060), and Scientific Research Project of Education Department of Jilin Province (Grant No. JJKH20220964KJ). Part of the research was carried out during the first author’s visit to Jilin University, the group in which he wishes to thank for its kind hospitality. We feel very grateful to anonymous reviewers for their time and valuable suggestions about our manuscript.

2. PRELIMINARIES

2.1. Notation and generalities. We work in \mathbb{R}^n and its open subsets. For a measurable set $E \subset \mathbb{R}^n$, we denote its n -dimensional Lebesgue measure by $|E|$. Given a function $h \in L^1_{loc}(\mathbb{R}^n)$, a measurable set E of finite measure and a number $s \in (0, \infty)$, we denote

$$\int_E h(x) dx := \frac{1}{|E|} \int_E h(x) dx, \quad \langle h \rangle_{E,s} := \left(\int_E |h(x)|^s dx \right)^{1/s}.$$

We write $\langle h \rangle_E := \langle h \rangle_{E,1}$.

By a cube we mean a cartesian product of n equally long intervals. Given a cube Q , we denote its side length by $\ell(Q) = |Q|^{1/n}$. The standard set of dyadic cubes is

$$\mathcal{D} = \{2^k([0, 1]^n + j) : k \in \mathbb{Z}, j \in \mathbb{Z}^n\}.$$

Given a cube $Q_0 \subset \mathbb{R}^n$, let Θ be the translation and scaling with $\Theta(Q_0) = [0, 1]^n$. We denote

$$\mathcal{D}(Q_0) = \{\Theta^{-1}(Q) : Q \in \mathcal{D}, \Theta^{-1}(Q) \subset Q_0\}.$$

We refer to $\mathcal{D}(Q_0)$ as dyadic subcubes of Q_0 .

Given a reference cube Q_0 , we define the standard and fractional maximal functions as

$$Mf(x) = \sup_{P \in \mathcal{D}(Q_0)} 1_P(x) \langle f \rangle_{3P}$$

and for $s \in (0, \infty)$

$$M_s f(x) = \sup_{P \in \mathcal{D}(Q_0)} 1_P(x) (3\ell(P))^s \langle f \rangle_{3P}.$$

The standard theory includes the bounds

$$\|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)} \leq C_n < \infty, \quad \|M\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq C_{p,n} < \infty \quad (2.1)$$

for all $p \in (1, \infty]$, see for instance Theorem 2.1.6 in [19], and the fractional maximal function bounds

$$\|M_s\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} \leq C_{s,n,p} < \infty \quad (2.2)$$

for

$$\frac{s}{n} = \frac{1}{p} - \frac{1}{q}, \quad q > \frac{n}{n-s},$$

see for instance Chapter 6 in [20]. We denote the upper and lower Sobolev conjugates as

$$p_* = \frac{pn}{n+p}, \quad p^* = \frac{pn}{n-p}$$

and we denote the Hölder conjugate as $p' = p/(p-1)$ when $p \in (1, \infty)$ and $1' = \infty$ and $\infty' = 1$.

2.2. Solutions. We axiomatize the properties of the coefficients of the equations that we study. The definition below is standard.

Definition 2.1. Consider an open and connected set Ω . Let $0 < \lambda \leq \Lambda < \infty$ be real numbers. We call a function $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ a (λ, Λ) -elliptic coefficient in Ω if the following conditions hold.

- $x \mapsto a(x, \xi)$ is measurable for all ξ , and $\xi \mapsto a(x, \xi)$ is continuously differentiable for all x .
- for all $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^n$, we have $a(x, 0) = 0$ and

$$|a(x, \xi) - a(x, \eta)| \leq \Lambda |\xi - \eta|. \quad (2.3)$$

This is referred to as *Lipschitz condition in gradient variable*.

- for all $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^n$, we have

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) \geq \lambda |\xi - \eta|^2. \quad (2.4)$$

This is referred to as *strong monotonicity*.

The next definition of a solution is standard.

Definition 2.2. Let a be a (λ, Λ) -elliptic coefficient in Ω . Let $F \in L^1_{loc}(\Omega; \mathbb{R}^n)$, $f \in L^1_{loc}(\Omega)$ and $h \in W^{1,2}(\Omega)$. We call a function $u \in W^{1,2}(\Omega)$ a solution to

$$\begin{aligned} \operatorname{div} a(x, \nabla u(x)) &= \operatorname{div} F(x) + f(x), & x \in \Omega \\ u(x) &= h(x), & x \in \partial\Omega \end{aligned} \quad (2.5)$$

if $u - h \in W_0^{1,2}(\Omega)$ and it holds for all test functions $\varphi \in C_c^\infty(\Omega)$

$$\int_{\Omega} a(x, \nabla u(x)) \cdot \nabla \varphi(x) \, dx = \int_{\Omega} F(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} f(x) \varphi(x) \, dx.$$

We remark that by approximation, the smoothness condition on test functions can always be relaxed down to one Sobolev derivative in the relevant $L^p(\Omega)$ space.

The key property of solutions to equations with zero right hand side is that their gradients satisfy a reverse Hölder inequality. The strength of the reverse Hölder inequality for the gradients of solutions to homogeneous equations determines the quality of sparse, weighted and L^p bounds that the general solutions satisfy. The exponent on the right hand side is usually taken to be 1 or 2 instead of the $1/2$ here, but it is a general property of reverse Hölder inequalities for solutions that if they hold for one exponent on the right hand side, then they hold for all them, see Appendix B of [4]. This is in strong contrast with the exponent on the left hand side, which cannot be changed freely.

Definition 2.3. Let Ω be a domain and let $w \in L^1_{loc}(\Omega)$ be non-negative. Let $q > 2$. We say w satisfies a q -weak reverse Hölder inequality in Ω if there exists a constant $C_{w,q}$ such that for all cubes $P \subset \mathbb{R}^n$

$$\left(\int_P 1_\Omega(x) w(x)^q dx \right)^{1/q} \leq C_{w,q} \left(\int_{2P} 1_\Omega(x) w(x)^{1/2} dx \right)^2.$$

If this only holds for P such that $2P \subset \Omega$, we say w satisfies a local q -weak reverse Hölder inequality in Ω . Similarly, if this only holds for P such that $P \cap \Omega \neq \emptyset$, we say w satisfies a boundary q -weak reverse Hölder inequality in Ω .

For the proof of the sparse bound, we will need a reverse Hölder inequality for gradients of differences of solutions rather than solutions themselves. For linear equations, there is of course no difference, but in the case of nonlinear equations, this seems to be an important point. To keep the exposition of linear and nonlinear cases unified, we set a definition to capture this discrepancy.

To state the next definition, we introduce a family of smoothed cubes. We let O_0 be a smooth domain such that

$$\frac{299}{100}[-1/2, 1/2]^n \subset O_0 \subset 3[-1/2, 1/2]^n.$$

Given any cube $Q \subset \mathbb{R}^n$, we let O_Q be the smooth domain such that if A_Q is the scaling and translation mapping $A_Q Q = [-1/2, 1/2]^n$, then $O_Q = A_Q^{-1} O_0$. Heuristically, O_Q is $3Q$ but with corners smoothed away.

Definition 2.4 (Admissible upper exponent). Let Ω be a domain. Let a be a (λ, Λ) -elliptic coefficient in Ω . Given a cube Q , define the class $\mathcal{U}(Q)$ as the family of pairs (u, v) such that $u, v \in W_0^{1,2}(\Omega)$ and

$$\operatorname{div} a(x, \nabla u(x)) = \operatorname{div} a(x, \nabla v(x))$$

holds in $O_Q \cap \Omega$ in the sense of Definition 2.2.

A number $q > 2$ is called an admissible upper exponent for a if for any cube Q and any pair $(u, v) \in \mathcal{U}(Q)$ the function

$$1_{2Q}|\nabla u - \nabla v|$$

satisfies a local or boundary q -weak reverse Hölder inequality in $\Omega \cap 2Q$. For an admissible upper exponent $q > 2$, we denote

$$N_{h,bdr}^\Omega(a, q) = \sup_Q \sup_{(u,v) \in \mathcal{U}(Q)} \sup_{P \subset 2Q: P \cap \Omega \neq \emptyset} \frac{\langle |\nabla u - \nabla v| \rangle_{P,q}}{\langle |\nabla u - \nabla v| \rangle_{2P,1/2}},$$

$$N_{h,loc}^\Omega(a, q) = \sup_Q \sup_{(u,v) \in \mathcal{U}(Q)} \sup_{P \subset 2Q: 2P \subset \Omega} \frac{\langle |\nabla u - \nabla v| \rangle_{P,q}}{\langle |\nabla u - \nabla v| \rangle_{2P,1/2}}.$$

where the supremum in Q is over all cubes.

Notice that $N_{h,bdr}^\Omega$ may be infinite a priori. For concrete applications of our results, verifying finiteness of this quantity is separated from the main sparse argument of Section 3. When proving the theorems in the introduction, we have three cases. For Theorem 1.1, we rely on global Meyers type estimates valid in Lipschitz domains, which we state and prove as Proposition 4.3. For Theorem 1.2, we use the boundary regularity results of Kinnunen and Zhou [25]. Finally, for Theorem 1.5, we can take advantage of the boundary regularity estimates of Escauriaza, Dong and Kim [14].

The second key property of solutions is an energy estimate. An energy estimate as below is always valid for $q = 2$, but in order to carry out a bootstrap argument, we write a generic definition. Unlike in the case of admissible upper exponent, the case of lower exponents is interesting only in the case of estimates holding up to the boundary. Local versions do not suffice for running our arguments.

Definition 2.5 (Admissible lower exponent). Let a be a (λ, Λ) -elliptic coefficient in Ω . A number $1 < q \leq 2$ is called an admissible lower exponent for a if there exists a finite number $N_l^\Omega(a, q)$ such that for all cubes P , for any source data $F \in L_{loc}^q(\Omega)$ and $f \in L_{loc}^{q_*}(\Omega)$, and for any solution $u \in W_0^{1,2}(O_P \cap \Omega)$ to (2.5) in $O_P \cap \Omega$ it holds

$$\|\nabla u\|_{L^q(O_P \cap \Omega)} \leq N_l^\Omega(a, q) \left(\int_{O_P \cap \Omega} |F(x)|^q dx + \left(\int_{O_P \cap \Omega} |f(x)|^{q_*} dx \right)^{q/q_*} \right)^{1/q}.$$

If $q_* \leq 1$, we require $f \equiv 0$.

3. A GENERAL RESULT

In this section, we prove the following general result.

Theorem 3.1 (Local sparse bound). *Let $\theta \in (0, 1)$, $0 < \lambda \leq \Lambda < \infty$ and Q_0 be a cube. Let a (λ, Λ) -elliptic coefficient a in $3Q_0$ be given. Let $q_l \in (1, 2]$ be an admissible lower exponent for a in the sense of*

Definition 2.5. Let $q_h \in (2, \infty)$ be an admissible upper exponent for a in the sense of Definition 2.4. Denote

$$A = N_l^\Omega(a, q_l), \quad B = N_{h,loc}^\Omega(a, q_h).$$

Let $p > q_l$ and assume that $F \in L^p(3Q_0; \mathbb{R}^n)$ and $f \in L^{p^*}(3Q_0)$. If f is not identically zero, assume $q_{l^*} > 1$. Let $u \in W_0^{1,2}(3Q_0)$ be a solution to

$$\operatorname{div} a(x, \nabla u(x)) = \operatorname{div} F(x) + f(x), \quad x \in 3Q_0$$

according to Definition 2.2. Let $g \in L^\infty(3Q_0)$ be non-negative.

Then there exists a $(1 - \theta)$ -sparse family \mathcal{P} of cubes P such that for all $P \in \mathcal{P}$ it holds $3P \subset 3Q_0$ and

$$\begin{aligned} \int_{Q_0} |\nabla u(x)| g(x) dx &\leq C \sum_{P \in \mathcal{P}} |P| \langle |F| \rangle_{3P, q_l} \langle g \rangle_{P, q'_h} \\ &\quad + C \sum_{P \in \mathcal{P}} |P| 3\ell(P) \langle |f| \rangle_{3P, q_{l^*}} \langle g \rangle_{P, q'_h} \end{aligned}$$

where

$$C = 9^{n/q_l} (A\Lambda + 1) AB \left(\frac{\|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)}}{\theta} \right)^{1/q_l}.$$

For the totality of this section, we fix the notation as in the hypothesis of Theorem 3.1. Clearly a will be a (λ, Λ) -elliptic coefficient in all subcubes $Q \subset 3Q_0$. For such a cube Q , we denote by u_Q the solution $u \in W_0^{1,2}(O_Q)$ to

$$\operatorname{div} a(x, \nabla u(x)) = \operatorname{div} F(x) + f(x)$$

in O_Q . We denote

$$L(Q) := \int_Q |\nabla u_Q(x)| g(x) dx.$$

The main ingredient of the proof is an iteration formula for this quantity.

Lemma 3.2. Let Q be a cube, $\theta \in (0, 1)$, and let notations and assumptions of Theorem 3.1 remain valid. Then there exists a family \mathcal{P} of pairwise disjoint cubes such that $3P \subset 3Q$ for all $P \in \mathcal{P}$; it holds

$$L(Q) \leq C |Q| (\langle |F| \rangle_{3Q, q_l} + 3\ell(Q) \langle |f| \rangle_{3Q, q_{l^*}}) \langle g \rangle_{Q, q'_h} + \sum_{P \in \mathcal{P}} L(P) \quad (3.1)$$

with

$$C = 9^{n/q_l} (A\Lambda + 1) AB \left(\frac{\|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)}}{\theta} \right)^{1/q_l}$$

and

$$|Q \setminus \bigcup \mathcal{P}| \geq (1 - \theta) |Q|.$$

Proof. For brevity, we write $u = 1_{O_Q}u_Q$. Then $u \in W^{1,2}(\mathbb{R}^n)$ as $u_Q \in W_0^{1,2}(O_Q)$. Consider the set

$$\Xi = \{x \in Q : M(|\nabla u|^{q_i})(x) > D^{q_i}\}.$$

Then

$$\Xi \cup \mathcal{N} \supset \{x \in Q : |\nabla u(x)|^{q_i} > D^{q_i}\}$$

for some $\mathcal{N} \subset \mathbb{R}^n$ with $|\mathcal{N}| = 0$. Denote

$$D = A \left(\frac{\|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)}}{3^{-n\theta}} \right)^{1/q_i} (\langle |F| \rangle_{3Q, q_i} + 3\ell(Q) \langle |f| \rangle_{3Q, q_i^*}).$$

We let \mathcal{P} be the family of maximal cubes $P \in \mathcal{D}(Q)$ such that

$$\langle |\nabla u|^{q_i} \rangle_{3P} > D^{q_i}.$$

Then $\cup \mathcal{P} = \Xi$ and for all $P \in \mathcal{P}$, it holds $\langle |\nabla u|^{q_i} \rangle_{3P} \leq 2^n D^{q_i}$. By (2.1) and the definition of the admissible lower exponent (Definition 2.5)

$$\begin{aligned} |\Xi| &\leq \frac{\|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)}}{D^{q_i}} \int_{3Q} |\nabla u(x)|^{q_i} dx \\ &\leq \frac{A^{q_i} \|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)}}{D^{q_i}} \left(\int_{3Q} |F(x)|^{q_i} dx + \left(\int_{3Q} |f(x)|^{q_i^*} dx \right)^{q_i/q_i^*} \right) \\ &\leq \theta |Q|. \end{aligned}$$

In addition, the family \mathcal{P} is pairwise disjoint by maximality. Hence it remains to prove the claimed inequality (3.1) for $L(Q)$.

We write

$$\begin{aligned} \int_Q |\nabla u(x)|g(x) dx &\leq \int_{Q \setminus \cup \mathcal{P}} |\nabla u(x)|g(x) dx \\ &\quad + \sum_{P \in \mathcal{P}} \int_P |\nabla u(x) - \nabla u_P(x)|g(x) dx + \sum_{P \in \mathcal{P}} \int_P |\nabla u_P(x)|g(x) dx \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

Now

$$\text{I} \leq D \int_{Q \setminus \cup \mathcal{P}} g(x) dx \leq D|Q| \langle g \rangle_{Q, q'}$$

by the fact $Q \setminus \cup \mathcal{P} = Q \setminus \Xi$ and by Hölder's inequality.

To estimate II, we first notice that by Hölder's inequality

$$\int_P |\nabla u_P(x) - \nabla u(x)|g(x) dx \leq \|\nabla u_P - \nabla u\|_{L^{q_h}(P)} \|g\|_{L^{q'_h}(P)}.$$

By construction, the equation

$$\operatorname{div} a(x, \nabla u_P(x)) = \operatorname{div} \tilde{F}(x) \tag{3.2}$$

with $\tilde{F}(x) = a(x, \nabla u(x))$ holds in O_P . Hence u and u_P are a pair in $\mathcal{U}(P)$ as in Definition 2.4. By the fact that q_h is an admissible upper

exponent in the sense of Definition 2.4, we invoke that definition to estimate

$$\begin{aligned} \|\nabla u_P - \nabla u\|_{L^{q_h}(P)} &\leq B|P|^{1/q_h} \langle |\nabla u_P - \nabla u| \rangle_{2P, q_l} \\ &\leq B|P|^{1/q_h} (\langle |\nabla u| \rangle_{2P, q_l} + \langle |\nabla u_P| \rangle_{2P, q_l}). \end{aligned}$$

Applying Definition 2.5 of admissible lower exponent to the equation (3.2) we bound further

$$\langle |\nabla u_P| \rangle_{2P, q_l} \leq (3/2)^{n/q_l} A\Lambda \langle |\nabla u| \rangle_{3P, q_l}.$$

Hence by maximality of the cubes P

$$\|\nabla u_P - \nabla u\|_{L^{q_l}(2P)} \leq 3^{n/q_l} (A\Lambda + 1)BD$$

and by Hölder's inequality

$$\text{II} \leq 3^{n/q_l} (A\Lambda + 1)BD \sum_{P \in \mathcal{P}} |P|^{1/q_h} \|g\|_{L^{q'_h}(P)} \leq 3^{n/q_l} (A\Lambda + 1)BD |Q| \langle g \rangle_{Q, q'_h}.$$

Finally, by definition

$$\text{III} = \sum_j L(P_j).$$

This concludes the proof. \square

Proof of Theorem 3.1. We construct the sparse family by a recursion. Let $\mathcal{S}_0 = \{Q_0\}$. For $j \geq 0$ given, we apply Lemma 3.2 to each $Q \in \mathcal{S}_j$ to recover a pairwise disjoint family $\mathcal{P}(Q)$ as in Lemma 3.2. Let $\mathcal{S}_{j+1} = \bigcup_{Q \in \mathcal{S}_j} \mathcal{P}(Q)$. By induction, we have constructed a \mathcal{S}_j for each $j \geq 0$. Denote $\mathcal{P} = \bigcup_{j=0}^{\infty} \mathcal{S}_j$. By Lemma 3.2, for all $j \geq 0$

$$\begin{aligned} \int_{Q_0} |\nabla u(x)|g(x) dx &= L(Q_0) \\ &\leq C \sum_{k=0}^j \sum_{P \in \mathcal{S}_k} |P| (\langle |F| \rangle_{3P, q_l} + 3\ell(Q) \langle |f| \rangle_{3Q, q_l^*}) \langle g \rangle_{P, q'_h} + \sum_{P \in \mathcal{S}_{j+1}} L(P). \end{aligned}$$

To complete the proof, we show that the family \mathcal{P} is sparse and that the right-most sum tends to zero as $j \rightarrow \infty$.

For $P \in \mathcal{S}_j$, we set

$$E_P := P \setminus \left(\bigcup_{Q \in \mathcal{S}_{j+1}} Q \right) = P \setminus \left(\bigcup_{k=j+1}^{\infty} \bigcup_{Q \in \mathcal{S}_k} Q \right)$$

where the second equality follows by the construction: a cube $Q \in \mathcal{S}_k$ with $k \geq j+1$ is a subcube of a cube Q' in \mathcal{S}_{j+1} . It is then clear that

$$\sum_{P \in \mathcal{P}} 1_{E_P} \leq 1.$$

Further, by Lemma 3.2, for $P \in \mathcal{S}_j$

$$|E_P| = \left| P \setminus \left(\bigcup_{Q \in \mathcal{S}_{j+1}} Q \right) \right| \geq (1 - \theta)|P|.$$

Hence \mathcal{S} is $(1 - \theta)$ -sparse.

By the definition of the sets E_P , for each $j \geq 0$

$$\begin{aligned} \left| \bigcup_{P \in \mathcal{S}_{j+1}} P \right| &= \left| \bigcup_{P \in \mathcal{S}_j} \bigcup_{\substack{P' \subset P \\ P' \in \mathcal{S}_{j+1}}} P' \right| = \left| \bigcup_{P \in \mathcal{S}_j} P \setminus E_P \right| \leq \sum_{P \in \mathcal{S}_j} |P \setminus E_P| \\ &= \sum_{P \in \mathcal{S}_j} (|P| - |E_P|) \leq \theta \sum_{P \in \mathcal{S}_j} |P| = \theta \left| \bigcup_{P \in \mathcal{S}_j} P \right| \end{aligned}$$

where the last equality used that the cubes P are pairwise disjoint by Lemma 3.2. Iterating the estimate in the display above, we conclude

$$\left| \bigcup_{P \in \mathcal{S}_j} P \right| \leq \theta^j |Q_0|.$$

Finally by Hölder's inequality and the definition of the lower admissible exponent (Definition 2.5)

$$\begin{aligned} L(P) &\leq \|\nabla u_P\|_{L^{q_i}(P)} \|g\|_{L^{q'_i}(P)} \\ &\leq C \|g\|_\infty |P| \left(\| |F| \|_{3P, q_i}^{q_i} + 9\ell(P)^{q_i} \| |f| \|_{3P, q_{i^*}}^{q_i} \right)^{1/q_i} \\ &\leq C \int_P (M(1_{3Q_0} |F|^{q_i})^{1/q_i} + M_1(1_{3Q_0} |f|^{q_{i^*}})^{1/q_{i^*}}). \end{aligned}$$

Summing and applying Hölder's inequality in the integral, we bound

$$\begin{aligned} &\sum_{P \in \mathcal{S}_{j+1}} L(P) \\ &\leq C \left| \bigcup_{P \in \mathcal{S}_{j+1}} P \right|^{1/p'} \left\| M(1_{3Q_0} |F|^{q_i})(x)^{1/q_i} + M_{q_{i^*}}(1_{3Q_0} |f|^{q_{i^*}})(x)^{1/q_{i^*}} \right\|_{L^p(Q_0)} \\ &\leq K \theta^j \end{aligned}$$

where the constant K is finite by the maximal function theorem, the fractional maximal function theorem and by the assumption on the source data. Sending $j \rightarrow \infty$, we conclude the proof. \square

Next we discuss a global version of the sparse bound. Under the current set of definitions, the proof is essentially identical as the difference between local and boundary estimates is captured by the question whether or not the constant $N_{h, bdr}^\Omega(a, q)$ (as opposed to $N_{h, loc}^\Omega(a, q)$) from Definition 2.4 is finite. To address the presence of the boundary of the domain, we hence only have to modify the notation. Given a cube $Q \subset 3Q_0$ and the reference domain $\Omega \subset Q_0$, we let U_Q be the smooth domain such that if A_Q is a scaling and translation with $A_Q Q = [-1/2, 1/2]^n$, then $U_Q = \Omega \cap A_Q^{-1}(O_0)$. We denote by u_Q^Ω the solution $u \in W_0^{1,2}(U_Q)$ with source data F and f . We denote

$$L^\Omega(Q) := \int_Q |\nabla u_Q^\Omega(x)| g(x) dx.$$

Replacing u_P by u_P^Ω in the proof of Theorem 3.1, we see that the same argument works mutatis mutandis with the constant B replaced by

$N_{h,bdr}^\Omega(a, q_h)$. To apply Theorem 3.3 to real problems, one still has to prove that $N_{h,bdr}^\Omega(a, q_h)$ appearing on the right hand side of the sparse bound is finite, and this is where more geometric considerations involving density of the complement of the domain or flattening the boundary appear. That is done in Section 4 and Section 5. More precisely, the geometric input comes through the application of the main theorem in [25] and Proposition 4.3. With this discussion, we state the final abstract theorem.

Theorem 3.3 (Global sparse bound). *Under the hypotheses of Theorem 3.1, when $\Omega \subset Q_0$ is a domain and the additional assumption*

$$N_{h,bdr}^\Omega(a, q_h) < \infty$$

holds, we have that

$$\begin{aligned} \int_{\Omega} |\nabla u(x)|g(x) dx \leq C \sum_{P \in \mathcal{P}} |P| \langle |F| \rangle_{3P, q_l} \langle g \rangle_{P, q'_h} \\ + C \sum_{P \in \mathcal{P}} |P| 3\ell(P) \langle |f| \rangle_{3P, q_{l^*}} \langle g \rangle_{P, q'_h} \end{aligned}$$

where

$$\begin{aligned} C = 9^{n/q_l} (N_l^\Omega(a, q_l)\Lambda + 1) N_l^\Omega(a, q_l) N_{h,bdr}^\Omega(a, q_h) \\ \times \left(\frac{\|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)}}{\theta} \right)^{1/q_l}. \end{aligned}$$

4. APPLICATIONS

4.1. Nonlinear equations. In this section, we apply the abstract Theorem 3.3 to two concrete partial differential equations. We start with the results for nonlinear equations. Our remaining task is to identify the admissible upper and lower exponents according to Definition 2.4 and Definition 2.5.

Proposition 4.1. *Let Q be a cube and Ω a domain. Let $0 < \lambda \leq \Lambda < \infty$, a (λ, Λ) -elliptic coefficient a be given, and let $u \in W_0^{1,2}(O_Q \cap \Omega)$, $F \in L^2(O_Q; \mathbb{R}^n)$ and $f \in L^{2^*}(O_Q)$ be as in Definition 2.2. Then*

$$\int_{O_Q \cap \Omega} |\nabla u(x)|^2 dx \leq \frac{2}{\lambda^2} \int_{O_Q} |F(x)|^2 dx + \frac{2c_n^2}{\lambda^2} \left(\int_{O_Q} |f(x)|^{2^*} dx \right)^{2/2^*}.$$

In other words, 2 is an admissible lower exponent for a in Ω with $N_l^\Omega(a, 2) \leq (1 + c_n)\sqrt{2}/\lambda$ where c_n is the constant from $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^}(\mathbb{R}^n)$ Sobolev inequality.*

Proof. We identify all the three functions u , f and F with their extensions by zero to globally defined functions, that is, we set $u(x) = f(x) = F(x) = 0$ for all $x \notin O_Q \cap \Omega$. Because $u \in W_0^{1,2}(O_Q \cap \Omega)$, we have that

the corresponding extension is in $W^{1,2}(\mathbb{R}^n)$. By strong monotonicity and the definition of solution

$$\begin{aligned} \lambda \int_{3Q} |\nabla u(x)|^2 dx &\leq \int_{3Q} a(x, \nabla u(x)) \cdot \nabla u(x) dx \\ &= \int_{3Q} F(x) \cdot \nabla u(x) dx + \int_{3Q} f(x)u(x) dx. \end{aligned}$$

By Hölder's inequality and by Sobolev inequality, we have for the zeroth order term on the right hand side

$$\begin{aligned} \int_{3Q} f(x)u(x) dx &\leq \|u\|_{L^{2^*}(3Q)} \|f\|_{L^{2^*}(3Q)} \leq c_n \|\nabla u\|_{L^2(3Q)} \|f\|_{L^{2^*}(3Q)} \\ &\leq \frac{\varepsilon}{2} \int_{3Q} |\nabla u(x)|^2 dx + \frac{c_n^2 \|f\|_{L^{2^*}(3Q)}^2}{2\varepsilon}, \end{aligned}$$

where the last step followed for any $\varepsilon > 0$ from Young's inequality. The constant c_n is the constant of Sobolev inequality. By Young's inequality again, we have for the first order term on the right hand side

$$\int_{3Q} F(x) \cdot \nabla u(x) dx \leq \frac{\varepsilon}{2} \int_{3Q} |\nabla u(x)|^2 dx + \frac{1}{2\varepsilon} \int_{3Q} |F(x)|^2 dx.$$

Choosing $\varepsilon = \lambda/2$, we conclude the proof. \square

A classical result of Meyers [33] shows that gradients of solutions to homogeneous problems satisfy a reverse Hölder inequality. For completeness, we provide below a short argument for the inequality exactly in the form in which we need it. Note that this estimate holds up to the boundary. We assume a Lipschitz condition on Ω to have control over $|3Q \cap \Omega^c|/|3Q|$ for cubes Q with $2Q \cap \Omega^c \neq \emptyset$. This gives further a control over the constant appearing in Sobolev inequality for functions u on $3Q$ vanishing in Ω^c .

Proposition 4.2. *Let Ω be a Lipschitz domain and let Q_0 be a cube. Let $0 < \lambda \leq \Lambda < \infty$, let a be a (λ, Λ) -elliptic coefficient, and let $u \in W_0^{1,2}(\Omega)$. Assume that $u \in W_0^{1,2}(\Omega)$ is a solution to*

$$-\operatorname{div} a(x, \nabla u(x)) = 0$$

in $O_{Q_0} \cap \Omega$ with some boundary data (not necessarily zero at $\partial O_{Q_0} \setminus \partial \Omega$). Then there exists $q = q(n, \Lambda/\lambda, \Omega) > 2$ and $C = C(q, n, \Lambda/\lambda, \Omega)$ such that

$$\left(\int_{Q_0} 1_\Omega(x) |\nabla u(x)|^q dx \right)^{1/q} \leq C \left(\int_{2Q_0} 1_\Omega(x) |\nabla u(x)|^{1/2} dx \right)^2.$$

Proof. We identify u with its extension by zero to a global Sobolev function. Given any cube Q such that $3Q \subset 3Q_0$, let $\varphi \in C^\infty(\mathbb{R}^n)$ be such that $\|\nabla \varphi\|_\infty \leq 2\ell(Q)$ and $1_Q \leq \varphi \leq 1_{2Q}$. Then $u\varphi^2 \in W_0^{1,2}(\Omega)$ is a valid test function. We obtain by strong monotonicity and product

rule

$$\begin{aligned} \lambda \int |\nabla u(x)|^2 \varphi(x)^2 dx &\leq \int a(x, \nabla u(x)) \cdot \nabla u(x) \varphi(x)^2 dx \\ &= \int a(x, \nabla u(x)) \cdot \nabla [u(x) \varphi(x)^2] dx \\ &\quad - 2 \int u(x) \varphi(x) a(x, \nabla u(x)) \cdot \nabla \varphi(x) dx \end{aligned}$$

where the first term is zero by the equation and the second term is bounded by

$$\frac{1}{\varepsilon} \int u(x)^2 |\nabla \varphi(x)|^2 dx + \varepsilon \Lambda^2 \int \varphi(x)^2 |\nabla u(x)|^2 dx.$$

Setting $\varepsilon = \lambda/(2\Lambda^2)$, we conclude

$$\int |\nabla u(x)|^2 \varphi(x)^2 dx \leq \frac{4\Lambda^2}{\lambda^2} \int u(x)^2 |\nabla \varphi(x)|^2 dx =: T.$$

If $2Q \subset \Omega$, we may replace u by $u - \langle u \rangle_{2Q}$ as $u - \langle u \rangle_{2Q}$ is a solution and $(u - \langle u \rangle_{2Q})\varphi$ is a test function. We may then apply Sobolev–Poincaré inequality on T . If $2Q \cap \Omega^c \neq \emptyset$, we may apply Sobolev inequality for Sobolev functions vanishing in an ample portion of $3Q$. Altogether, we obtain the bound

$$\left(\int_Q 1_\Omega(x) |\nabla u(x)|^2 dx \right)^{1/2} \leq \frac{2c_{n,\Omega}\Lambda}{\lambda} \left(\int_{3Q} 1_\Omega(x) |\nabla u(x)|^{2^*} dx \right)^{1/2^*}$$

where $c_{n,\Omega}$ is larger of the constants of Sobolev–Poincaré inequalities with vanishing at the boundary or with mean value zero. This estimate holds for all cubes Q with $3Q \subset 3Q_0$. Consequently, we may apply Gehring’s lemma (Theorem 3.22 in [6]) together with the Appendix B of [4] to conclude that there exists $q = q(\Lambda/\lambda, n, \Omega)$ and $C = C(q)$ such that for all cubes Q with $3Q \subset 3Q_0$

$$\left(\int_Q 1_\Omega(x) |\nabla u(x)|^q dx \right)^{1/q} \leq C \left(\int_{2Q} 1_\Omega(x) |\nabla u(x)|^{1/2} dx \right)^2.$$

This is the claimed inequality. \square

We first remark that the number q in the above proposition is an admissible upper exponent in the sense of Definition 2.4 for linear coefficient functions a in Ω with $N_{h,bdr}^\Omega(a, q) \leq C(n, q, \Lambda/\lambda, \Omega)$. Indeed, the difference of two solutions to the same linear inhomogeneous equation is a solution to the homogeneous linear equation with the same coefficient matrix, and so we can apply the inequality to the difference appearing in the Definition 2.4. This proves that Definition 2.4 applies to the difference of a pair of solutions to any fixed linear equation. For the case of nonlinear equations, the difference of two solutions to the same inhomogeneous equation need not be a solution to the homogeneous equation with the same coefficient function. However, in case the

nonlinear equation admits a linearization, we can show that the difference is a solution to a homogeneous equation with similar structure. Precisely, we can prove the following.

Proposition 4.3. *Let $0 < \lambda \leq \Lambda < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let a be a (λ, Λ) -elliptic coefficient in Ω . Then, in the sense of Definition 2.4, there exists an admissible upper exponent $q = q(n, \Lambda/\lambda, \Omega) > 2$ and a finite constant $C = C(n, q, \Lambda/\lambda, \Omega)$ such that $N_{h,bdr}^\Omega(a, q) \leq C$.*

Proof. Let u and v be as in the definition of admissible exponent. Then

$$\operatorname{div} a(x, \nabla u(x)) - \operatorname{div} a(x, \nabla v(x)) = 0$$

in the weak sense. In case a is linear, we immediately see that $u - v$ is a solution and the claim follows from the Meyers' estimate, Proposition 4.2. In case a is not necessarily linear, we use the fact that a is continuously differentiable in the second variable. Then

$$\begin{aligned} a(x, \nabla u) - a(x, \nabla v) &= \int_0^1 \nabla_\xi a(x, t\nabla u(x) + (1-t)\nabla v(x)) (\nabla u(x) - \nabla v(x)) dt, \end{aligned}$$

and we note that $u - v$ is solution to

$$\operatorname{div} A \nabla (u - v) = 0$$

where

$$A(x) = \int_0^1 \nabla_\xi a(x, t\nabla u(x) + (1-t)\nabla v(x)) dt$$

is uniformly elliptic matrix with entries measurable in x . Indeed, $x \mapsto t\nabla u(x) + (1-t)\nabla v(x)$ is measurable and the Jacobian matrix of a is continuous. Hence the composition is measurable. We can invoke Meyers' estimate from Proposition 4.2 to complete the proof. \square

Proof of Theorem 1.1. By Proposition 4.1, 2 is an admissible lower exponent. By Proposition 4.3, there exists an upper admissible exponent $q > 2$ so that $N_{h,bdr}^\Omega(a, q)$ is finite. By Theorem 3.3, the claim follows. \square

4.2. Linear equations. In the linear setting, we dispose over several powerful tools that we have not been able to use in the case of nonlinear equations. As we focus on the linear problem, we fix a matrix $A : \Omega \rightarrow \mathbb{R}^{n \times n}$ such that for all $x, \xi \in \mathbb{R}^n$

$$a(x, \xi) = A(x)\xi.$$

Then our assumptions on the coefficient a convert into the standard ellipticity and boundedness assumptions on the matrix A with measurable coefficients.

Proof of Theorem 1.2. By Theorem 1.5 in Kinnunen–Zhou [25], we see that any $q > 2$ is an admissible upper exponent and $N_{h,bdr}^\Omega(A, q)$ is finite, that is, the estimates hold up to the boundary. By Theorem 2.1 in Di Fazio [13], we see that any $p > 1$ is an admissible lower exponent. The claim follows by Theorem 3.3. \square

Proof of Corollary 1.3. Given $w \in A_2$, there exists $\varepsilon = \varepsilon([w]_{A_2}, n) \in (0, 1)$ such that it holds $w \in A_{2-\varepsilon}$ (Corollary 9.2.6 [19]) and $w \in \text{RH}_{1+\varepsilon}$ with $[w]_{\text{RH}_{1+\varepsilon}} = C(n, [w]_{A_2})$ (Theorem 9.2.2 [19]). Choose $q' > 1$ so that $2/q' > 2 - \varepsilon$ and $(q/q')' = 1/(2 - q') < 1 + \varepsilon$. Concretely, we can set $q' = 2/(2 - \varepsilon^2)$. By Theorem 1.2, a (q', q') sparse estimate holds. Using the fact

$$\|f\|_{L^2(\mathbb{R}^n, wdx)} = \sup_{\|g\|_{L^2(w^{-1}dx)} \leq 1} \left| \int f(x)g(x) dx \right|,$$

we can invoke Proposition 6.1, to conclude from the sparse estimate above that

$$\|1_\Omega \nabla u\|_{L^2(\mathbb{R}^n, wdx)} \leq C(n, A, \Omega, [w]_{A_2}) \|F\|_{L^2(\mathbb{R}^n, wdx)}.$$

As this bound holds for all weights $w \in A_2$, we can apply the Rubio de Francia extrapolation theorem (Theorem 9.5.3 in [20]). We hence conclude that for all p and all $w \in A_p$

$$\|\nabla u\|_{L^p(\Omega, wdx)} \leq C(n, p, A, \Omega, [w]_{A_p}) \|F\|_{L^p(\mathbb{R}^n, wdx)}.$$

Restricting the attention to F vanishing in Ω^c and using density of $C_c^\infty(\Omega)$ in $L^p(\Omega, wdx)$, we conclude the proof. \square

5. CASE OF DINI COEFFICIENTS

In this section, we discuss the case where $A : \Omega \rightarrow \mathbb{R}^{n \times n}$ is assumed to be of Dini mean oscillation as defined in the introduction. We only see this assumption through the results quoted from [14]. The following Proposition follows by Lemma 2.11 of [14] and the well-known properties of reverse Hölder inequalities, see Appendix B in [4].

Proposition 5.1. *Let Ω be a C^2 -domain and let $v \in W_0^{1,2}(\Omega)$ be a solution to*

$$\operatorname{div} A(x) \nabla v(x) = 0$$

in $O_P \cap \Omega$ and set $v = 0$ in Ω^c . Then

$$\|\nabla v\|_{L^\infty(P \cap \Omega)} \leq C \left(\int_{2P} 1_\Omega(x) |\nabla v(x)|^{1/2} dx \right)^2 \quad (5.1)$$

for $C = C(\lambda, \Lambda, n, \text{Dini}_A, \Omega)$.

The second strong estimate we need is Theorem 1.9 from [14].

Proposition 5.2. *Let Ω be a C^2 -domain and let P be a cube. Let $F \in L^2(O_P \cap \Omega)$. Let $v \in W_0^{1,2}(O_P \cap \Omega)$ be a solution to*

$$\operatorname{div} A(x) \nabla v(x) = \operatorname{div} F(x)$$

in $O_P \cap \Omega$. Assume $\omega_A(r) \leq c|\log r|^{-2}$ for some c and all $r \in (0, 1/2)$. Then for all $\mu > 0$

$$|\{x \in O_P \cap \Omega : |\nabla v(x)| > \mu\}| \leq \frac{C}{\mu} \int_{O_P \cap \Omega} |F(x)| dx \quad (5.2)$$

for $C = C(\lambda, \Lambda, n, \text{Dini}_A, \Omega)$.

Using the proposition above, we can prove a weak type (1,1) bound for an auxiliary operator. This operator takes the place of the auxiliary maximal function of Lerner and Ombrosi [32]. Given a cube Q , we denote by u_Q the function such that as a restriction $u_Q \in W_0^{1,2}(O_Q \cap \Omega)$ is a solution to

$$\text{div } A(x) \nabla u(x) = \text{div } F(x)$$

in $O_Q \cap \Omega$ and $u_Q = 0$ in $(O_Q \cap \Omega)^c$.

Proposition 5.3. Fix a cube Q . For a cube $P \subset O_Q \cap \Omega$, we write $w_P = u_Q - u_P$. We define an auxiliary function for all $x \in 3Q$ as

$$S(x) = \sup_{P \in \mathcal{D}(3Q)} 1_P(x) \sup_{y, y' \in P} |\nabla w_P(y) - \nabla w_P(y')|.$$

Then there exists a constant $C = C(\lambda, \Lambda, n, \text{Dini}_A, \Omega)$ such that for all $\mu > 0$

$$|\{x \in O_Q \cap \Omega : S(x) > \mu\}| \leq \frac{C}{\mu} \int_{O_Q \cap \Omega} |F(x)| dx. \quad (5.3)$$

Proof. As always, we extend all the functions appearing in the proof by zero to be globally defined functions. Note that w_P solves a homogeneous equation in $O_P \cap \Omega$. By (5.1), for any $x \in O_Q \cap \Omega$

$$\begin{aligned} S(x) &\leq C \sup_{P \in \mathcal{D}(3Q)} 1_P(x) \left(\int_{2P} |\nabla w_P(y)|^{1/2} dy \right)^2 \\ &\leq C \sup_{P \in \mathcal{D}(3Q)} 1_P(x) \left(\int_{2P} |\nabla u_P(y)|^{1/2} dy \right)^2 + CM(1_{O_Q \cap \Omega} |\nabla u|^{1/2})^2(x). \end{aligned}$$

By Kolmogorov's inequality (Exercise 2.1.5 in [19]) and Proposition 5.2 applied in O_P , the first term is bounded by maximal function of F . Hence

$$\begin{aligned} \|S\|_{L^{1,\infty}(O_Q \cap \Omega)} &\leq C \|M(1_{O_Q \cap \Omega} F)\|_{L^{1,\infty}(O_Q \cap \Omega)} \\ &\quad + C \|M(1_{O_Q \cap \Omega} |\nabla u|^{1/2})^2\|_{L^{1,\infty}(O_Q \cap \Omega)}. \end{aligned}$$

Because the maximal function is bounded $L^{2,\infty}(\mathbb{R}^n) \rightarrow L^{2,\infty}(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)$, we conclude by an application of Proposition 5.2 in O_Q

$$\begin{aligned} \|S\|_{L^{1,\infty}(O_Q \cap \Omega)} &\leq C \|M(1_{O_Q \cap \Omega} F)\|_{L^{1,\infty}(O_Q \cap \Omega)} + C \|\nabla u\|_{L^{1,\infty}(O_Q \cap \Omega)} \\ &\leq C \int_{O_Q \cap \Omega} |F(x)| dx. \end{aligned}$$

This completes the proof. \square

Now we are in a position to prove the strong iteration lemma for u solving an equation with Dini continuous coefficients. We denote again

$$L(Q) := \int_Q |\nabla u_Q(x)| g(x) dx.$$

Lemma 5.4. *Let $\theta \in (0, 1)$ and let Q be a cube. Let $g \in L^\infty(3Q)$. Then there exists a family \mathcal{P} of pairwise disjoint cubes $P \subset 3Q$ such that*

$$L(Q) \leq C|Q|(|F|)_{3Q,1} \langle g \rangle_{Q,1} + \sum_{P \in \mathcal{P}} L(P) \quad (5.4)$$

with $C = C(\lambda, \Lambda, n, \text{Dini}_A, \Omega, \theta)$ and

$$|Q \setminus \bigcup \mathcal{P}| \geq (1 - \theta)|Q|.$$

Proof. For brevity, we write $u = u_Q$. Consider the set

$$\Xi = \{x \in Q : \max(|\nabla u(x)|, S(x), M(1_{3Q}F)(x)) > D\}$$

where $D = 12^{n+1} \max(C_w, C_S, \|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)}) \langle |F| \rangle_{3Q,1} / \theta$ and C_w is the constant from (5.2) and C_S is the constant from (5.3). We let \mathcal{P} be the family of maximal cubes $P \in \mathcal{D}(Q)$ such that

$$\frac{|P \cap \Xi|}{|P|} > 2^{-n-1}.$$

Then

$$\bigcup_{P \in \mathcal{P}} P \supset \Xi. \quad (5.5)$$

Now

$$|\bigcup \mathcal{P}| \leq \sum_{P \in \mathcal{P}} |P| \leq 2^{n+1} \sum_{P \in \mathcal{P}} |P \cap \Xi| = 2^{n+1} |\bigcup_{P \in \mathcal{P}} P \cap \Xi| \leq 2^{n+1} |\Xi|$$

and by (5.2), (5.3) and the weak (1, 1) estimate of the maximal function,

$$|\Xi| \leq \frac{3(C_w + C_S + \|M\|_{L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)})}{D} \int_{3Q} |F(x)| dx \leq \theta |Q|.$$

Hence it remains to prove the claimed inequality (5.4) for $L(Q)$.

We write

$$\begin{aligned} \int_Q |\nabla u(x)| g(x) dx &\leq \int_{Q \setminus \bigcup \mathcal{P}} |\nabla u(x)| g(x) dx \\ &+ \sum_{P \in \mathcal{P}} \int_P |\nabla u(x) - \nabla u_P(x)| g(x) dx + \sum_{P \in \mathcal{P}} \int_P |\nabla u_P(x)| g(x) dx \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

Now by definition of the cubes P

$$\text{I} \leq D \int_{Q \setminus \bigcup \mathcal{P}} g(x) dx \leq D|Q| \langle g \rangle_{Q,1}.$$

Next we estimate II. By maximality of P , it always holds

$$\frac{|P \cap \Xi|}{|P|} \leq \frac{1}{2}. \quad (5.6)$$

Let $N = 4 \cdot 3^n C_w$. Recall that u_P is identically zero in $(O_P \cap \Omega)^c$. Hence by (5.2)

$$\begin{aligned} & |\{x \in P : |\nabla u_P(x)| > ND\}| \\ & \leq |\{x \in O_P \cap \Omega : |\nabla u_P(x)| > ND\}| \leq \frac{C_w}{DN} \int_{3P} |F(x)| dx \\ & \leq \frac{3^n C_w}{DN} |P| \inf_{x \in P} M(1_{3Q} F)(x) \leq \frac{3^n C_w}{N} |P| \leq \frac{1}{4} |P| \end{aligned}$$

where we used the fact $|P \setminus \Xi| \geq |P|/2$, following by (5.6), to bound the maximal function. Hence there exists $x_P \in P \setminus \Xi$ with

$$\max(|\nabla u_P(x_P)|, |\nabla u(x_P)|) \leq ND.$$

Denote $w_P = u_P - u$. We write

$$\begin{aligned} \int_P |\nabla u_P(x) - \nabla u(x)| g(x) dx &= \int_P |\nabla w_P(x)| g(x) dx \\ &\leq |\nabla w_P(x_P)| \int_P g(x) dx + \int_P |\nabla w_P(x) - \nabla w_P(x_P)| g(x) dx. \end{aligned}$$

Here

$$\begin{aligned} \int_P |\nabla w_P(x) - \nabla w_P(x_P)| g(x) dx &\leq \inf_{x \in P} S(x) \langle g \rangle_{P,1} |P| \\ &\leq |P| S(x_P) \langle g \rangle_{P,1} \leq D |P| \langle g \rangle_{P,1} \end{aligned}$$

and

$$|\nabla w_P(x_P)| \leq |\nabla u_P(x_P)| + |\nabla u(x_P)| \leq 2ND.$$

This concludes the estimate for II.

Finally, by definition

$$\text{III} = \sum_j L(P_j).$$

This concludes the proof. \square

Proof of Theorem 1.4. The result follows by iterating Lemma 5.4. See the iteration at the end of the proof of Theorem 3.1, which is an identical argument. \square

Proof of Corollary 1.5. This follows at once by Theorem 1.4 and Proposition 6.1 with $r = 1 = s'$. Indeed for $\sigma = w^{1-p'}$ and any $g \in L^\infty(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n, \sigma dx)$, we get

$$\begin{aligned} \int 1_\Omega |\nabla u(x)| |g(x)| dx &\leq C \sum_P |P| \langle 1_{3Q_0} |F| \rangle_{3P,1} \langle 1_{3Q_0} g \rangle_{3P,1} \\ &\leq C [w]_{A_p}^{\max(1, \frac{1}{p-1})} \|1_{3Q_0} F\|_{L^p(\mathbb{R}^n, w dx)} \|g\|_{L^{p'}(\mathbb{R}^n, \sigma dx)}. \end{aligned}$$

The claim follows by taking supremum over the family of all g as above and satisfying $\|g\|_{L^{p'}(\mathbb{R}^n, \sigma dx)} \leq 1$. \square

6. SPARSE FORMS

We recall here how the sparse bounds imply weighted bounds. Recall that a locally integrable $w \geq 0$ is in RH_s with $s \geq 1$ if there is a constant $[w]_{RH_s}$ such that for all cubes $P \subset \mathbb{R}^n$

$$\left(\int_P w(x)^s dx \right)^{1/s} \leq [w]_{RH_s} \int_P w(x) dx.$$

The following proposition is essentially Proposition 6.4 in [5]. Although the exact wording differs slightly from that in [5], an inspection of the proof quickly shows that this statement follows from the very same proof. For the reader's convenience, we outline the short proof of the case $r = s' = 1$ below.

Proposition 6.1. *Assume $1 \leq r \leq 2 < s \leq \infty$ and $\theta \in (0, 1)$. Let $p \in (r, s)$, with the understanding $\infty' = 1$ and $1' = \infty$. Then there exists a finite constant C such that the following holds. If \mathcal{P} is a θ -sparse family, then for all non-negative test functions $f, g \in C_c(\mathbb{R}^n)$*

$$\begin{aligned} \sum_P |P| \langle f \rangle_{3P, r} \langle g \rangle_{3P, s'} \\ \leq C ([w]_{A_{p/r}} [w]_{RH_{(s/r)'}})^\beta \|f\|_{L^p(\mathbb{R}^n, w dx)} \|g\|_{L^{p'}(\mathbb{R}^n, w^{1-p'} dx)} \end{aligned}$$

where

$$\beta = \max\left(\frac{1}{p-r}, \frac{s-1}{s-p}\right).$$

We do not need the exact quantitative form except for the proof of Corollary 1.5. In that case, we have $r = s' = 1$. We reproduce the simple proof modulo quoting some basic results in this special case. First, by the Three Lattice Theorem (Theorem 3.1 in [31]), it suffices to bound

$$\sum_P |P| \langle f \rangle_{P, 1} \langle g \rangle_{P, 1} = \int \left(\sum_P 1_P(x) \langle f \rangle_{P, 1} \right) g(x) dx$$

with all cubes P in the sparse family being dyadic. By the Rubio de Francia extrapolation theorem (Theorem 9.5.3 in [20]), it suffices to deal with the case $p = 2$. Denoting $\sigma = w^{-1}$, we write

$$\langle f \rangle_{P, 1} = \frac{\sigma(P)}{|P|} \left(\frac{1}{\sigma(P)} \int_P f(x) \sigma(x)^{-1} \sigma(x) dx \right)$$

and

$$\langle g \rangle_{P, 1} = \frac{w(P)}{|P|} \left(\frac{1}{w(P)} \int_P g(x) w(x)^{-1} w(x) dx \right).$$

Here for a measurable set E and a non-negative locally integrable function v , we denote

$$v(E) = \int_E v(x) dx.$$

Define

$$M^\sigma f(x) := \sup_{P \in \mathcal{D}} 1_P(x) \langle f \rangle_P^\sigma := \sup_{P \in \mathcal{D}} \frac{1_P(x)}{\sigma(P)} \int_P f(x) \sigma(x) dx.$$

Then the sparse form can be bounded as

$$\begin{aligned} & \sum_P |P| \langle f \rangle_{P,1} \langle g \rangle_{P,1} \\ & \leq \sum_P |E_P| \frac{w(P)\sigma(P)}{|P|^2} \inf_{x \in E_P} [M^\sigma(f\sigma^{-1})(x) \cdot M^w(gw^{-1})(x)] \\ & \leq [w]_{A_2} \sum_P \int_{E_P} M^\sigma(f\sigma^{-1})(x) M^w(gw^{-1})(x) dx \\ & = [w]_{A_2} \int M^\sigma(f\sigma^{-1})(x) w(x)^{-1/2} \cdot M^w(gw^{-1})(x) w(x)^{1/2} dx \\ & \leq [w]_{A_2} \|M^\sigma(f\sigma^{-1})\|_{L^2(\mathbb{R}^n, \sigma dx)} \|M^w(gw^{-1})\|_{L^2(\mathbb{R}^n, w dx)} \\ & \leq 4[w]_{A_2} \|f\sigma^{-1}\|_{L^2(\mathbb{R}^n, \sigma dx)} \|gw^{-1}\|_{L^2(\mathbb{R}^n, w dx)} \\ & = 4[w]_{A_2} \|f\|_{L^2(\mathbb{R}^n, w dx)} \|g\|_{L^2(\mathbb{R}^n, \sigma dx)}. \end{aligned}$$

The penultimate inequality used the inhomogeneous dyadic maximal inequality, which holds for all weights with constant 2 (see for instance Theorem 15.1 from [31]).

REFERENCES

- [1] K. Adimurthi, T. Mengesha, and N. C. Phuc. Gradient weighted norm inequalities for linear elliptic equations with discontinuous coefficients. *Appl. Math. Optim.*, 83(1):327–371, 2021. [4](#)
- [2] D. Beltran and L. Cladek. Sparse bounds for pseudodifferential operators. *J. Anal. Math.*, 140(1):89–116, 2020. [2](#)
- [3] C. Benea and C. Muscalu. Sparse domination via the helicoidal method. *Rev. Mat. Iberoam.*, 37(6):2037–2118, 2021. [2](#)
- [4] F. Bernicot, T. Coulhon, and D. Frey. Gaussian heat kernel bounds through elliptic Moser iteration. *J. Math. Pures Appl. (9)*, 106(6):995–1037, 2016. [8](#), [16](#), [18](#)
- [5] F. Bernicot, D. Frey, and S. Petermichl. Sharp weighted norm estimates beyond Calderón-Zygmund theory. *Anal. PDE*, 9(5):1079–1113, 2016. [1](#), [2](#), [22](#)
- [6] A. Björn and J. Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011. [16](#)
- [7] B. V. Boyarskiĭ. Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients. *Mat. Sb. N.S.*, 43(85):451–503, 1957. [3](#)
- [8] M. Bulíček, L. Diening, and S. Schwarzacher. Existence, uniqueness and optimal regularity results for very weak solutions to nonlinear elliptic systems. *Anal. PDE*, 9(5):1115–1151, 2016. [4](#)
- [9] S.-S. Byun and L. Wang. Elliptic equations with BMO coefficients in Reifenberg domains. *Comm. Pure Appl. Math.*, 57(10):1283–1310, 2004. [2](#), [3](#)
- [10] L. A. Caffarelli and I. Peral. On $W^{1,p}$ estimates for elliptic equations in divergence form. *Comm. Pure Appl. Math.*, 51(1):1–21, 1998. [5](#)

- [11] J. M. Conde-Alonso and G. Rey. A pointwise estimate for positive dyadic shifts and some applications. *Math. Ann.*, 365(3-4):1111–1135, 2016. [2](#)
- [12] A. Culiuc, F. Di Plinio, and Y. Ou. Domination of multilinear singular integrals by positive sparse forms. *J. Lond. Math. Soc. (2)*, 98(2):369–392, 2018. [2](#)
- [13] G. Di Fazio. L^p estimates for divergence form elliptic equations with discontinuous coefficients. *Boll. Un. Mat. Ital. A (7)*, 10(2):409–420, 1996. [4](#), [18](#)
- [14] H. Dong, L. Escauriaza, and S. Kim. On C^1 , C^2 , and weak type-(1, 1) estimates for linear elliptic operators: part II. *Math. Ann.*, 370(1-2):447–489, 2018. [6](#), [9](#), [18](#)
- [15] H. Dong and D. Kim. On L_p -estimates for elliptic and parabolic equations with A_p weights. *Trans. Amer. Math. Soc.*, 370(7):5081–5130, 2018. [4](#)
- [16] H. Dong and S. Kim. On C^1 , C^2 , and weak type-(1, 1) estimates for linear elliptic operators. *Comm. Partial Differential Equations*, 42(3):417–435, 2017. [6](#)
- [17] F. Duzaar and G. Mingione. Gradient estimates via non-linear potentials. *Amer. J. Math.*, 133(4):1093–1149, 2011. [2](#)
- [18] F. W. Gehring. The L^p -integrability of the partial derivatives of a quasiconformal mapping. *Acta Math.*, 130:265–277, 1973. [3](#)
- [19] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008. [7](#), [18](#), [19](#)
- [20] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2009. [7](#), [18](#), [22](#)
- [21] T. P. Hytönen. The sharp weighted bound for general Calderón-Zygmund operators. *Ann. of Math. (2)*, 175(3):1473–1506, 2012. [1](#)
- [22] T. Iwaniec. Projections onto gradient fields and L^p -estimates for degenerated elliptic operators. *Studia Math.*, 75(3):293–312, 1983. [2](#)
- [23] T. Iwaniec and C. Sbordone. Riesz transforms and elliptic PDEs with VMO coefficients. *J. Anal. Math.*, 74:183–212, 1998. [4](#)
- [24] J. Kinnunen and S. Zhou. A local estimate for nonlinear equations with discontinuous coefficients. *Comm. Partial Differential Equations*, 24(11-12):2043–2068, 1999. [4](#)
- [25] J. Kinnunen and S. Zhou. A boundary estimate for nonlinear equations with discontinuous coefficients. *Differential Integral Equations*, 14(4):475–492, 2001. [4](#), [9](#), [14](#), [18](#)
- [26] T. Kuusi and G. Mingione. Linear potentials in nonlinear potential theory. *Arch. Ration. Mech. Anal.*, 207(1):215–246, 2013. [2](#)
- [27] M. T. Lacey. An elementary proof of the A_2 bound. *Israel J. Math.*, 217(1):181–195, 2017. [1](#)
- [28] M. T. Lacey. Sparse bounds for spherical maximal functions. *J. Anal. Math.*, 139(2):613–635, 2019. [2](#), [5](#)
- [29] A. K. Lerner. On an estimate of Calderón-Zygmund operators by dyadic positive operators. *J. Anal. Math.*, 121:141–161, 2013. [1](#)
- [30] A. K. Lerner. On pointwise estimates involving sparse operators. *New York J. Math.*, 22:341–349, 2016. [1](#), [5](#)
- [31] A. K. Lerner and F. Nazarov. Intuitive dyadic calculus: the basics. *Expo. Math.*, 37(3):225–265, 2019. [2](#), [22](#), [23](#)
- [32] A. K. Lerner and S. Ombrosi. Some remarks on the pointwise sparse domination. *J. Geom. Anal.*, 30(1):1011–1027, 2020. [1](#), [5](#), [19](#)
- [33] N. Meyers. An L^p -estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 17:189–206, 1963. [3](#), [15](#)

- [34] N. G. Meyers and A. Elcrat. Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions. *Duke Math. J.*, 42:121–136, 1975. [3](#)
- [35] S. Petermichl. The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic. *Amer. J. Math.*, 129(5):1355–1375, 2007. [1](#), [2](#)
- [36] S. Petermichl and A. Volberg. Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular. *Duke Math. J.*, 112(2):281–305, 2002. [1](#)
- [37] D. Sarason. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.*, 207:391–405, 1975. [3](#)

OLLI SAARI: DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT POLITÈCNICA DE CATALUNYA, AVINGUDA DIAGONAL 647, 08028 BARCELONA, CATALUNYA, SPAIN

INSTITUTE OF MATHEMATICS OF UPC-BARCELONATECH, PAU GARGALLO 14, 08028 BARCELONA, CATALUNYA, SPAIN

CENTRE DE RECERCA MATEMÀTICA, EDIFICI C, CAMPUS BELLATERRA, 08193 BELLATERRA, CATALUNYA, SPAIN

HUA-YANG WANG: ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, P.R. CHINA

UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100049, P.R. CHINA

Email address: wanghuayang@amss.ac.cn

YUANHONG WEI: SCHOOL OF MATHEMATICS, JILIN UNIVERSITY, CHANGCHUN 130012, P.R. CHINA

Email address: weiyuanhong@jlu.edu.cn