

# SHARP INEQUALITIES FOR ONE-SIDED MUCKENHOUPHT WEIGHTS

PAUL HAGELSTEIN, IOANNIS PARISSIS, AND OLLI SAARI

ABSTRACT. Let  $A_\infty^+$  denote the class of one-sided Muckenhoupt weights, namely all the weights  $w$  for which  $M^+ : L^p(w) \rightarrow L^{p,\infty}(w)$  for some  $p > 1$ , where  $M^+$  is the forward Hardy-Littlewood maximal operator. We show that  $w \in A_\infty^+$  if and only if there exist numerical constants  $\gamma \in (0, 1)$  and  $c > 0$  such that

$$w(\{x \in \mathbb{R} : M^+ \mathbf{1}_E(x) > \gamma\}) \leq cw(E)$$

for all measurable sets  $E \subset \mathbb{R}$ . Furthermore, letting

$$C_w^+(\alpha) := \sup_{0 < w(E) < +\infty} \frac{1}{w(E)} w(\{x \in \mathbb{R} : M^+ \mathbf{1}_E(x) > \alpha\})$$

we show that for all  $w \in A_\infty^+$  we have the asymptotic estimate  $C_w^+(\alpha) - 1 \lesssim (1 - \alpha)^{\frac{1}{c[w]_{A_\infty^+}}}$  for  $\alpha$  sufficiently close to 1 and  $c > 0$  a numerical constant, and that this estimate is best possible. We also show that the reverse Hölder inequality for one-sided Muckenhoupt weights, previously proved by Martín-Reyes and de la Torre, is sharp, thus providing a quantitative equivalent definition of  $A_\infty^+$ . Our methods also allow us to show that a weight  $w \in A_\infty^+$  satisfies  $w \in A_p^+$  for all  $p > e^{c[w]_{A_\infty^+}}$ .

## 1. INTRODUCTION

We are interested in topics related to one-sided maximal operators on Euclidean spaces. Our focus is on the one-dimensional case and the main operator under study in this paper is the forward one-sided Hardy-Littlewood maximal operator defined by

$$M^+ f(x) := \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)| dt, \quad x \in \mathbb{R}, \quad f \in L^1_{\text{loc}}(\mathbb{R}).$$

By a weight  $w$  we always mean a non-negative, locally integrable function on the real line. The weights  $w$  for which  $M^+ : L^p(w) \rightarrow L^{p,\infty}(w)$  have been identified and extensively studied. See for example [9, 10, 11, 12, 14, 16]. Thus it is well known that the appropriate one-sided Muckenhoupt

---

2010 *Mathematics Subject Classification*. Primary: 42B25, Secondary: 42B35.

*Key words and phrases*. one sided maximal function, tauberian condition, Muckenhoupt weight.

P. H. is partially supported by a grant from the Simons Foundation (#208831 to Paul Hagelstein).

I. P. is supported by grant MTM2014-53850 of the Ministerio de Economía y Competitividad (Spain), grant IT-641-13 of the Basque Government, and IKERBASQUE.

O. S. is supported by the Academy of Finland and the Väisälä Foundation.

class can be defined by

$$[w]_{A_p^+} := \sup_{a < b < c} \frac{w(a, b)}{|(a, c)|} \left( \frac{\sigma(b, c)}{|(a, c)|} \right)^{p-1},$$

where  $\sigma := w^{-\frac{1}{p-1}}$ , and we have that  $M^+ : L^p(w) \rightarrow L^{p, \infty}(w)$ ,  $1 < p < \infty$ , if and only if  $w \in A_p^+$ . For  $p = 1$  we also have the endpoint result that  $M^+ : L^1(w) \rightarrow L^{1, \infty}(w)$  if and only if

$$[w]_{A_1^+} := \left\| \frac{M^- w}{w} \right\|_{L^\infty(\mathbb{R})} < +\infty,$$

where  $M^- f(x) := \sup_{h>0} \frac{1}{h} \int_{x-h}^x |f(t)| dt$ ; see for example [11, 16].

The corresponding one-sided  $A_\infty^+$  class has also been studied and one can define for example

$$A_\infty^+ := \bigcup_{p>1} A_p^+;$$

see [10]. The class  $A_\infty^+$  can be defined in many other equivalent ways. Here we adopt the definition through a Fujii-Wilson-type constant which amounts to demanding that

$$[w]_{A_\infty^+} := \sup_{a < b} \frac{1}{w(a, b)} \int_{(a, b)} M^-(w \mathbf{1}_{(a, b)}) < +\infty.$$

We note here that there is no standard definition for the  $A_\infty^+$  constant of one-sided Muckenhoupt weights  $w$ . The definition above appears in [12, Definition 1.7] where it is shown that  $w \in A_\infty^+$  if and only if  $[w]_{A_\infty^+} < +\infty$ . A similar definition appears in [7]. We note also that for two-sided weights the constant above was introduced independently by Fujii [2] and Wilson [18, 19]. In the two-sided case the constant above was shown to be the appropriate quantity in order to prove sharp weighted norm inequalities for singular integral and maximal operators; see for example [6]. Similar results for the one-sided maximal operator appear in [12].

In this paper we pursue a characterization of  $A_\infty^+$  in terms of *tauberian constants*, in the spirit of [3, 4]. Given a locally integrable function  $w \in A_\infty^+$  and  $\alpha \in (0, 1)$  we define

$$C_w^+(\alpha) := \sup_{0 < w(E) < +\infty} \frac{1}{w(E)} w(\{x \in \mathbb{R} : M^+ \mathbf{1}_E(x) > \alpha\}).$$

We call  $C_w^+(\alpha)$  the *sharp weighted tauberian constant* of  $M^+$ . It is obvious that  $C_w^+(\alpha) < +\infty$  for all  $\alpha \in (0, 1)$  whenever  $M^+ : L^p(w) \rightarrow L^{p, \infty}(w)$ . In this paper we show that an apparently much weaker converse to this statement holds, namely that  $C_w^+(\alpha_0) < +\infty$  for a single value  $\alpha_0 \in (0, 1)$  already implies that  $M^+ : L^p(w) \rightarrow L^p(w)$  for sufficiently large  $p > 1$  and thus that  $w \in A_\infty^+$ . This is the content of our main theorem.

**Theorem 1.1.** *Let  $w$  be a non-negative, locally integrable function on the real line. The following are equivalent:*

- (i) *We have that  $w \in A_\infty^+$ .*
- (ii) *There exists  $\delta > 0$  such that  $C_w^+(\alpha) - 1 \lesssim (1 - \alpha)^\delta$  as  $\alpha \rightarrow 1^-$ .*
- (iii) *There exists  $\alpha_0 \in (0, 1)$  such that  $C_w^+(\alpha_0) < +\infty$ .*

The study of asymptotic estimates of the type  $C_w^+(\alpha) - 1 \lesssim (1 - \alpha)^\delta$  as  $\alpha \rightarrow 1^-$  has a small history. In [17], Solyanik proved corresponding estimates for the usual Hardy-Littlewood maximal function with respect to axes parallel cubes in  $\mathbb{R}^n$ , both in its centered and non-centered version, as well as for the strong maximal function. In [5] the first two authors continued these investigations and introduced the term *Solyanik estimate* to indicate the validity of  $\lim_{\alpha \rightarrow 1^-} C(\alpha) = 1$ , whenever  $C(\alpha)$  denotes the sharp tauberian constant with respect to some geometric maximal operator. Finally, in [4], Solyanik estimates are shown to hold in the classical (two-sided) weighted setting and in fact they characterize the usual  $A_\infty$  class of Muckenhoupt weights.

Going back to our main result above and assuming that  $w \in A_\infty^+$  we can give a very precise version of the Solyanik estimate (ii).

**Corollary 1.2.** *Let  $w \in A_\infty^+$  be a one-sided Muckenhoupt weight on the real line. There exists a numerical constant  $c > 0$  such that*

$$C_w^+(\alpha) - 1 \lesssim (1 - \alpha)^{(c[w]_{A_\infty^+})^{-1}} \quad \text{for all } 1 > \alpha > 1 - e^{-c[w]_{A_\infty^+}}$$

*with the implicit constant independent of everything. Furthermore this estimate is optimal up to numerical constants; if  $w$  is a locally integrable function that satisfies*

$$C_w^+(\alpha) - 1 \lesssim (1 - \alpha)^{\frac{1}{\beta}} \quad \text{whenever } 1 > \alpha > 1 - e^{-\beta}$$

*for some constant  $\beta > 1$  then  $w \in A_\infty^+$  and  $[w]_{A_\infty^+} \lesssim \beta$ .*

We note in passing that the sharpness direction in Corollary 1.2 above relies on exhibiting a sharp reverse Hölder inequality for one-sided Muckenhoupt weights. The validity of the reverse Hölder inequality is actually proved in [12]. We show in Theorem 3.2 that the reverse Hölder inequality of [12] is best possible, which in turn allows to prove the optimality of the Solyanik estimate in the corollary above.

Secondly, it is of some importance to note that our method of proof together with the corollary above allows us to conclude a quantitative embedding of  $A_\infty^+$  into  $A_p^+$ .

**Corollary 1.3.** *Let  $w \in A_\infty^+$  be a one-sided Muckenhoupt weight on the real line and denote by  $[w]_{A_\infty^+}$  its Fujii-Wilson constant. Then there exists a numerical constant  $c > 0$  such that  $w \in A_p^+$  for all  $p > e^{c[w]_{A_\infty^+}}$  and  $[w]_{A_p^+} \leq \exp(\exp(c[w]_{A_\infty^+}))$ .*

We close this introductory section with a few comments concerning our motivation for studying these estimates. In [7, 15] the third author has considered higher dimensional weighted norm inequalities for one-sided maximal operators. These arise naturally in the study of solutions of appropriate doubly nonlinear partial differential equations. Other approaches to defining and studying higher dimensional one-sided maximal operators appear for example in [1], [8], and [13]. A common feature of all these studies is that, in the higher dimensional case, the connection between the one-sided classes  $A_p^+$  and  $A_\infty^+$  remains elusive. This should be contrasted to the one-dimensional case where we have a full analogue of the two-sided Muckenhoupt theory.

It is customary for most of the one-dimensional papers studying  $A_\infty^+$  in the literature to set everything up with respect to some other measure  $g$  and thus study the classes  $A_p^+(g)$ . This setup is particularly suited for some symmetry arguments that allow one to show that a weight in  $A_\infty^+$

belongs to  $A_p^+$  for some  $p > 1$ . This approach does not seem to be available in higher dimensions as many general results that hold for arbitrary measures in one-dimension fail in higher dimensions. The current paper avoids this setup and thus proposes another self-contained strategy for proving that  $\bigcup_{p>1} A_p^+ = A_\infty^+$ . We plan to pursue the higher dimensional analogues in a future work.

## 2. NOTATION

We use the letters  $C, c$  to denote generic positive constants which might change even in the same line of text. We write  $A \lesssim B$  if  $A \leq CB$  and  $A \approx B$  if  $A \lesssim B$  and  $B \lesssim A$ . Throughout the text  $w$  is a nonnegative, locally integrable function on the real line and we write  $w(a, b) := \int_{(a,b)} w(t) dt$ . Finally, given  $\beta \in (0, 1)$  and an interval  $(a, b) \subset \mathbb{R}$  we say that a function  $f : (a, b) \rightarrow \mathbb{R}$  lies in the local Hölder class  $C^\beta(a, b)$  if for all compact  $K \subset (a, b)$  and all  $x, y \in K$  we have that  $|f(x) - f(y)| \lesssim_K |x - y|^\beta$  for all  $x, y \in K$ .

## 3. PRELIMINARIES

In this section we collect classical results about one-sided Muckenhoupt weights that we will need throughout the paper.

The first is a version of the classical rising sun lemma adjusted to our setup and is a minor modification of [16, Lemma 2.1].

**Lemma 3.1.** *Let  $\lambda > 0$  and  $f \geq 0$  with compact support. Then*

$$\{x \in \mathbb{R} : M^+f(x) > \lambda\} = \bigcup_j (a_j, b_j)$$

where the intervals  $\{(a_j, b_j)\}_j$  are pairwise disjoint and for every  $j$  we have that  $f_x^{b_j} f > \lambda$  for all  $x \in (a_j, b_j)$ . In particular  $f_{a_j}^{b_j} f = \lambda$  for all  $j$ .

*Proof.* Let  $E_\lambda := \{x \in \mathbb{R} : M^+f(x) > \lambda\}$ . By [16], we have that  $E_\lambda = \bigcup_j (a_j, b_j)$  where  $\{(a_j, b_j)\}_j$  is a disjoint collection such that  $f_{a_j}^{b_j} f = \lambda$  for every  $j$  and such that  $f_x^{b_j} f \geq \lambda$  whenever  $x \in (a_j, b_j)$  for some  $j$ . Now, for every  $x \in (a_j, b_j)$  there exists  $r > x$  such that  $f_x^r f > \lambda$ . If  $r = b_j$  then we are done. If  $r > b_j$  then since  $f_{b_j}^r f \leq \lambda$  we can conclude again that  $f_x^{b_j} f > \lambda$ . Finally, if  $r < b_j$  we consider the maximal  $s \geq r$  such that  $f_r^s f \geq \lambda$ . Necessarily  $s \geq b$  due to the maximality of  $s$ . From this and the fact that  $f_{b_j}^s f \leq \lambda$  we conclude that  $f_r^{b_j} f \geq \lambda$ . Thus  $\int_x^{b_j} f = \int_x^r f + \int_r^{b_j} f > \lambda(b_j - x)$  and we are done.  $\square$

Whenever we have a one-sided weight  $w \in A_\infty^+$  we will need to use quantitative estimates of the type

$$\frac{w(E)}{w(a, c)} \lesssim \left( \frac{|E|}{|(b, c)|} \right)^\delta$$

whenever  $a < b < c$  and  $E \subseteq (a, b)$ . In fact it is well known that the estimate above provides an equivalent definition for the class  $A_{\infty}^+$ ; see for example [10]. Another equivalent way to define  $A_{\infty}^+$  goes through the validity of appropriate reverse Hölder inequalities.

The following theorem summarizes these equivalences. Note that the direct implications (i) and (ii) below are directly taken from [12]. The optimality of these estimates for  $w \in A_{\infty}^+$ , contained in (iii), appears to be new as the authors in [12] didn't pursue this direction.

**Theorem 3.2.** *Let  $w$  be a nonnegative, locally integrable function on  $\mathbb{R}$ . Then the following hold.*

(i) *If  $w \in A_{\infty}^+$  then for  $0 < \epsilon \leq \frac{1}{2[w]_{A_{\infty}^+}}$  and for all  $a < b < c$  we have the one-sided reverse Hölder inequality*

$$|(b, c)|^{\epsilon} \int_{(a, b)} w^{1+\epsilon} \leq 2 \left( \int_{(a, c)} w \right)^{1+\epsilon}.$$

(ii) *If  $w \in A_{\infty}^+$  then for  $0 < \epsilon \leq \frac{1}{2[w]_{A_{\infty}^+}}$  and for all  $a < b < c$  and for all measurable sets  $E \subseteq (a, b)$  we have*

$$\frac{w(E)}{w(a, c)} \leq 2 \left( \frac{|E|}{|(b, c)|} \right)^{\frac{\epsilon}{1+\epsilon}} \leq 2 \left( \frac{|E|}{|(b, c)|} \right)^{\frac{1}{3[w]_{A_{\infty}^+}}}.$$

(iii) *Conversely, if the conclusion of (i) or (ii) holds for some  $0 < \epsilon < 1$  then  $w \in A_{\infty}^+$  and  $[w]_{A_{\infty}^+} \lesssim \frac{1}{\epsilon}$ ; thus (i), (ii) are best possible up to numerical constants.*

*Proof.* The statement and proof of (i) is [12, Theorem 3.4]. Part (ii) follows trivially by an application of the standard Hölder inequality.

Let us move to the proof of (iii). First we observe that the validity of (ii) for some  $0 < \epsilon < 1$  implies the reverse Hölder inequality

$$|(b, c)|^{\frac{\epsilon}{2}} \int_{(a, b)} w^{1+\epsilon/2} \leq 6 \left( \int_{(a, c)} w \right)^{1+\epsilon/2}$$

To see this we can assume that  $w(a, c)/|(b, c)| = 1$ . Setting  $E_{\lambda} := \{x \in (a, b) : w(x) > \lambda\}$  we have

$$\frac{|E_{\lambda}|}{|(b, c)|} \leq \frac{1}{\lambda} \frac{w(E_{\lambda})}{w(a, c)} \leq 2 \frac{1}{\lambda} \frac{|E_{\lambda}|^{\frac{\epsilon}{1+\epsilon}}}{|(b, c)|^{\frac{\epsilon}{1+\epsilon}}}$$

and thus we have proved the estimate

$$\frac{|E_{\lambda}|}{|(b, c)|} \leq 2^{1+\epsilon} \frac{1}{\lambda^{1+\epsilon}}.$$

Using (ii) again this implies  $\frac{w(E_\lambda)}{w(a,c)} \leq 2^{1+\epsilon} \frac{1}{\lambda^\epsilon}$ . Now

$$\begin{aligned} \int_{(a,b)} w^{1+\epsilon/2} &= \int_{(a,b)} w^{\frac{\epsilon}{2}} w \leq w(a,b) + \frac{\epsilon}{2} \int_1^\infty \lambda^{\epsilon/2-1} w(E_\lambda) d\lambda \\ &\leq w(a,b) + 2^\epsilon \epsilon \int_1^\infty \frac{w(a,c)}{\lambda^{1+\epsilon/2}} d\lambda \leq 5w(a,c) = 5|(b,c)|. \end{aligned}$$

This shows the claimed reverse Hölder inequality.

Now we show that a reverse Hölder inequality with exponent  $1 + \epsilon$  implies that  $[w]_{A_\infty^+} \lesssim 1/\epsilon$ . For this fix  $a < b < c$  and set  $r := 1 + \epsilon$ . We write  $(a, b) = \cup_{j=1}^\infty I_j$  where  $I_j = (x_j, x_{j+1}]$ , and  $x_0 := a$ ,  $x_1 := \frac{a+b}{2}$  and  $x_j := \frac{x_{j-1}+b}{2}$ .

Assuming that a reverse Hölder inequality, as in (i), holds with exponent  $r = 1 + \epsilon$  we can use the bound  $\|M^-\|_{L^r \rightarrow L^r} \lesssim r'$  to estimate

$$\begin{aligned} \int_{(a,b)} M^-(w \mathbf{1}_{(a,b)}) &\leq \sum_{j=1}^\infty \int_{(a,b)} M^-(w \mathbf{1}_{I_j}) = \sum_{j=1}^\infty \int_{\cup_{k \geq j} I_k} M^-(w \mathbf{1}_{I_j}) \\ &\lesssim r' \sum_{j=1}^\infty \left| \bigcup_{k \geq j} I_k \right|^{\frac{1}{r'}} \left( \int_{I_j} w^r \right)^{\frac{1}{r}} \\ &\approx r' \sum_{j=1}^\infty |I_j|^{\frac{1}{r'}} \left( \int_{I_j \cup I_{j+1}} w \right) |I_{j+1}|^{-\frac{1}{r'}} \lesssim r' w(a,b) \end{aligned}$$

and thus  $[w]_{A_\infty^+} \lesssim r' \approx \frac{1}{\epsilon}$  as we wanted.  $\square$

#### 4. PROOF OF THE MAIN THEOREM

We divide the proof of the main theorem into two parts. In the first part we show Corollary 1.2 which also shows that (i) implies (ii) in Theorem 1.1.

*Proof of Corollary 1.2.* Let  $E \subset \mathbb{R}$  be a compact subset of the real line with  $0 < w(E) < +\infty$  and let  $\alpha > 0$ . We use Lemma 3.1 to write

$$\{x \in \mathbb{R} : M^+ \mathbf{1}_E(x) > \alpha\} = \bigcup_j (a_j, b_j),$$

where  $f_x^{b_j} \chi_E > \alpha$  for all  $x \in (a_j, b_j)$  and  $f_{a_j}^{b_j} \chi_E = \alpha$  for each  $j$ . Let  $(a, b) \in \{(a_j, b_j)\}_j$ . By Lemma 3.1 we have that  $f_a^b \mathbf{1}_E = \alpha$  and  $f_x^b \mathbf{1}_E > \alpha$  for all  $x \in (a, b)$ . Using an idea from [9] we choose an increasing sequence  $\{x_k\}_{k=0}^\infty$  such that  $x_0 := a$ ,  $(a, b) = \cup_{k \geq 1} (x_{k-1}, x_k]$ , and

$\int_{x_{k-1}}^{x_k} \mathbf{1}_E = \int_{x_k}^b \mathbf{1}_E$  for all  $k \geq 1$ . Then we can estimate for all  $k \geq 1$

$$\begin{aligned} w((x_{k-1}, x_k] \setminus E) &= \frac{w((x_{k-1}, x_k] \setminus E)}{w(x_{k-1}, x_{k+1})} w(x_{k-1}, x_{k+1}) \\ &\lesssim \left( \frac{|(x_{k-1}, b) \setminus E|}{|(x_k, x_{k+1})|} \right)^{\frac{1}{3[w]_{A_\infty^+}}} w(x_{k-1}, x_{k+1}) \end{aligned}$$

by (ii) of Theorem 3.2. By Lemma 3.1 we can conclude that  $|(x_{k-1}, b) \setminus E| \leq \frac{1-\alpha}{\alpha} \int_{x_{k-1}}^b \mathbf{1}_E$ . Remembering the definition of the sequence  $\{x_k\}_k$  we can further calculate

$$\int_{x_{k-1}}^b \mathbf{1}_E = \int_{x_{k-1}}^{x_k} \mathbf{1}_E + \int_{x_k}^b \mathbf{1}_E = 2 \int_{x_k}^b \mathbf{1}_E.$$

By a single recursion of this formula and another use of the definition of the sequence  $\{x_k\}_k$  we thus have

$$\int_{x_{k-1}}^b \mathbf{1}_E = 4 \int_{x_{k+1}}^b \mathbf{1}_E = 4 \int_{x_k}^{x_{k+1}} \mathbf{1}_E \leq 4(x_{k+1} - x_k).$$

We have proved that

$$w((x_{k-1}, x_k] \setminus E) \lesssim (1 - \alpha)^{(3[w]_{A_\infty^+})^{-1}} w(x_{k-1}, x_{k+1}).$$

Summing over  $k \geq 1$  we conclude that for every  $j$  we have

$$w((a_j, b_j) \setminus E) \lesssim (1 - \alpha)^{(3[w]_{A_\infty^+})^{-1}} w(a_j, b_j).$$

Summing over  $j$  we get the desired estimate

$$C_w^+(\alpha) - 1 \lesssim (1 - \alpha)^{(c[w]_{A_\infty^+})^{-1}}$$

whenever  $\alpha > 1 - e^{-c[w]_{A_\infty^+}}$ , for some numerical constant  $c > 1$ .

We now proceed to exhibit the optimality of the Solyanik estimate just proved. For this let us assume that we have an estimate  $C_w^+(\alpha) - 1 \lesssim (1 - \alpha)^{\frac{1}{\beta}}$  for  $\alpha > 1 - e^{-\beta}$ . We will prove that for all  $a < b < c$  and measurable  $E \subset (a, b)$  we have the estimate

$$\frac{w(E)}{w(a, c)} \lesssim \left( \frac{|E|}{|(b, c)|} \right)^{\frac{1}{\beta}}$$

By Theorem 3.2 this will imply that  $w \in A_\infty^+$  and  $[w]_{A_\infty^+} \lesssim \beta$  as claimed.

We now fix real numbers  $a < b < c$  and consider a measurable set  $E \subseteq (a, b)$ . We set  $E' := (a, c) \setminus E$  and consider two cases.

If  $|E|/|(b, c)| \geq e^{-\beta}$  then

$$\frac{w(E)}{w(a, c)} \leq 1 = ee^{-1} \leq e \left( \frac{|E|}{|(b, c)|} \right)^{\frac{1}{\beta}}$$

and we are done.

In the complementary case we have for all  $x \in (a, b)$

$$\begin{aligned} M^+ \mathbf{1}_{E'}(x) &\geq \frac{|E' \cap (x, c)|}{|(x, c)|} = \frac{|(b, c)| + |(x, b)| - |(x, b) \cap E|}{|(x, c)|} \geq 1 - \frac{|(x, b) \cap E|}{|(x, c)|} \\ &\geq 1 - \frac{|E|}{|(b, c)|} > 1 - e^{-\beta}. \end{aligned}$$

Obviously  $M^+ \mathbf{1}_{E'} = 1$  on  $(b, c)$  and thus  $(a, c) \subseteq \{x \in \mathbb{R} : M^+ \mathbf{1}_{E'}(x) \geq 1 - |E|/|b, c|\}$ . As  $1 - |E|/|(b, c)| > 1 - e^{-\beta}$  we can use the assumed Solyanik estimate to conclude that

$$w(a, c) \leq C_w^+(1 - |E|/|(b, c)|)w(E') = C_w^+(1 - |E|/|(b, c)|)(w(a, c) - w(E))$$

and thus

$$w(E) \leq \left(C_w^+(1 - |E|/|(b, c)|) - 1\right)w(a, c) \lesssim \left(\frac{|E|}{|(b, c)|}\right)^{\frac{1}{\beta}} w(a, c)$$

as we wanted.  $\square$

It is trivial that (ii) implies (iii) in Theorem 1.1 so we move on to prove that (iii) implies (i). It will clearly suffice to show the following.

**Proposition 4.1.** *Suppose that  $w$  is a non-negative, locally integrable function that satisfies  $C_w^+(\alpha_0) < +\infty$  for some  $\alpha_0 \in (0, 1)$ . Then  $M^+$  is of restricted weak type  $(p, p)$  with respect to  $w$  for  $p = \log C_w^+(\alpha_0) / \log(1/\alpha_0)$ , with constant  $C_w^+(\alpha_0)^{\frac{1}{p}}$ .*

The proof of Proposition 4.1 relies on the notion of the *Halo extension* of a set  $E$ , defined as follows. Given  $\lambda \in (0, 1)$  the Halo extension of  $E$  is

$$\mathcal{H}_\lambda^+(E) := \{x \in \mathbb{R} : M^+ \mathbf{1}_E(x) > \lambda\}.$$

We also set  $\mathcal{H}_\lambda^{+,0}(E) := E$  and for a positive integer  $k > 1$

$$\mathcal{H}_\lambda^{+,k}(E) := \mathcal{H}_\lambda^+(\mathcal{H}_\lambda^{+,k-1}(E)).$$

The heart of the matter is the following lemma.

**Lemma 4.2.** *Let  $0 < \lambda < \alpha < 1$  and  $E \subset \mathbb{R}$  be a measurable set with  $0 < w(E) < +\infty$ . Then*

$$\mathcal{H}_\lambda^+(E) \subseteq \mathcal{H}_\alpha^{+,N}(E),$$

where

$$N = \left\lceil \frac{\log \frac{1}{\lambda}}{\log \frac{1}{\alpha}} \right\rceil.$$

Here  $\lceil x \rceil$  denotes the smallest integer which is no less than  $x$ .

*Proof.* Let  $(a, b)$  be one of the component intervals of  $\mathcal{H}_\lambda^+(E)$  provided by Lemma 3.1. The same lemma allows us to write  $\mathcal{H}_\alpha^+(E)$  as the union of disjoint intervals  $\cup_j (\xi_j, \eta_j)$ , where  $\int_{\xi_j}^{\eta_j} \chi_E = \alpha$ .

Since  $\lambda < \alpha < 1$  we have that  $a, b \notin (\xi_j, \eta_j)$  for any  $j$ . Indeed, if say  $a \in (\xi_j, \eta_j)$  then there exists some  $h > 0$  such that

$$\alpha h < \int_a^{a+h} \mathbf{1}_E \leq \lambda h$$

since  $a \notin \mathcal{H}_\lambda^+(E)$ . This however contradicts the condition  $\lambda < \alpha$ . Similarly we see that  $b \notin (\xi_j, \eta_j)$ . Thus  $\mathcal{H}_\alpha^+(E) \cap (a, b) = \bigcup_{j \in J} (\xi_j, \eta_j)$ . For  $x \in (a, b) \setminus \mathcal{H}_\alpha^+(E)$  let  $J_x := \{j \in J : \xi_j > x\}$ . Then if  $x \in (a, b) \setminus \mathcal{H}_\alpha^+(E)$  we can calculate

$$|\mathcal{H}_\alpha^+(E) \cap (x, b)| = \sum_{j \in J_x} |(\xi_j, \eta_j)| = \frac{1}{\alpha} \left| \bigcup_{j \in J_x} (\xi_j, \eta_j) \cap E \right| = \frac{1}{\alpha} |E \cap (x, b)| > \frac{\lambda}{\alpha} |(x, b)|,$$

the last inequality following by Lemma 3.1. Thus  $(a, b) \subseteq \mathcal{H}_{\frac{\lambda}{\alpha}}^+(\mathcal{H}_\alpha^+(E))$  and accordingly  $\mathcal{H}_\lambda^+(E) \subseteq \mathcal{H}_{\frac{\lambda}{\alpha}}^+(\mathcal{H}_\alpha^+(E))$ .

Let  $K \geq 1$  be the smallest integer such that  $\lambda/\alpha^K > \alpha$ . Iterating the estimate above we conclude that

$$\mathcal{H}_\lambda^+(E) \subseteq \mathcal{H}_{\frac{\lambda}{\alpha^K}}^+(\mathcal{H}_\alpha^{+,K}(E)) \subseteq \mathcal{H}_\alpha^{+,K+1}(E).$$

Note that  $K$  satisfies  $\lambda/\alpha^{K-1} < \alpha$  and thus  $K+1 = \lceil \log(1/\lambda)/\log(1/\alpha) \rceil$ .  $\square$

We can now give the proof of Proposition 4.1.

*Proof of Proposition 4.1.* Let  $\lambda \in (0, 1)$ . Since  $C_w^+(\alpha_0) < +\infty$  we trivially get that for all  $\lambda > \alpha_0$  and every  $E \subset \mathbb{R}$

$$w(\{x \in \mathbb{R} : M^+ \mathbf{1}_E(x) > \lambda\}) \leq C_w^+(\alpha_0) w(E) \leq \frac{C_w^+(\alpha_0)}{\lambda^q} w(E)$$

for any  $q \geq 1$ . It thus suffices to consider the case  $0 < \lambda < \alpha_0 < 1$ . Let  $1 > \alpha > \alpha_0$ . Then for any set  $S \subset \mathbb{R}$  we have

$$w(\mathcal{H}_\alpha^+(S)) \leq C_w^+(\alpha_0) w(S).$$

By Lemma 4.2 applied for  $\alpha > \lambda > 0$  we can conclude that

$$w(\mathcal{H}_\lambda^+(E)) \leq C_w^+(\alpha_0)^N w(E)$$

with  $N = \lceil \log \frac{1}{\lambda} / \log \frac{1}{\alpha} \rceil \leq \log \frac{1}{\lambda} / \log \frac{1}{\alpha} + 1$ . We get

$$w(\{x \in \mathbb{R} : M^+ \mathbf{1}_E(x) > \lambda\}) \leq C_w^+(\alpha_0) \left( \frac{1}{\lambda} \right)^{\frac{\log C_w^+(\alpha_0)}{\log \frac{1}{\alpha}}} w(E)$$

for all  $\alpha > \alpha_0$ . Letting  $\alpha \rightarrow \alpha_0^+$  we conclude that  $M^+$  is of restricted weak type  $(p, p)$  with respect to  $w$  for  $p = \log C_w^+(\alpha_0) / \log(1/\alpha_0)$ , with constant  $C_w^+(\alpha_0)^{\frac{1}{p}}$ .  $\square$

We conclude this section with the proof of Corollary 1.3

*Proof of Corollary 1.3.* Assuming that  $w \in A_{\infty}^+$  with constant  $[w]_{A_{\infty}^+}$ , Theorem 1.1 implies that  $C_w^+(\alpha_0) \lesssim 1$  for  $\alpha_0 = 1 - e^{-c[w]_{A_{\infty}^+}}$ , and  $c > 1$  is a numerical constant. Now Proposition 4.1 implies that  $M^+$  is of restricted weak type  $(p, p)$  with respect to  $w$  for  $p \approx (\log(1/\alpha_0))^{-1} \approx e^{c[w]_{A_{\infty}^+}}$  and thus  $w \in A_p^+$  for  $p > e^{c[w]_{A_{\infty}^+}}$ . By carefully examining the interpolation constants and using the estimate

$$\|M^+\|_{L^p(w) \rightarrow L^p(w)} \gtrsim [w]_{A_p^+}^{\frac{1}{p}}$$

which is contained in [12, Theorem 1.4] we can conclude that  $[w]_{A_p^+} \leq e^{e^{c[w]_{A_{\infty}^+}}}$  for some numerical constant  $c > 0$ . See also [4, p. 21] for the details of this calculation.  $\square$

## 5. LOCAL HÖLDER CONTINUITY

The methods of this paper easily imply that  $C_w^+$  is locally Hölder continuous on  $(0, 1)$  whenever  $w \in A_{\infty}^+$ .

**Corollary 5.1.** *There exists a numerical constant  $c > 1$  such that for all  $w \in A_{\infty}^+$  we have  $C_w^+ \in C^{\frac{1}{c[w]_{A_{\infty}^+}}}(0, 1)$ .*

*Proof.* Let  $E \subset \mathbb{R}$  and  $0 < \lambda < \alpha < 1$ . By the proof of Lemma 4.2 we have that

$$\mathcal{H}_{\lambda}^+(E) \subseteq \mathcal{H}_{\frac{\lambda}{\alpha}}^+(\mathcal{H}_{\alpha}^+(E)).$$

We conclude that

$$0 < C_w^+(\lambda) - C_w^+(\lambda/\alpha) \leq C_w^+(\lambda/\alpha)(C_w^+(\alpha) - 1).$$

Using Corollary 1.2 we get that for all  $0 < x < y < 1$  we have

$$|C_w^+(x) - C_w^+(y)| \lesssim C_w^+(y) \left( \frac{y-x}{x} \right)^{\frac{1}{c[w]_{A_{\infty}^+}}}$$

for some numerical constant  $c > 0$ . The local Hölder continuity of  $C_w^+$  follows immediately from the estimate above.  $\square$

## ACKNOWLEDGMENT

We are indebted to Francisco Javier Martín-Reyes for enlightening discussions related to the subject of the paper.

## REFERENCES

- [1] L. Forzani, F. J. Martín-Reyes, and S. Ombrosi, *Weighted inequalities for the two-dimensional one-sided Hardy-Littlewood maximal function*, Trans. Amer. Math. Soc. **363** (2011), no. 4, 1699–1719, DOI 10.1090/S0002-9947-2010-05343-7. MR2746661 (2011k:42034)  $\uparrow$ 3
- [2] N. Fujii, *Weighted bounded mean oscillation and singular integrals*, Math. Japon. **22** (1977/78), no. 5, 529–534. MR0481968 (58 #2058)  $\uparrow$ 2
- [3] P. Hagelstein, T. Luque, and I. Parissis, *Tauberian conditions, Muckenhoupt weights, and differentiation properties of weighted bases*, Trans. Amer. Math. Soc. **367** (2015), no. 11, 7999–8032. MR3391907  $\uparrow$ 2

- [4] P. Hagelstein and I. Parissis, *Weighted Solyanik estimates for the Hardy-Littlewood maximal operator and embedding of  $A_\infty$  into  $A_p$* , to appear in *J. Geom. Anal.* (2015), available at [1405.6631](#). ↑[2](#), [3](#), [10](#)
- [5] P. Hagelstein and I. Parissis, *Solyanik estimates in harmonic analysis*, Special functions, partial differential equations, and harmonic analysis, Springer Proc. Math. Stat., vol. 108, Springer, Cham, 2014, pp. 87–103. [MR3297656](#) ↑[3](#)
- [6] T. Hytönen and C. Pérez, *Sharp weighted bounds involving  $A_\infty$* , *Anal. PDE* **6** (2013), no. 4, 777–818. [MR3092729](#) ↑[2](#)
- [7] J. Kinnunen and O. Saari, *On weights satisfying parabolic Muckenhoupt conditions*, *Nonlinear Anal.* **131** (2016), 289–299. ↑[2](#), [3](#)
- [8] A. K. Lerner and S. Ombrosi, *A boundedness criterion for general maximal operators*, *Publ. Mat.* **54** (2010), no. 1, 53–71. [MR2603588](#) ([2011b:42064](#)) ↑[3](#)
- [9] F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, *Proc. Amer. Math. Soc.* **117** (1993), no. 3, 691–698. [MR1111435](#) ([93d:42016](#)) ↑[1](#), [6](#)
- [10] F. J. Martín-Reyes, L. Pick, and A. de la Torre,  *$A_\infty^+$  condition*, *Canad. J. Math.* **45** (1993), no. 6, 1231–1244. [MR1247544](#) ([94m:42042](#)) ↑[1](#), [2](#), [5](#)
- [11] F. J. Martín-Reyes, P. Ortega Salvador, and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, *Trans. Amer. Math. Soc.* **319** (1990), no. 2, 517–534. [MR986694](#) ([90j:42047](#)) ↑[1](#), [2](#)
- [12] F. J. Martín-Reyes and A. de la Torre, *Sharp weighted bounds for one-sided maximal operators*, *Collect. Math.* **66** (2015), no. 2, 161–174. [MR3338703](#) ↑[1](#), [2](#), [3](#), [5](#), [10](#)
- [13] S. Ombrosi, *Weak weighted inequalities for a dyadic one-sided maximal function in  $\mathbb{R}^n$* , *Proc. Amer. Math. Soc.* **133** (2005), no. 6, 1769–1775, DOI 10.1090/S0002-9939-05-07830-5. [MR2120277](#) ([2005k:42055](#)) ↑[3](#)
- [14] P. Ortega Salvador, *Weighted inequalities for one-sided maximal functions in Orlicz spaces*, *Studia Math.* **131** (1998), no. 2, 101–114. [MR1636403](#) ([99g:42022](#)) ↑[1](#)
- [15] O. Saari, *Parabolic BMO and global integrability of supersolutions to doubly nonlinear parabolic equations*, to appear in *Rev. Mat. Iberoam.* (2014), available at [1408.5760](#). ↑[3](#)
- [16] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, *Trans. Amer. Math. Soc.* **297** (1986), no. 1, 53–61. [MR849466](#) ([87k:42018](#)) ↑[1](#), [2](#), [4](#)
- [17] A. A. Solyanik, *On halo functions for differentiation bases*, *Mat. Zametki* **54** (1993), no. 6, 82–89, 160 (Russian, with Russian summary); English transl., *Math. Notes* **54** (1993), no. 5-6, 1241–1245 (1994). [MR1268374](#) ([95g:42033](#)) ↑[3](#)
- [18] J. M. Wilson, *Weighted inequalities for the dyadic square function without dyadic  $A_\infty$* , *Duke Math. J.* **55** (1987), no. 1, 19–50. [MR883661](#) ([88d:42034](#)) ↑[2](#)
- [19] J. M. Wilson, *Weighted Littlewood-Paley theory and exponential-square integrability*, *Lecture Notes in Mathematics*, vol. 1924, Springer, Berlin, 2008. [MR2359017](#) ([2008m:42034](#)) ↑[2](#)

P.H.: DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, WACO, TEXAS 76798  
*E-mail address:* [paul\\_hagelstein@baylor.edu](mailto:paul_hagelstein@baylor.edu)

I.P.: DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DEL PAIS VASCO, APTDO. 644, 48080 BILBAO, SPAIN AND IKERBASQUE, BASQUE FOUNDATION FOR SCIENCE, BILBAO, SPAIN  
*E-mail address:* [ioannis.parissis@ehu.es](mailto:ioannis.parissis@ehu.es)

O.S.: DEPARTMENT OF MATHEMATICS AND SYSTEMS ANALYSIS, AALTO UNIVERSITY, FI-00076 AALTO, FINLAND  
*E-mail address:* [olli.saari@aalto.fi](mailto:olli.saari@aalto.fi)