PHASE SPACE LOCALIZING OPERATORS

MARCO FRACCAROLI, OLLI SAARI, AND CHRISTOPH THIELE

Abstract. We construct phase space localizing operators in all dimensions. These are frequency localized variants of the conditional expectation operator related to a dyadic stopping time. Our construction is an improvement over the so-called phase plane projections of Muscalu, Tao and the third author in one dimension. The motivation for such operators comes from time-frequency analysis. They are used in particular to prove uniform estimates for multilinear modulation invariant operators.

1. Introduction

Given a dyadic cube U and a finite partition $\mathcal{P}$ of U into smaller dyadic cubes, the corresponding conditional expectation operator is the orthogonal projection in $L^2(U)$ onto the subspace of functions measurable with respect to the sigma algebra $\Sigma_{\mathcal{P}}$ generated by $\mathcal{P}$. The conditional expectation g of a function $f \in L^2(U)$ can be written as

$$g = \sum_{I \in \mathcal{P}} A_I f, \tag{1.1}$$

$$A_I f(x) = 1_I(x) \frac{1}{|I|} \int_I f(y)\, dy. \tag{1.2}$$

The conditional expectation g satisfies

$$A_I(f - g) = 0 \tag{1.3}$$

whenever a dyadic cube I is in $\Sigma_{\mathcal{P}}$ and

$$1_I g - A_I g = 0 \tag{1.4}$$

whenever I is not in Σ_P. If we define

$$S := \sup_{I \in \Sigma_P} \|A_I f\|_\infty, \tag{1.5}$$

then the conditional expectation g satisfies

$$\|g\|_\infty \leqslant S. \tag{1.6}$$

Date: February 20, 2024.

2020 *Mathematics Subject Classification.* 42B15, 42C15.

Key words and phrases. Phase space localization, time-frequency analysis, modulation invariant operators, uniform estimates.

In the context of Fourier analysis, the sharp cutoffs in the definition of A_I destroy frequency information and are undesirable. Instead of A_I as above, one works with phase space localized operators of the type

$$A_I f = \rho_I^{-\alpha} \times (\phi_{|I|} * f). \tag{1.7}$$

Here ρ_I is a weight growing with relative distance to I, $\alpha > 0$ is sufficiently large, and $\phi_{|I|}$ is a convolution kernel whose Fourier transform is supported in the ball of radius $|I|^{-1}$ about the origin. There is long-standing interest in constructing operators with similar properties as the conditional expectation but based on the smoother operators (1.7). The obvious attempt to define a map from f to g as in (1.1) but with the operators A_I as in (1.7), an operator that is sometimes used in similar context, for example in slightly modified form in [2], fails for our purpose. Such a map does not satisfy strong enough replacements for (1.3), (1.4), (1.5) and (1.6).

The purpose of the present paper is to construct in Theorem 1.1 a function g based on a partition $\mathcal{P}$ and a function f so that suitable modifications of (1.3), (1.4), (1.5) and (1.6) hold, namely, in this order, (1.10), (1.11), (1.8) and (1.9). In place of unrealistically strong vanishing properties as in (1.3), (1.4), one has very strong estimates (1.10) and (1.11) in terms of estimates at least as strong as Carleson measure estimates. Such Carleson measure estimates are also called sparseness or outer $L^\infty(\ell^1)$ bounds ([4], [5]) in different parts of the literature. Note that we allow for a ratio 2^m between $|I|$ and the scale of ϕ in (1.7). Uniformity of our estimates in the parameter m is important for applications to uniform estimates for multilinear forms. A precedent and motivating example for our phase space localizing operators appears as so-called phase plane projections in [14] in dimension $d = 1$. We generalize the result to higher dimensions, simplify the construction, and strengthen the statement.

To state the main result in detail, we fix a dimension $d \geqslant 1$. For a finite axis-parallel cube $I \subset \mathbb{R}^d$ and $r \geqslant 1$, let rI denote the axis-parallel cube with the same center but r times the side length. Define for a point $y \in \mathbb{R}^d$ and a Borel set $F \subset \mathbb{R}$ the mollified distance

$$\rho_I(y) = \inf\{r > 1 : y \in (2r-1)I\}, \quad \rho_I(F) = \inf_{y \in F} \rho_I(y).$$

For an integer j and a real number $\alpha > d$, let Φ_j^α be the set of continuous functions ϕ on $\mathbb{R}^d$ with

$$|\phi| \leqslant 2^{-dj} \rho_{[0,2^j]^d}^{-\alpha}$$

and $\widehat{\phi}(\xi) = 0$ if $|\xi| \geqslant 2^{-j}$, where the Fourier transform is defined as

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^d} \psi(x) e^{-2\pi i x \cdot \xi}\, dx.$$

Let Ψ_j^α be the set of functions $\psi \in \Phi_j^\alpha$ with $\widehat{\psi}(\xi) = 0$ for $|\xi| \leqslant 2^{-j-2}$.

For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, let $Q_{j,k}$ be the dyadic cube consisting of all points $y \in \mathbb{R}^d$ with $y_n \in [2^j k_n, 2^j(k_n+1))$ for $1 \leqslant n \leqslant d$, where y_n and k_n are the components of y and k respectively. Let $\mathcal{D}_j$ be the set of cubes $Q_{j,k}$ with $k \in \mathbb{Z}^d$. Let $\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$ be the set of all dyadic cubes. For a finite collection M of pairwise disjoint cubes contained in a cube U, let M_U be the set of cubes $J \in \mathcal{D}$ such that there exists $I \in M$ with $I \subset J \subset U$.

Theorem 1.1 (Main theorem)**.** *Let $1 \leqslant p \leqslant \infty$ and $1/p + 1/p' = 1$. Let $\alpha > d$ be real. There exists $C = C_{p,d,\alpha} > 0$ such that for all $m \in \mathbb{N}$ the following holds.*

Let $i_0 \in \mathbb{Z}$ and $U \in \mathcal{D}_{i_0}$. Let M be a finite non-empty collection of pairwise disjoint cubes contained in U. There exists a linear mapping sending each locally bounded measurable function f on $\mathbb{R}^d$ with $\|f\rho_U^{-\alpha}\|_\infty < \infty$ to a Borel function g supported in $5U$ with the following properties. Denote

$$S \coloneqq \sup_{i\in\mathbb{Z}} \sup_{I\in\mathcal{D}_i\cap M_U} \sup_{\phi\in\Phi^{4\alpha}_{i-m-2}} 2^{-id/p}\|\rho_I^{-\alpha}\phi * f\|_p. \tag{1.8}$$

Then

$$\|g\|_p \leqslant CS2^{i_0 d/p}, \tag{1.9}$$

and for every $j \leqslant i_0$ and every $J \in \mathcal{D}_j$ it holds

$$\sum_{i\leqslant i_0} \sum_{\substack{I\in\mathcal{D}_i\cap M_U\\ I\subset J}} \sup_{\phi\in\Phi^{4\alpha}_{i-m}} 2^{id/p'}\|\rho_I^{-3\alpha}\phi * (f-g)\|_p \leqslant CS2^{jd}, \tag{1.10}$$

and if no $J' \in \mathcal{D}_j$ with $\rho_J(J') \leqslant 1$ contains an element of M_U, it holds

$$\sum_{i\leqslant i_0} \sup_{\substack{I\in\mathcal{D}_i\setminus M_U\\ I\subset J}} \sup_{\psi\in\Psi^{4\alpha}_{i-m}} 2^{-id/p}\|\rho_I^{-3\alpha}\psi * g\|_p \leqslant CS\|1_U\rho_J^{-\alpha}\|_\infty. \tag{1.11}$$

To emphasize dependence on parameters, denote the quantity in (1.8) by $S_{\alpha,p}$. One obtains estimates similar to (1.9), (1.10) and (1.11) with $S_{\alpha,p}$ replaced by $S_{\beta,q}$ with different parameters by using the following easy proposition.

Proposition 1.2. *Let $1 \leqslant p, q \leqslant \infty$ and $\alpha > d$. There exists $C > 0$ such that the following holds. Let i_0, U, M f, m as in Theorem 1.1. If $p \leqslant q$, then*

$$S_{\alpha(1+\frac{q-p}{qp}),p} \leqslant CS_{\alpha,q}. \tag{1.12}$$

If $q \leqslant p$, then

$$S_{\alpha,p} \leqslant CS_{\alpha,q}^{q/p}\|f\rho_U^{-\alpha}\|_\infty^{1-q/p} \tag{1.13}$$

and

$$S_{\alpha,\infty} \leqslant C2^{dm}S_{\alpha,1}. \tag{1.14}$$

The estimate (1.13) loses control in the size by virtue of the power q/p and is thus more useful for q/p close to one. The estimate (1.14) loses uniformity in the parameter m and is more useful for small values of m. Using Proposition 1.2 together with Theorem 1.1, we recover all the estimates that one might want to infer from Proposition 7.4 of [14]. We emphasize that the square sum term in the definition of the size in that paper is not needed to estimate the phase space localization errors in Theorem 1.1.

Concerning the technical aspects of the present paper, our construction starts with a telescoping sum involving characteristic functions of cubes as in [14]. We then correct the jump discontinuities appearing in the construction with a family of coronas based on smooth restrictions of certain primitives of Littlewood–Paley pieces. The correction part constructed here is different from that in [14] and more amenable to our setup. It provides vanishing of higher moments, which is needed when $d > 1$. It also provides control over derivatives of the phase space projection, which is needed to prove (1.11) when $d > 1$. As a consequence, we can treat general trees with a unified construction without distinguishing lacunary and non-lacunary trees as in [14]. In dimension $d = 1$, our construction provides a simple and streamlined substitute to [14].

Concerning applications of our main theorem, we recall that the precedent for our main theorem, the one-dimensional phase plane projections appearing in [14], was used to prove uniform estimates for multilinear singular integrals with modulation invariance. In such modulation invariant setting, one typically conjugates a Theorem like 1.1 with modulations. In more detail, one applies the theorem to the functions $M_{-\eta}f$, where the modulation M_η is defined by

$$M_\eta f(x) := f(x)e^{2\pi i x\cdot\eta}.$$

Let g_1 and g_2 be obtained by applying the theorem to $M_{-\eta_1}f$ and $M_{-\eta_2}f$ respectively with possibly different collections of cubes . If η_1 and η_2 are sufficiently far from each other, measured in terms of the geometry of the cubes, then $M_{\eta_1}g_1$ and $M_{\eta_2}g_2$ are almost orthogonal to each other thanks to the good frequency localization in Theorem 1.1. This is crucial in the application to modulation invariant operators. In the companion paper [6], we use Theorem 1.1 to obtain uniform bounds for multilinear singular integrals with modulation symmetries in the plane and space.

For further background on related uniform bounds in $d = 1$, we refer to [17], [7], [10], [12] and [18] for uniform bounds for bilinear Hilbert transforms in various regions of exponents and to [13], [8], [1], [11] and [16] on bilinear multipliers closely related to uniform bounds for the bilinear Hilbert transform. We also point out that the classical dyadic conditional expectation is used similarly together with Walsh–Fourier modulations in the context of Walsh models for multilinear singular integrals; see for example [15] and [9]. Finally, we mention that phase plane projections in time-frequency analysis are not only utilized for establishing uniform bounds; see [3] for an application to vector valued estimates.

Acknowledgement. The authors were funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under the project numbers 390685813 (EXC 2047: Hausdorff Center for Mathematics) and 211504053 (CRC 1060: Mathematics of Emergent Effects). The first author was supported by the Basque Government through the BERC 2022-2025 program and by the Ministry of Science and Innovation: BCAM Severo Ochoa accreditation CEX2021-001142-S / MICIN / AEI / 10.13039/501100011033. The second author was supported by Generalitat de Catalunya (2021 SGR 00087), Ministerio de Ciencia e Innovación and the European Union – Next Generation EU (RYC2021-032950-I), (PID2021-123903NB-I00), the Spanish State Research Agency through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).

2. Construction of the phase space projection

Let $\alpha > d$. The letter C will denote a sufficiently large positive number that may be implicitly re-adjusted from inequality to inequality and that may depend on α and d. We denote the Lebesgue measure of a measurable set E by $|E|$. For a multi-index $l \in \mathbb{N}^d$, we denote

$$|l| = \sum_{n=1}^{d} l_n, \quad l! = \prod_{n=1}^{d} l_n!,$$

and for $k, l \in \mathbb{N}^d$, we write $k \leqslant l$ if $k_n \leqslant l_n$ for all $1 \leqslant n \leqslant d$.

Let $m \in \mathbb{N}$. By dilation and translation that observe the dyadic grid, it suffices to prove Theorem 1.1 for $i_0 = 0$ and U the unit cube $[0,1)^d$, a normalization that we henceforth assume. Let M and f be given as in the theorem. Define for $j \in \mathbb{Z}$

$$T = M_U, \quad T_j^0 := T \cap \mathcal{D}_j, \quad E_j^0 := \bigcup T_j^0.$$

Define further for an integer $k \geqslant 1$

$$T_j^k := \{I \in \mathcal{D}_j : \rho_I(E_j^0) \leqslant k\}, \quad E_j^k := \bigcup T_j^k,$$

and for an integer $k \geqslant 0$

$$B_j^k := T_j^{k+1} \backslash T_j^k.$$

Define F_j^k to be $\mathbb{R}^d \backslash E_j^k$.

Let τ be a Schwartz function on $\mathbb{R}^d$ such that $\widehat{\tau}(\xi) = 0$ if $|\xi| \geqslant 2$ and $\widehat{\tau}(\xi) = 1$ if $|\xi| \leqslant 1$. Consider the approximation of unity formed by the family of convolution kernels

$$\tau_j(x) = 2^{-jd}\tau(2^{-j}x).$$

For the purpose of telescoping, define the difference kernels

$$\psi_j = \tau_{j-1} - \tau_j.$$

We observe $\widehat{\psi}_j(\xi) = 0$ if $|\xi| \geqslant 2^{2-j}$ or $|\xi| \leqslant 2^{-j}$.

2.1. **Cone decomposition.** We decompose ψ_j using the open cones

$$O_n = \{\xi \in \mathbb{R}^d : 2d|\xi_n|^2 > |\xi|^2\}$$

for $1 \leqslant n \leqslant d$. Any point $\xi \in \mathbb{R}^d$ not covered by any of the cones satisfies

$$d|\xi|^2 = \sum_{n=1}^{d} |\xi|^2 \geqslant \sum_{n=1}^{d} 2d|\xi_n|^2 = 2d|\xi|^2,$$

and thus $\xi = 0$. It follows that there is for each n a smooth function $\widehat{\chi}_n$ with $\widehat{\chi}_n(\xi) = 0$ if $\xi \notin O_n$ such that

$$\sum_{n=1}^{d} \widehat{\chi}_n(\xi) = 1$$

if $1 \leqslant |\xi| \leqslant 4$. Define

$$\chi_{n,j}(x) = 2^{-dj}\chi_n(2^{-j}x), \quad \psi_{n,j} = \psi_j * \chi_{n,j}.$$

In particular, we observe

$$\psi_j = \sum_{n=1}^{d} \psi_{n,j}.$$

2.2. **Construction.** The function $\widehat{\chi}_n$ vanishes on the set $2d|\xi_n|^2 \leqslant 1$. Hence we find Schwartz functions $\theta_{n,j}$ that satisfy

$$\partial_n^{d+1}\theta_{n,j} = \psi_{n,j}.$$

The function $\widehat{\theta}_{n,j}$ has the same support as $\widehat{\psi}_{n,j}$, in particular $\widehat{\theta}_{n,j} = 0$ unless $2^{-j} \leqslant |\xi| \leqslant 2^{2-j}$. Let κ be a smooth function mapping $\mathbb{R}^d$ to $[0, \infty)$ with $\kappa(x) = 1$ for $|x| \leqslant 2^{-10}$ and $\kappa(x) = 0$ for $|x| \geqslant 2^{-9}$. For $j \in \mathbb{Z}$ and $j \leqslant 0$, we set $\kappa_j(x) = 2^{-jd}\kappa_j(2^{-j}x)$. Define

$$\sigma_{n,j} := (-1_{E_j^1} + \kappa_{j-m} * \sum_{I \in \mathcal{D}_{j-m-3}, \rho_I(E_j^1) \leqslant 2} 1_I) \times (\theta_{n,j-m} * f). \tag{2.1}$$

The function $\sigma_{n,j}$ is supported on the set of points which have positive distance at most 2^{j-m-1} from E_j^1, in particular it is supported on E_j^2 and vanishes on E_j^1.

The following lemma follows immediately from the construction.

Lemma 2.1. *For each non-positive $j \in \mathbb{Z}$, $1 \leqslant n \leqslant d$, the function*

$$G_{n,j} := (\theta_{n,j-m} * f)(x) 1_{E_j^1} + \sigma_{n,j} \tag{2.2}$$

is infinitely differentiable in $\mathbb{R}^d$. For a multi-index $\beta \in \mathbb{N}^d$, one has for almost almost every x

$$\partial^\beta G_{n,j}(x) = (\partial^\beta \theta_{n,j-m} * f)(x) 1_{E_j^1}(x) + \partial^\beta \sigma_{n,j}(x), \tag{2.3}$$

where $\partial^\beta \sigma_{n,j}$ is the classical derivative at points outside the boundary of E_j^1 and remains undefined at points of the boundary of E_j^1.

Define

$$g_{n,j} := \partial_n^{d+1} G_{n,j}. \tag{2.4}$$

Finally, we set

$$\chi = \kappa_{-m} * \sum_{I \in \mathcal{D}_{-m-3}, \rho_I(E_0^1) \leqslant 2} 1_I,$$

so that $\chi(x) = 1$ for $x \in E_0^1$ and $\chi(x) = 0$ for all $x \in F_0^2$.
Define

$$g := (\tau_{-m} * f)\chi + \sum_{n=1}^{d} \sum_{j \leqslant 0} g_{n,j}. \tag{2.5}$$

In particular, g is zero outside $5U$.
We note for later use that

$$\tau_j \in C\Phi_{j-1}^{4\alpha}, \tag{2.6}$$
$$\psi_{n,j} \in C\Psi_{j-2}^{4\alpha}, \tag{2.7}$$
$$\partial^l \theta_{n,j} \in C 2^{j(d+1-|l|)} \Psi_{j-2}^{4\alpha}, \quad |l| \leqslant 3d+3, \tag{2.8}$$
$$\|\partial^l \chi\|_\infty \leqslant C 2^{m|l|}, \quad |l| \leqslant 3d+3, \tag{2.9}$$

for some C depending on the above choices of the Schwartz functions. We shall assume these functions are chosen so as to nearly minimize this C, hence C can be considered depending only on d and α.

3. Proof of inequality (1.9)

The construction of g was independent of p. To avoid notational differences between finite and infinite p in this section, we shall assume $1 \leqslant p < \infty$ and prove bounds uniformly in p. The case $p = \infty$ then follows by a limiting process as p tends to ∞.

We begin the proof of (1.9) by writing

$$g = h + \sum_{n=1}^{d} k_n$$

with

$$h := (\tau_{-m} * f)\chi + \sum_{j\leqslant 0}(\psi_{j-m} * f)1_{E_j^1}, \tag{3.1}$$

$$k_n := \sum_{j\leqslant 0} \partial_n^{d+1}\sigma_{n,j}. \tag{3.2}$$

By the triangle inequality, it suffices to prove $\|h\|_p \leqslant CS$ and $\|k_n\|_p \leqslant CS$ for each n separately. We begin with h.

Definition 3.1. *Let $j < 0$ be an integer. We define $\mathcal{I}_j$ to be the set of dyadic cubes in $\mathcal{D}_j$ which are contained in E_{j+1}^1 but not contained in E_j^1. We set $\mathcal{I} = \bigcup_{i<0}\mathcal{I}_i$.*

Lemma 3.2. *The family $\mathcal{I}$ partitions E_0^1.*

Proof. Let $x \in E_0^1$. By nesting of the sets E_j^1 and finiteness of M, there is a minimal $j < 0$ such that $x \in E_{j+1}^1$. The cube $I \in \mathcal{D}_j$ containing x is then in $\mathcal{I}_j$. Hence $\mathcal{I}$ covers E_0^1.

To see that the cubes in $\mathcal{I}$ are pairwise disjoint, assume to get a contradiction that $I \subset J$ for some $i < j < 0$ and $I \in \mathcal{I}_i$ and $J \in \mathcal{I}_j$. Then $I \subset E_{i+1}^1$ and thus $I \subset E_j^1$. By the dyadic structure, $J \subset E_j^1$. This contradicts the choice of J. □

Lemma 3.3. *Let $I \in \mathcal{I}$. Then*

$$\|h1_I\|_p \leqslant CS|I|^{1/p}. \tag{3.3}$$

Proof. Fix $i < 0$ and $I \in \mathcal{I}_i$. As $I \subset E_{i+1}^1$ but I is disjoint from E_i^1, we have by telescoping

$$h1_I = (\tau_{i+1-m} * f)1_I.$$

Moreover, there is a $K \in T_{i+1}^0$ with $\rho_K(I) \leqslant 2$. We have

$$\tau_{i+1-m} \in C\Phi_{i-m}^{4\alpha},$$

and thus by the definition of S in (1.8),

$$\|h1_I\|_p \leqslant C\|\rho_K(\tau_{i+1-m} * f)\|_p \leqslant CS2^{id/p}.$$

□

As $\mathcal{I}$ is a partition of $E_0^1 = 3U$ by Lemma 3.2 and as h is supported on $3U$, we obtain with Lemma 3.3

$$\|h\|_p^p \leqslant \sum_{I\in\mathcal{I}}\|h1_I\|_p^p \leqslant CS\sum_{I\in\mathcal{I}}2^{id} \leqslant CS.$$

This completes the proof of the bound for h.

It remains to prove the bound for k_n. Note that $\partial_n^{d+1}\sigma_{n,i}$ is supported in $\bigcup B_i^1$.

Lemma 3.4. *For each $i \leqslant 0$, $1 \leqslant n \leqslant d$, $I \in B_i^1$ and each multi-index $l \in \mathbb{N}^d$ with $|l| \leqslant 2d+2$ it holds*

$$\|1_I\partial^l\sigma_{n,i}\|_p^p \leqslant C^p 2^{p(i-m)(d+1-|l|)}S^p|I|. \tag{3.4}$$

Proof. We define
$$\tilde{E}_i = \sum_{I\in\mathcal{D}_{i-m-3},\rho_I(E_i^1)\leqslant 2} 1_I.$$
We apply the Leibniz rule in the interior of I to estimate the left hand side of (3.4) by
$$C^p \sum_{l_1+l_2=l} \int_I |\partial^{l_1}(1_{\tilde{E}_i} * \kappa_{i-m})(x)|^p |\partial^{l_2}(\theta_{n,i-m} * f)(x)|^p \, dx.$$
We estimate
$$1_I(x)|\partial^{l_1}(1_{\tilde{E}_i} * \kappa_{i-m})(x)|^p \leqslant C2^{p(m-i)|l_1|}1_I(x)$$
by a rescaled version of (2.9). Using the control (2.8) on $\partial^{l_2}\theta_{n,i-m}$, we see that
$$\int_I |\partial^{l_2}(\theta_{n,i-m} * f)(y)|^p \, dy \leqslant C^p 2^{p(i-m)(d+1-|l_2|)} S^p,$$
where we also used the definition of S for a cube $J \in T_i^0$ with $\rho_J(I) \leqslant 2$. This concludes the proof of the lemma. □

Lemma 3.5. *For $1 \leqslant n \leqslant d$ and k_n from (3.2), it holds*
$$\|k_n\|_p \leqslant CS. \tag{3.5}$$

Proof. We estimate
$$\|k_n\|_p^p \leqslant \sum_{i\leqslant 0}\sum_{I\in B_i^1} \|1_I\partial_n^{d+1}\sigma_{n,i}\|_p^p \leqslant \sum_{i\leqslant 0}\sum_{I\in B_i^1} C^p S^p |I|$$
where we applied Lemma 3.4 with $|l| = d + 1$. To conclude (3.5), it now suffices to show that the cubes in $\bigcup_{i\leqslant 0} B_i^1$ are pairwise disjoint and contained in $3U$. Containment in $3U$ is clear by definition. To see disjointness, first note that the cubes in B_i^1 are pairwise disjoint for each i. Now let $i < k \leqslant 0$ and assume $J \in B_i^1$ and $K \in B_k^1$ with $J \subset K$. Then there is a point x in the closure of E_i^0 such that $\rho_J(x) \leqslant 1$. But then $\rho_K(x) \leqslant 1$ and x must be in the closure of E_k^0, which is a contradiction to $K \in B_k^1$. This proves disjointness and completes to proof of the lemma. □

4. Proof of Inequality (1.10)

In this section, we fix $1 \leqslant p \leqslant \infty$. In particular, notation in this section applies to $p = \infty$ as well.

We begin with a preparatory lemma that allows us to commute a weight past certain convolution and truncation operators. Let 1_F denote the characteristic function of a set F.

Lemma 4.1. *For all $i \in \mathbb{Z}$, all finite cubes $I \in \mathcal{D}_i$, all Borel sets $F \subset \mathbb{R}^d$, all functions ϕ with*
$$|\phi(x)| \leqslant 2^{(m-i)d}\rho_{[0,2^{i-m}]^d}(x)^{-4\alpha},$$

and for all functions g in $L^p(\mathbb{R}^d)$, we have

$$\|\rho_I^{-3\alpha}(\phi * (1_F g))\|_p \leqslant C\|\rho_I^{-\alpha} g\|_p \rho_I(F)^{-2\alpha}. \tag{4.1}$$

We remark that given $k \leqslant 0$ and $\phi \in \Phi^{4\alpha}_{i-m+k}$, the function $2^{kd}\phi$ satisfies the assumption of the lemma.

Proof. By simultaneous translation of I, F, and g, if necessary, we may assume that I contains the origin. Let $J \in \mathcal{D}$ be a cube in $\mathcal{D}_{i-m}$ containing zero, so that

$$|\phi| \leqslant C2^{(m-i)d}\rho_J^{-4\alpha}.$$

For every $x \in \mathbb{R}^d$, we estimate

$$|\rho_I^{-3\alpha}(\phi * (1_F g))(x)| \leqslant C2^{(m-i)d} \int \rho_I^{-3\alpha}(x)\rho_J^{-4\alpha}(x-y)1_F(y)|g|(y)\,dy. \tag{4.2}$$

For $|y| \geqslant 2|x|$, we estimate

$$\rho_I(x) \geqslant 1, \quad \rho_J(x-y) \geqslant C\rho_I(y).$$

For $|y| \leqslant 2|x|$, we estimate

$$\rho_J(x-y) \geqslant 1, \quad \rho_I(x) \geqslant C\rho_I(y).$$

We thus obtain for (4.2) an upper bound by

$$C2^{(m-i)d} \int \rho_J^{-\alpha}(x-y)(\rho_I^{-2\alpha}1_F)(y)(\rho_I^{-\alpha}|g|)(y)\,dy.$$

Using Young's convolution inequality and the uniformly bounded L^1 norm of $2^{(m-i)d}\rho_J^{-\alpha}$, one obtains the lemma. □

We write almost everywhere

$$f - g = (\tau_{-m} * f)(1-\chi) + \sum_{j\leqslant 0}(\psi_{j-m} * f)1_{F_j^1} - \sum_{n=1}^{d} \partial_n^{d+1}\sigma_{n,j}, \tag{4.3}$$

which follows from

$$f = (\tau_{-m} * f) + \sum_{j\leqslant 0}\psi_{j-m} * f,$$

$$g = (\tau_{-m} * f)\chi + \sum_{j\leqslant 0}(\psi_{j-m} * f)1_{E_j^1} + \sum_{n=1}^{d} \partial_n^{d+1}\sigma_{n,j}.$$

Lemma 4.2. *For all integers $i \leqslant j \leqslant 0$ and all $I \in T_i^0$, we have*

$$\|\rho_I^{-3\alpha}\phi_{i-m} * ((\tau_{-m} * f)(1-\chi))\|_p \leqslant CS\rho_I(F_0^1)^{-2\alpha}, \tag{4.4}$$

$$\|\rho_I^{-3\alpha}\phi_{i-m} * ((\psi_{j-m} * f)1_{F_j^1})\|_p \leqslant CS2^{jd/p}\rho_I(F_j^1)^{-2\alpha}, \tag{4.5}$$

$$\|\rho_I^{-3\alpha}\phi_{i-m} * \left(\partial_n^{d+1}\sigma_{n,j}\right)\|_p \leqslant CS2^{jd/p}\rho_I(F_j^1)^{-2\alpha}. \tag{4.6}$$

The right-hand sides of both of the inequalities (4.5) *and* (4.6) *are bounded by*

$$CS2^{-id/p'}2^{(\alpha-d/p)(i-j)}\|1_I \sum_{K\in B_i^0} \rho_K^{-\alpha}\|_1. \tag{4.7}$$

Proof. We begin with (4.4). As $1-\chi$ is supported in F_0^1, we use Lemma 4.1 with $\phi_{i-m} \in \Phi_{i-m}^{4\alpha}$ and then $I \subset U$ to estimate the left-hand-side of (4.4) by

$$C\rho_I(F_0^1)^{-2\alpha}\|\rho_I^{-\alpha}(\tau_{-m} * f)\|_p \leqslant C\rho_I(F_0^1)^{-2\alpha}\|\rho_U^{-\alpha}(\tau_{-m} * f)\|_p.$$

With the definition of S, using the control (2.6) of τ_{-m} and $U \in T$, this proves (4.4).

To estimate the left-hand-side of (4.5), choose a cube $J \in \mathcal{D}_j$ that contains I. Using Lemma 4.1 as above, we estimate the left-hand-side of (4.5) by

$$C\rho_I(F_j^1)^{-2\alpha}\|\rho_I^{-\alpha}(\psi_{j-m} * f)\|_p \leqslant C\rho_I(F_j^1)^{-2\alpha}\|\rho_J^{-\alpha}(\psi_{j-m} * f)\|_p.$$

Using the control (2.7) of ψ_{j-m} and $J \in T$, this proves (4.5).

As $\sigma_{n,j}$ is supported in $E_j^2 \cap F_j^1$, we estimate the left hand side of (4.6) again with Lemma 4.1 by a constant multiple of

$$\rho_I(F_j^1)^{-2\alpha}\|\rho_I^{-\alpha}\sum_{K\in B_j^1} 1_K\partial_n^{d+1}\sigma_{n,j}\|_p \leqslant CS\rho_I(F_j^1)^{-2\alpha}2^{jd/p}\sum_{K\in B_j^1}\sup_{x\in K}\rho_I^{-\alpha}(x).$$

In the last inequality, we used equation (3.4) from Lemma 3.4 and that each $K \in B_j^1$ is contained in F_j^1. This proves (4.6), because $i \leqslant j$ and thus for every integer $n \geqslant 0$ there are at most $C(1+|n|)^{d-1}$ cubes $K \in B_j^1$ such that

$$|n| + 1 \leqslant \inf_{x\in K}\rho_I(x) \leqslant |n| + 2,$$

and thus

$$\sum_{K\in B_j^1}\sup_{x\in K}\rho_I^{-\alpha}(x) \leqslant C\sum_{n\in\mathbb{Z}}(1+|n|)^{d-\alpha-1} \leqslant C.$$

We turn to showing the upper bound (4.7). By nesting, $\rho_I(F_j^1) \geqslant \rho_I(F_i^1)$. On the other hand, there is a cube in B_j^0 between I and F_j^1. Hence $\rho_I(F_j^1) \geqslant 2^{j-i}$, and we estimate the right-hand side of (4.5) and (4.6) by

$$CS2^{jd/p}2^{-\alpha(j-i)}\rho_I(F_i^1)^{-\alpha} \leqslant CS2^{id/p}2^{-(\alpha-d/p)(j-i)}\rho_I(F_i^1)^{-\alpha}.$$

As there is a cube in B_i^0 as close to I as to F_i^1, we have

$$\rho_I(F_i^1)^{-\alpha} \leqslant \inf_{x\in I}\sum_{K\in B_i^0}\rho_K(x)^{-\alpha} \leqslant 2^{-id}\|1_I\sum_{K\in B_i^0}\rho_K^{-\alpha}\|_1.$$

This proves (4.7) and completes the proof of the lemma. □

Lemma 4.3. *We have for all integers $j \leqslant i \leqslant 0$, every dyadic cube $I \in T_i^0$ and every $1 \leqslant n \leqslant d$,*

$$\|\rho_I^{-3\alpha}\phi_{i-m} * \Big((\psi_{n,j-m} * f)1_{F_j^1} - \partial_n^{d+1}\sigma_{n,j}\Big)\|_p \leqslant CS2^{(j-i)(1+d/p)}2^{-id/p'} \sum_{J\in B_j^0} \|\rho_I^{-\alpha}1_J\|_1. \quad (4.8)$$

Proof. Note that the term in brackets on the left-hand-side of (4.8) has a smooth extension to $\mathbb{R}^d$ by Lemma 2.1. By a $(d+1)$-fold integration by parts and the sufficient decay of $\partial_n^k\phi_{i-m}$ for all $1 \leqslant k \leqslant d$, we rewrite and then estimate the left hand side of (4.8) as follows:

$$\|\rho_I^{-3\alpha}(\partial_n^{d+1}\phi_{i-m}) * \Big((\theta_{n,j-m} * f)1_{F_j^1} - \sigma_{n,j}\Big)\|_p$$
$$\leqslant \|\rho_I^{-3\alpha}(\partial_n^{d+1}\phi_{i-m} * ((\theta_{n,j-m} * f)1_{F_j^1}))\|_p \quad (4.9)$$
$$+ \sum_{J\in B_j^1} \|\rho_I^{-3\alpha}(\partial_n^{d+1}\phi_{i-m} * (1_J\sigma_{n,j}))\|_p. \quad (4.10)$$

We first estimate (4.10). As $\partial_n^{d+1}\phi_{i-m} \in 2^{(m-i)(d+1)}\Phi_{m-i}^{4\alpha}$, we estimate each summand in (4.10) first with Lemma 4.1 and then apply equation (3.4) from Lemma 3.4 to obtain an upper bound by

$$C2^{(m-i)(d+1)}\|\rho_I^{-\alpha}1_J\sigma_{n,j}\|_p$$
$$\leqslant CS2^{(m-i)(d+1)}2^{(j-m)(d+1)}2^{jd/p}\|\rho_I^{-\alpha}1_J\|_\infty$$
$$\leqslant CS2^{(j-i)(1+d/p)}2^{-id/p'}\|\rho_I^{-\alpha}1_J\|_1.$$

We used $j < i$ in the last inequality to conclude that $\rho_I^{-\alpha}$ is essentially constant on J. The desired estimate for (4.10) follows from summing over $J \in B_j^1$.

To estimate (4.9), we write it as

$$\|\rho_I^{-3\alpha}\partial_n^{d+1}\phi_{i-m} * \tau_{j-m} * ((\theta_{n,j-m} * f)1_{F_j^1})\|_p,$$

because $\widehat{\tau}_{j-m+1}$ is constant one on the support of $\widehat{\partial_n^{d+1}\phi}_{i-m}$ when $i \geqslant j+1$. By Lemma 4.1, this is estimated by

$$C2^{(m-i)(d+1)}\|\rho_I^{-\alpha}(\tau_{j-m+1} * ((\theta_{n,j-m} * f)1_{F_j^1}))\|_p$$
$$\leqslant C2^{(m-i)(d+1)} \sum_{K\in\mathcal{D}_j} \rho_I^{-\alpha}(K)\|1_K(\tau_{j-m+1} * ((\theta_{n,j-m} * f)1_{F_j^1}))\|_p.$$

Fix $K \in \mathcal{D}_j$. First assume $K \subset E_j^1$. Because $1_K \leqslant \rho_K^{-3\alpha}$, it follows by Lemma 4.1 that

$$\|1_K(\tau_{j-m+1} * ((\theta_{n,j-m} * f)1_{F_j^1}))\|_p \leqslant C\rho_K(F_j^1)^{-2\alpha}\|\rho_K^{-\alpha}(\theta_{n,j-m} * f)\|_p$$
$$\leqslant CS2^{(j-m)(d+1)}2^{jd/p}\rho_K(F_j^1)^{-2\alpha} \leqslant CS2^{(j-m)(d+1)}2^{jd/p} \sum_{J\in B_j^1} \rho_J(K)^{-2\alpha}.$$

The penultimate inequality followed by $K \subset E_j^1$, and thus there existing a cube $K' \in T_j$ with distance at most 2^{j+1} from K, and applying the definition of S to K'. The last inequality followed by choosing a J in B_j^1 closest to K so that

$$\rho_K(F_j^1)^{-2\alpha} \leqslant C\rho_K(J)^{-2\alpha} = C\rho_J(K)^{-2\alpha},$$

where the last identity follows from equal side length of J and K.

Assume then $K \subset F_j^1$. As $\widehat{\theta}_{n,j-m}$ vanishes on the support of $\widehat{\tau}_{j-m+1}$ and $1_{F_j^1} + 1_{E_j^1} = 1$, we have

$$\|1_K(\tau_{j-m+1} * ((\theta_{n,j-m} * f)1_{F_j^1}))\|_p = \|1_K(\tau_{j-m+1} * ((\theta_{n,j-m} * f)1_{E_j^1}))\|_p \leqslant C\rho_K(E_j^1)^{-2\alpha}\|\rho_K^{-1}(\theta_{n,j-m} * f)\|_p \quad (4.11)$$

Now let K' be a cube in B_j^0 closest to K, so that

$$\rho_K(E_j^1)^{-1} \leqslant C\rho_K(K')^{-1}.$$

With the triangle inequality, we conclude

$$\rho_{K'}(x) \leqslant C\rho_K(x)\rho_K(E_j^1)$$

so that

$$\rho_K(x) \geqslant \rho_{K'}(x)\rho_K(E_j^1)^{-1} \geqslant \rho_{K'}(x)\rho_K(E_j^1)^{-\alpha}$$

and hence we can estimate (4.11) by

$$\rho_K(K')^{-\alpha}\|\rho_{K'}^{-1}(\theta_{n,j-m} * f)\|_p \leqslant CS2^{(j-m)(d+1)}2^{jd/p}\sum_{J\in B_j^0}\rho_J(K)^{-\alpha}.$$

Putting the above estimates together, we estimate (4.9) by

$$\begin{aligned} CS2^{(j-i)(d+1)}2^{jd/p}&\sum_{K\in\mathcal{D}_j}\rho_I^{-\alpha}(K)\sum_{J\in B_j^1}\rho_J(K)^{-\alpha} \\ &\leqslant CS2^{(j-i)(d+1)}2^{-jd/p'}\|\rho_I^{-\alpha}\sum_{J\in B_j^1}\rho_J^{-\alpha}\|_1 \\ &\leqslant CS2^{(j-i)(d/p+1)}2^{jd/p}\sum_{J\in B_j^1}\|\rho_I^{-\alpha}1_J\|_\infty \\ &\leqslant CS2^{(j-i)(d/p+1)}2^{-id/p'}\sum_{J\in B_j^1}\|\rho_I^{-\alpha}1_J\|_1. \end{aligned}$$

Here we used $j \leqslant i$ in the estimation of integrals. □

We are now ready to prove inequality (1.10). We claim that it suffices to prove inequality (1.10) under the additional assumption $J \in T$. Assume we have proven the inequality under this additional assumption. Let J be arbitrary and consider the collection $\mathcal{J}$ of maximal dyadic cubes I satisfying $I \subset J$ and $I \in T$. Applying the assumed inequality with J replaced by an element of $\mathcal{J}$ and summing over the disjoint cubes in $\mathcal{J}$ we obtain the inequality for the given J. Hence we can make the auxiliary assumption.

Following the decomposition (4.3) and splitting the sum on the left hand side of (1.10) along the cases $k < i$ and $i \leqslant k$, we may apply Lemma 4.2 to the terms with $k < i$ and Lemma 4.3 to the terms with $i \leqslant k$. Hence we are left with estimating

$$CS \sum_{i\leqslant j} \sum_{I\in T_i^0, I\subset J} \sum_{i\leqslant k\leqslant 0} 2^{\epsilon(i-k)} \sum_{K\in B_i^0} \|1_I \rho_K^{-\alpha}\|_1 \tag{4.12}$$

$$+CS \sum_{i\leqslant j} \sum_{I\in T_i^0, I\subset J} \sum_{k<i} 2^{\epsilon(k-i)} \sum_{K\in B_k^0} \|\rho_I^{-\alpha} 1_K\|_1, \tag{4.13}$$

where $\epsilon = \min(\alpha - d/p, 1 + d/p)$, by the right hand side of equation (1.10). We estimate (4.12) and (4.13) separately.

We first estimate (4.12) from above by

$$\begin{aligned} CS \sum_{i\leqslant j} \sum_{I\in T_i^0, I\subset J} \|1_I \sum_{K\in B_i^0} \rho_K^{-\alpha}\|_1 &\leqslant CS \sum_{i\leqslant j} \|1_J \sum_{K\in B_i^0} \rho_K^{-\alpha}\|_1 \\ &\leqslant CS \sum_{i\leqslant j} \left(\sum_{K\in B_i^0, K\subset 3J} 2^{id} + \|1_J \sum_{K\in B_i^0, K\not\subset 3J} \rho_K^{-\alpha}\|_1 \right) \\ &\leqslant CS2^{jd} + CS \sum_{i\leqslant j} 2^{(\alpha-d)(i-j)/2} 2^{jd} \leqslant CS2^{jd}. \end{aligned}$$

Then we estimate (4.13) by

$$\begin{aligned} CS \sum_{k<j} \sum_{k<i\leqslant j} \sum_{I\in T_i^0, I\subset J} 2^{-\epsilon|k-i|} \sum_{K\in B_k^0} \|\rho_I^{-\alpha} 1_K\|_1 &\leqslant CS \sum_{k<j} \sum_{k<i\leqslant j} 2^{-\epsilon|k-i|} \sum_{K\in B_k^0} \|\rho_J^{-\alpha} 1_K\|_1 \\ &\leqslant CS \sum_{k<j} \sum_{K\in B_k^0} \|\rho_J^{-\alpha} 1_K\|_1 \leqslant CS\|\rho_J^{-\alpha}\|_1 \leqslant CS2^{jd}. \end{aligned}$$

These are the desired estimates for (4.12) and (4.13). The proof of inequality (1.10) is complete.

5. Proof of Inequality (1.11)

In this section, we assume $1 \leqslant p < \infty$. We will prove estimates with constants independent of p and thus the estimates will extend to the case $p = \infty$.

We first prove estimates for cubes that are far away from T.

Lemma 5.1. *Let $i \leqslant 0$ and $I \in \mathcal{D}_i$ satisfy $I \not\subset 7U$. Then*

$$2^{-id/p} \|\rho_I^{-3\alpha} \psi * g\|_p \leqslant CS2^{i\alpha} \|1_U \rho_I^{-\alpha}\|_\infty. \tag{5.1}$$

Proof. As $g = 1_{5U} g$, we can apply Lemma 4.1 to obtain

$$\|\rho_I^{-3\alpha}\psi * g\|_p \leqslant C\rho_I(5U)^{-2\alpha}\|g\rho_I^{-\alpha}\|_p.$$

We use the L^p bound (1.9) on g, which we have already established, and that the support of g is contained in $5U$, to conclude

$$\|g\rho_I^{-\alpha}\|_p \leqslant \|g\|_p\|1_{5U}\rho_I^{-\alpha}\|_\infty \leqslant CS\|1_U\rho_I^{-\alpha}\|_\infty.$$

In the last inequality, we replaced $5U$ by U, thanks to the assumptions on size and location of I. As $i \leqslant 0$ and $I \not\subset 5U$, inequality (5.1) then follows from $\rho_I(5U)^{-2\alpha} \leqslant 2^{2i\alpha} \leqslant 2^{i\alpha}2^{id/p}$. □

Let $\mathcal{P}$ be the collection of maximal dyadic cubes K such that $K \subset 7U$ but $3K$ does not contain any cube from T. By maximality, the cubes in $\mathcal{P}$ are pairwise disjoint. They also have side-length at most 1.

Lemma 5.2. *Let $i \leqslant k \leqslant 0$, $I \in \mathcal{D}_i$ with $I \subset 7U$ and $\psi \in \Psi^{4\alpha}_{i-m}$. If there exists $K \in \mathcal{P} \cap \mathcal{D}_k$ such that $I \subset K$, then*

$$2^{-id/p}\|\rho_I^{-3\alpha}\psi * g\|_p \leqslant CS\left(\rho_I((\tfrac{3}{2}K)^c)^{-2\alpha} + 2^{i-k}\right). \tag{5.2}$$

Proof. We write g as sum of small scales and large scales,

$$g_s = \sum_{j=-\infty}^{k-1}\left((\psi_{j-m} * f)1_{E_j^1} + \sum_{n=1}^{d}\partial_n^{d+1}\sigma_{n,j}\right), \tag{5.3}$$

$$g_l = (\tau_{-m} * f)\chi + \sum_{j=k}^{0}\left((\psi_{j-m} * f)1_{E_j^1} + \sum_{n=1}^{d}\partial_n^{d+1}\sigma_{n,j}\right). \tag{5.4}$$

By the triangle inequality, it suffices to prove analogs of (5.2) for the summands separately.

Small scales. Let F be the union of dyadic cubes $B \in \mathcal{D}_{k-2}$ with $\rho_B(E_{k-1}^1) \leqslant 1$. The term corresponding to the small scales satisfies $g_s = 1_F g_s$, and we have by Lemma 4.1

$$\|\rho_I^{-3\alpha}(\psi * g_s)\|_p \leqslant C\rho_I(F)^{-2\alpha}\|\rho_I^{-\alpha}g_s\|_p.$$

As the interior of K is disjoint from E_k^1, we conclude the interiors of $\frac{3}{2}K$ and F are disjoint so that

$$\rho_I(F)^{-2\alpha} \leqslant \rho_I((\tfrac{3}{2}K)^c)^{-2\alpha}.$$

Our estimate of g_s will be complete once we show

$$\|\rho_I^{-\alpha}g_s\|_p \leqslant 2^{id/p}S.$$

For h and k_n as defined in (3.1) and (3.2), it holds

$$g_s = -1_{E_{k-1}^1}(\tau_{k-1-m} * f) + h + \sum_{n=1}^{d}k_n. \tag{5.5}$$

We estimate the three summands separately.

We start by noticing that for any $J \in \mathcal{D}_j$ with $j \leqslant k-1$ and $J \subset E_j^1$ it holds $\rho_J(I) \geqslant 2$. Suppose that such a cube J is given. If $j > i$, then

$$\rho_I(J) \geqslant C 2^{j-i} \rho_J(I) \geqslant C \sup_{y \in J} \rho_I(y)$$

and for $j \leqslant i$

$$\rho_I(J) \geqslant C \sup_{y \in J} \rho_I(y) \tag{5.6}$$

is immediate. We conclude that for all cubes J as above the inequality (5.6) holds.

First term in (5.5). For each cube $J \in \mathcal{D}_{k-1}$ with $J \subset E_{k-1}^1$, we note that

$$\|\rho_I^{-\alpha} 1_J (\tau_{k-1-m} * f)\|_p^p \leqslant \rho_I(J)^{-\alpha p} \|1_J(\tau_{k-1-m} * f)\|_p^p \leqslant C^p S^p \rho_I(J)^{-\alpha p} |J|. \tag{5.7}$$

In the last inequality, we have estimated the L^p norm using the definition of S and that there is an cube near J in T_{k-1}^0. As $\rho_J(I) \geqslant 2$, we can invoke (5.6). Summing over the disjoint cubes $J \in \mathcal{D}_{k-1}$ with $J \subset E_{k-1}^1$ and using (5.7), we hence obtain

$$\|\rho_I^{-\alpha} 1_{E_{k-1}^1} (\tau_{k-1-m} * f)\|_p^p \leqslant C^p S^p \int \rho_I(y)^{-\alpha p} \, dy \leqslant C^p S^p |I|.$$

This gives the desired bound for the first term in (5.5).

Second term in (5.5). We turn to the second term in (5.5). We use the decomposition given in Lemma 3.2

$$1_{E_{k-1}^1} = \sum_{J \in \mathcal{I}_{<k-1}} 1_J.$$

Fix a cube $J \in \mathcal{I}_j$ with $j < k-1$. By Lemma 3.3

$$\|\rho_I^{-\alpha} 1_J h\|_p^p \leqslant C^p \rho_I(J)^{-\alpha p} S^p |J|.$$

Using now the fact $\rho_J(I) \geqslant 2$ and its consequence (5.6), we estimate

$$\sum_{J \in \mathcal{I}_{<k-1}} C^p \rho_I(J)^{-\alpha p} S^p |J| \leqslant C^p S^p \int \rho_I(x)^{-\alpha p} \, dx \leqslant C S^p |I|.$$

This proves the desired estimate for h.

Third term in (5.5). To bound the term k_n in (5.5), we apply (3.4) in Lemma 3.4 and estimate

$$\sum_{n=1}^{d} \|\rho_I^{-\alpha} k_n\|_p^p \leqslant C^p \sum_{j=-\infty}^{k-1} \sum_{n=1}^{d} \sum_{J \in B_j^1} \rho_I(\partial E_j^1 \cap \partial J)^{-\alpha p} \|1_J \partial_n^{d+1} \sigma_{n,j}\|_p^p$$

$$\leqslant C^p \sum_{j=-\infty}^{k-1} \sum_{J \in B_j^1} \rho_I(\partial E_j^1 \cap \partial J)^{-\alpha p} 2^{jd} S^p \tag{5.8}$$

For each $J \in B_j^1$ there exists $\tilde{J} \subset E_j^1$ with

$$\rho_I(\partial E_j^1 \cap \partial J) \geqslant C\rho_I(\partial E_j^1 \cap \partial J \cap \partial \tilde{J}).$$

We can then use $\rho_{\tilde{J}}(I) \geqslant 2$ to apply (5.6) to $\tilde{J}$ and to estimate

$$\rho_I(\partial E_j^1 \cap \partial J) \geqslant C \sup_{y \in \tilde{J}} \rho_I(y) \geqslant C \sup_{y \in 4^{-1}J} \rho_I(y).$$

This together with disjointness of $J \in B_j^1$ for all j implies

$$C^p \sum_{j=-\infty}^{k-1} \sum_{J \in B_j^1} \rho_I(\partial E_j^1 \cap \partial J)^{-\alpha p} 2^{jd} \leqslant C^p \int \rho_I(x)^{-\alpha p}\, dx \leqslant C^p |I|$$

so that the right hand side of (5.8) is bounded by $C^p|I|S^p$ as claimed. This concludes the proof of the bound for the small scales (5.3).

Large scales. We turn to estimate (5.4). Consider first the sum

$$\sum_{n=1}^{d} \sum_{j=k}^{0} \left((\psi_{n,j-m} * f) 1_{E_j^1} + \partial_n^{d+1} \sigma_{n,j} \right). \tag{5.9}$$

Following section 2.1, we form the cone decomposition

$$\psi = \sum_{\nu=1}^{d} \psi_\nu,$$

where the Fourier transform of ψ_ν vanishes whenever $2d|\xi_\nu|^2 \leqslant |\xi|^2$. We notice

$$2^{-(d+1)(i+m)} \partial_\nu^{-d-1} \psi_\nu \in C\Phi_{i-m}^{4\alpha}, \quad 2^{(d+1)(j-m)} \partial_\nu^{d+1} \psi_{n,j-m} \in C\Psi_{j-m}^{4\alpha}$$

where ∂_ν^{-d-1} is used to denote the Fourier multiplier operator with symbol $(2\pi i \xi_\nu)^{-d-1}$.

The summands in (5.9) are $d+1$ times differentiable by Lemma 2.1, and integrating by parts we obtain

$$\begin{aligned} &\|\rho_I^{-3\alpha} \psi_\nu * \left((\psi_{n,j-m} * f) 1_{E_j^1} + \partial_n^{d+1} \sigma_{n,j} \right) \|_p \\ &\quad = \|\rho_I^{-3\alpha} \partial_\nu^{-d-1} \psi_\nu * \left((\partial_\nu^{d+1} \psi_{n,j-m} * f) 1_{E_j^1} + \partial_\nu^{d+1} \partial_n^{d+1} \sigma_{n,j} \right) \|_p \\ &\qquad \leqslant \|\rho_I^{-3\alpha} \partial_\nu^{-d-1} \psi_\nu * ((\partial_\nu^{d+1} \psi_{n,j-m} * f) 1_{E_j^1}) \|_p \\ &\qquad\qquad + \|\rho_I^{-3\alpha} \partial_\nu^{-d-1} \psi_\nu * \partial_\nu^{d+1} \partial_n^{d+1} \sigma_{n,j} \|_p. \end{aligned} \tag{5.10}$$

By Lemma 4.1, the first term is bounded by

$$C 2^{(d+1)(i-j)} \|\rho_I^{-\alpha} (2^{(d+1)(j-m)} \partial_\nu^{d+1} \psi_{n,j-m} * f)\|_p.$$

Because $j \geqslant k$, we can find $\widehat{I}$ with $\widehat{I} \in \mathcal{D}_j$ and $\widehat{I} \supset K \supset I$. We know by maximality that $9K$ contains a cube from T, and hence $\rho_{\widehat{I}}(E_j^0) \leqslant 2$.

Because $\rho_I^{-\alpha} \leqslant \rho_{\hat{I}}^{-\alpha}$, we conclude that

$$\sum_{j=k}^{0} 2^{(d+1)(i-j)} C \|\rho_I^{-\alpha}(2^{(d+1)(j-m)}\partial_\nu^{d+1}\psi_{n,j-m} * f)\|_p \leqslant C 2^{(i-k)((1-1/p)d+1)+id/p} S$$

which is the claimed upper bound for the first term on the right hand side of (5.10).

To estimate the second term on the right hand side of (5.10), we note that by Lemma 4.1

$$\sum_{j=k}^{0} \|\rho_I^{-3\alpha}\partial_\nu^{-d-1}\psi_n * \partial_\nu^{d+1}\partial_n^{d+1}\sigma_{n,j}\|_p \leqslant C\sum_{j=k}^{0} 2^{(d+1)(i-m)} \left(\sum_{J\in B_j^1} \rho_I(J)^{-2\alpha p}\|1_J\partial_\nu^{d+1}\partial_n^{d+1}\sigma_{n,j}\|_p^p\right)^{1/p}. \quad (5.11)$$

Let $\hat{I} \in \mathcal{D}_j$ be the dyadic cube with $I \subset \hat{I}$. Then $\rho_I(J) \geqslant \rho_{\hat{I}}(J)$. By Lemma 3.4

$$\|1_J\partial_\nu^{d+1}\partial_n^{d+1}\sigma_{n,j}\|_p \leqslant C2^{(m-j)(d+1)}2^{jd/p}S,$$

and consequently we estimate the right hand side of (5.11) by

$$CS\sum_{j=k}^{0} 2^{(d+1)(i-j)}\|\rho_{\hat{I}}^{-2\alpha}\|_p \leqslant CS\sum_{j=k}^{0} 2^{(d+1)(i-j)+jd/p} \leqslant C2^{(d(1-1/p)+1)(i-k)}S2^{id/p}$$

which is the desired upper bound.

First term in (5.4). The remaining term in (5.4)

$$\|\rho_I^{-3\alpha}\psi * ((\tau_{-m} * f)\chi)\|_p$$

is estimated identically to the first term in (5.10) as χ is smooth. As we have given estimates for the contributions from each of the terms in (5.3) and (5.4), the proof of the lemma is complete. □

Lemma 5.3. *Let $i \leqslant 0$ and $I \in \mathcal{D}_i$. If $I \subset 7U$ and $I \not\subset K$ for all $K \in \mathcal{P}$, then there is $J \in T_i$ with $\rho_I(J) \leqslant 1$.*

Proof. By definition, $3I$ contains an element J from T. Assuming J is the maximal element contained in $3I$, we see that either $J \in T_{i+1}$ and $I \subset J$ or $J \in T_i$ and $\rho_I(J) \leqslant 1$. □

Now we can complete the proof of inequality (1.11). If $J \cap 7U = \varnothing$, we use Lemma 5.1 to estimate the left hand side of (1.11) by

$$CS \sum_{i \leqslant 0} \sup_{\substack{I \in \mathcal{D}_i \\ I \cap 7U = \varnothing \\ I \subset J}} |I|^{\alpha/d} \|1_U \rho_I^{-\alpha}\|_\infty \leqslant CS \|1_U \rho_J^{-\alpha}\|_\infty,$$

which is the desired right hand side.

If $J \cap 7U \neq \varnothing$, we find a $K \in \mathcal{P}$ such that $J \subset K$ and we use Lemma 5.2 to estimate the left hand side of (1.11) by

$$CS \sum_{i \leqslant i_0} \sup_{\substack{I \in \mathcal{D}_i \setminus T \\ I \subset J}} \rho_I((\tfrac{3}{2}K)^c)^{-2\alpha} + (|I|/|K|)^{1/d} \leqslant CS. \tag{5.12}$$

This concludes the proof.

6. Proof of Proposition 1.2

Assume first $1 \leqslant p \leqslant q \leqslant \infty$. By Hölder's inequality, we have

$$\|\rho_I^{-\alpha(1+\frac{q-p}{qp})} \phi * f\|_p \leqslant \|\rho_I^{-\alpha\frac{q-p}{qp}}\|_{\frac{qp}{q-p}} \|\rho_I^{-\alpha} \phi * f\|_q,$$

and thus for some universal constant C

$$S_{\alpha(1+\frac{q-p}{qp}),p} \leqslant C S_{\alpha,q},$$

which proves (1.12).

Assume now $1 \leqslant q \leqslant p \leqslant \infty$. We use logarithmic convexity

$$S_{\alpha,p} \leqslant S_{\alpha,q}^{q/p} S_{\alpha,\infty}^{1-q/p}.$$

To bound $S_{\alpha,\infty}$, we proceed similarly as in Lemma 4.1 and note that for $i \leqslant i_0$, $I \in D_i \cap M_U$, and $\phi \in \Phi_{i-m-2}^{4\alpha}$,

$$\|\rho_I^{-\alpha}(\phi * f)\|_\infty \leqslant \|\rho_I^{-\alpha}(\phi * \rho_U^{\alpha})\|_\infty \|\rho_U^{-\alpha} f\|_\infty$$

and

$$\|\rho_I^{-\alpha}(\phi * \rho_U^{\alpha})\|_\infty \leqslant C \|\rho_I^{-\alpha} \rho_U^{\alpha}\|_\infty \leqslant C.$$

This proves (1.13).

Finally, if one is willing to lose uniformity in the parameter m, one can use a local Bernstein's inequality. Let φ be a Schwartz function on $\mathbb{R}^d$ so that $\widehat{\varphi}(\xi) = 0$ for $|\xi| > 1$ and $\varphi(x) \geqslant c_d > 0$ for $x \in U$ for a dimensional constant c_d. Denote $\varphi_J(x) = \varphi(2^{-j}(x - c(J)))$ for $j \in \mathbb{Z}$ and $J \in \mathcal{D}_j$ and $c(J)$ the center of J. Then for $I \in \mathcal{D}_i$ and $\phi \in \Phi_{i-m-2}^{4\alpha}$

$$\begin{aligned}\|\rho_I^{-\alpha} \phi * f\|_\infty &\leqslant C \sum_{J \in \mathcal{D}_i} \rho_I(J)^{-\alpha} \|\varphi_J \phi * f\|_\infty \\ &\leqslant C 2^{d(m-i)} \sum_{J \in \mathcal{D}_i} \rho_I(J)^{-\alpha} \|\varphi_J \phi * f\|_1 \leqslant C 2^{d(m-i)} \|\rho_I^{-\alpha} \phi * f\|_1.\end{aligned}$$

Here we used that the Fourier transform of $\varphi_J \phi * f$ is supported in a ball of radius 2^{3+m-i} and Bernstein's inequality. This completes the proof of (1.14).

References

[1] C. Demeter and S. Z. Gautam. Bilinear Fourier restriction theorems. *J. Fourier Anal. Appl.*, 18(6):1265–1290, 2012.

[2] F. Di Plinio and A. Fragkos. The weak-type Carleson theorem via wave packet estimates, 2022. Available at arXiv:2204.08051.

[3] F. Di Plinio, K. Li, H. Martikainen, and E. Vuorinen. Banach-valued multilinear singular integrals with modulation invariance. *Int. Math. Res. Not. IMRN*, 2022(7):5256–5319, 2022.

[4] Y. Do and C. Thiele. L^p theory for outer measures and two themes of Lennart Carleson united. *Bull. Amer. Math. Soc. (N.S.)*, 52(2):249–296, 2015.

[5] M. Fraccaroli. Duality for outer $L^p_\mu(\ell^r)$ spaces and relation to tent spaces. *J. Fourier Anal. Appl.*, 27(4):Paper No. 67, 48, 2021.

[6] M. Fraccaroli, O. Saari, and C. Thiele. Uniform bounds for bilinear symbols with linear K-quasiconformally embedded singularity, 2024. Available at arXiv:2402.11661.

[7] L. Grafakos and X. Li. Uniform bounds for the bilinear Hilbert transforms. I. *Ann. of Math. (2)*, 159(3):889–933, 2004.

[8] L. Grafakos and X. Li. The disc as a bilinear multiplier. *Amer. J. Math.*, 128(1):91–119, 2006.

[9] V. Kovač, C. Thiele, and P. Zorin-Kranich. Dyadic triangular Hilbert transform of two general functions and one not too general function. *Forum Math. Sigma*, 3:Paper No. e25, 27, 2015.

[10] X. Li. Uniform bounds for the bilinear Hilbert transforms. II. *Rev. Mat. Iberoam.*, 22(3):1069–1126, 2006.

[11] X. Li. Uniform estimates for some paraproducts. *New York J. Math.*, 14:145–192, 2008.

[12] X. Li and L. Xiao. Uniform estimates for bilinear Hilbert transforms and bilinear maximal functions associated to polynomials. *Amer. J. Math.*, 138(4):907–962, 2016.

[13] C. Muscalu. *L(p) estimates for multilinear operators given by singular symbols.* ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–Brown University.

[14] C. Muscalu, T. Tao, and C. Thiele. Uniform estimates on multi-linear operators with modulation symmetry. *J. Anal. Math.*, 88:255–309, 2002.

[15] R. Oberlin and C. Thiele. New uniform bounds for a Walsh model of the bilinear Hilbert transform. *Indiana Univ. Math. J.*, 60(5):1693–1712, 2011.

[16] O. Saari and C. Thiele. Paraproducts for bilinear multipliers associated with convex sets. *Math. Ann.*, 385(3-4):2013–2036, 2023.

[17] C. Thiele. A uniform estimate. *Ann. of Math. (2)*, 156(2):519–563, 2002.

[18] G. Uraltsev and M. Warchalski. The full range of uniform bounds for the bilinear Hilbert transform, 2022. Available at arXiv:2205.09851.

BCAM - Basque Center for Applied Mathematics, Mazarredo, 14 E48009 Bilbao, Basque Country, Spain.

Email address: mfraccaroli@bcamath.org

Departament de Matemàtiques, Universitat Politècnica de Catalunya, Avinguda Diagonal 647, 08028 Barcelona, Catalunya, Spain

Centre de Recerca Matemàtica, Edifici C, Campus Bellaterra, 08193 Bellaterra, Catalunya, Spain

Email address: olli.saari@upc.edu

Mathematical Institute, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany
Email address: thiele@math.uni-bonn.de