$L^p - L^q$ LOCAL SMOOTHING ESTIMATES FOR THE WAVE EQUATION VIA k-BROAD FOURIER RESTRICTION

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Abstract. We explore the connection between k-broad Fourier restriction estimates and sharp regularity L^p-L^q local smoothing estimates for the solutions of the wave equation in $\mathbb{R}^n\times\mathbb{R}$ for all $n\geq 3$ via a Bourgain–Guth broad-narrow analysis. An interesting feature is that local smoothing estimates for $e^{it\sqrt{-\Delta}}$ are not invariant under Lorentz rescaling.

1. Introduction

Let u denote the solution of the Cauchy problem for the wave equation in $\mathbb{R}^n\times\mathbb{R}$

$$\begin{cases} (\partial_t^2-\Delta)u(x,t)=0 \\ u(x,0):=f(x), \quad \partial_t u(x,0):=0. \end{cases}$$

It is well known that u can be written in terms of the half-wave propagator

$$e^{it\sqrt{-\Delta}}f(x):=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+t|\xi|)}\widehat{f}(\xi)\,\mathrm{d}\xi,$$

which satisfies the *fixed-time* bounds [29, 26]

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p_{-\bar{s}_p}(\mathbb{R}^n)}\lesssim_t \|f\|_{L^p(\mathbb{R}^n)}, \qquad \bar{s}_p:=(n-1)\Big|\frac{1}{2}-\frac{1}{p}\Big|, \tag{1.1}$$

for any $1<p<\infty$ and any $t>0$, where the implicit constant is locally bounded in t. Here L^p_s denotes the Bessel potential space. Whilst these bounds are sharp for each fixed t, Sogge [32] observed that there exists some $\sigma>0$ such that

$$\Big(\int_1^2 \|e^{it\sqrt{-\Delta}}f\|^p_{L^p_{-\bar{s}_p+\sigma}(\mathbb{R}^n)}\,\mathrm{d}t\Big)^{1/p}\lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{1.2}$$

holds for all $2<p<\infty$. This regularity gain in L^p satisfied by $e^{it\sqrt{-\Delta}}$ after a local integration in time is commonly referred to as the *local smoothing phenomenon* of the wave equation. It is conjectured [32] that (1.2) holds for all $\sigma<\sigma_p$ where

$$\sigma_p:=\begin{cases} 1/p & \text{if} \quad \frac{2n}{n-1}\leq p<\infty, \\ \bar{s}_p & \text{if} \quad 2<p\leq\frac{2n}{n-1}. \end{cases}$$

The *local smoothing conjecture*[1] is at its strongest when $p=\frac{2n}{n-1}$. The remaining cases follow by interpolation against the fixed-time estimates (1.1). More precisely,

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[1] It is also expected that endpoint regularity results with $\sigma=1/p$ should hold if $p>2n/(n-1)$: see [18] for results in this direction if $n\geq 4$. Similarly, the forthcoming Conjecture 1.1 could hold for endpoint regularity cases $\sigma=\sigma_{p,q}$. Such endpoint cases will not be considered in this paper.

one interpolates (1.2) with the energy conservation identity

$$\|e^{it\sqrt{-\Delta}}f\|_{L^2(\mathbb{R}^n\times[1,2])} = \|f\|_{L^2(\mathbb{R}^n)} \tag{1.3}$$

and the L^∞ estimate (see for instance [33, Chapter IX, §4])

$$\|e^{it\sqrt{-\Delta}}f\|_{L^\infty_{-(n-1)/2-\varepsilon}(\mathbb{R}^n\times[1,2])} \lesssim_t \|f\|_{L^\infty(\mathbb{R}^n)}, \tag{1.4}$$

which holds for all $\varepsilon > 0$.

The local smoothing conjecture has been studied in numerous papers ever since it was first posed in [32], see for instance [27, 39, 19, 9, 10, 18, 23, 4, 20]. When $n = 2$, sharp results follow by the work of Guth, Wang and Zhang [15]. They prove a reverse square function estimate, which then implies the conjecture by the method of [27]. When $n \geq 3$, the conjecture holds for all $p \geq \frac{2(n+1)}{n-1}$ by the Bourgain–Demeter decoupling theorem [4] and the method of Wolff [39]. See also [7] for partial results in the range $2 \leq p \leq \frac{2(n+1)}{n-1}$. Verification of the full local smoothing conjecture would imply affirmative answers to a number of other important open problems such as the Bochner–Riesz conjecture, the Fourier restriction conjecture and the Kakeya conjecture; see [36] for further background.

This note focuses on an $L^p - L^q$ variant of the local smoothing conjecture. The fixed-time estimate [25, 34, 6]

$$\|e^{it\sqrt{-\Delta}}f\|_{L^\infty_{-(n+1)/2-\varepsilon}(\mathbb{R}^n)} \lesssim_t \|f\|_{L^1(\mathbb{R}^n)}, \quad \varepsilon > 0, \tag{1.5}$$

along with the complex interpolation method can be used to upgrade (1.1) to the L^p-improving inequality

$$\|e^{it\sqrt{-\Delta}}f\|_{L^q_{-\bar{s}_{p,q}}(\mathbb{R}^n)} \lesssim_t \|f\|_{L^p(\mathbb{R}^n)}, \quad \bar{s}_{p,q} := \begin{cases} \bar{s}_q + \frac{1}{p} - \frac{1}{q} & \text{if} \quad q \geq p' \\ \bar{s}_p + \frac{1}{p} - \frac{1}{q} & \text{if} \quad q \leq p' \end{cases}, \tag{1.6}$$

valid for any $1 < p \leq q < \infty$ and any $t > 0$, and where the implicit constant is locally bounded in t. Here $p' = p/(p-1)$. Similarly, any local smoothing estimate (1.2) can be interpolated with (1.5) to obtain $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n \times [1,2])$ estimates for $q \geq p$. This motivates the following conjecture [31, 38].

Conjecture 1.1 ($L^p - L^q$ local smoothing conjecture)**.** *For $n \geq 2$, the inequality*

$$\Big(\int_1^2 \|e^{it\sqrt{-\Delta}}f\|^q_{L^q_{-\bar{s}_{p,q}+\sigma}(\mathbb{R}^n)}\,\mathrm{d}t\Big)^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{1.7}$$

holds for all $\sigma < \sigma_{p,q}$ if $1 < p \leq q < \infty$ and $p' < q$, where

$$\sigma_{p,q} := \begin{cases} \frac{1}{q} & \textit{if} \quad \frac{1}{q} \leq \frac{n-1}{n+1}\frac{1}{p'} \\ \frac{(n-1)}{2}\left(\frac{1}{p'} - \frac{1}{q}\right) & \textit{if} \quad \frac{1}{q} \geq \frac{n-1}{n+1}\frac{1}{p'} \end{cases}.$$

By the preceding discussion, validity of the conjecture for $q = p$ implies the cases $q > \max\{p, p'\}$ by interpolation with (1.5); thus Conjecture 1.7 is a consequence of the $L^p - L^p$ result in [15] when $n = 2$. When $q > p$, Conjecture 1.1 is at its strongest on the *critical line*

$$\frac{1}{q} = \frac{n-1}{n+1}\frac{1}{p'};$$

validity of the sharp regularity estimates (1.7) for a pair (p^*, q^*) there immediately implies, by interpolation with (1.3), (1.4) and (1.5), sharp regularity $L^p - L^q$ local smoothing estimates for $(1/p, 1/q) \in \mathfrak{Q}_{p^*,q^*} \backslash (\overline{P_0P_1} \cup \overline{P_1P_2})$, where $\mathfrak{Q}_{p^*,q^*}$ is the closed quadrangle with vertices

$$P_0 = (0,0), \qquad P_1 = (1,0), \qquad P_2 = (1/2,1/2), \qquad P_* = (1/p^*, 1/q^*).$$

This region of validity can further be extended to a hexagon with additional vertices at $(1/p_1, 1/p_1)$ and $(1/p_2, 1/p_2)$ if the conjecture is known to hold on the line $p = q$ for some $\frac{2n}{n-1} < p_1 < \infty$, $2 < p_2 < \frac{2n}{n-1}$.

It is of fundamental importance that sharp regularity estimates on the critical line can be obtained despite the full conjecture being open. As a prime example we mention the Strichartz estimate [35]

$$\|e^{it\sqrt{-\Delta}} f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^{n+1})} \lesssim \|f\|_{\dot{L}^2_{\frac{1}{2}}(\mathbb{R}^n)}, \tag{1.8}$$

which corresponds to the endpoint case $\sigma = \sigma_{p,q}$ in (1.7) on the critical line for $q = \frac{2(n+1)}{n-1}$. Sharp $L^p - L^q$ local smoothing estimates beyond (1.8) were first studied by Schlag and Sogge [31] when $n = 2$. Further improvements beyond $q = \frac{2(n+1)}{n-1}$ and in any dimension $n \geq 2$ were obtained in [38, 21, 22] using the Wolff–Tao bilinear Fourier restriction estimates [40, 37] for the cone, which can be interpreted as local smoothing estimates via Plancherel's theorem. These bilinear estimates and their conjectured k-linear counterparts are of the form

$$\| \prod_{j=1}^{k} |e^{it\sqrt{-\Delta}} f_j|^{1/k} \|_{L^p(B_R)} \lesssim_\varepsilon R^\varepsilon \prod_{j=1}^{k} \|f_j\|^{1/k}_{L^2(\mathbb{R}^n)}, \quad p \geq \bar{p}_{n,k} := \frac{2(n+k+1)}{n+k-1} \tag{1.9}$$

where $2 \leq k \leq n+1$; $\operatorname{supp} \widehat{f_j} \subseteq \{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 2\}$ for all $1 \leq j \leq k$; the sets $\{\frac{\xi}{|\xi|} : \xi \in \operatorname{supp} \widehat{f_j}\}$ are separated; $B_R \subseteq \mathbb{R}^{n+1}$ denotes a ball of radius R and the estimates are supposed to hold for all $\varepsilon > 0$ and all $R \geq 1$. The only known cases are $k = 2$ [40, 37], $k = n$ [1] and $k = n+1$ [3]. The remaining cases $3 \leq k < n$ are open up to some partial positive results for $p \geq \frac{2k}{k-1}$ [3].

As the exponents $\bar{p}_{n,k}$ decrease with k, it is natural to explore if higher orders of multilinearity imply further progress on Conjecture 1.1. This line of investigation was considered by Lee [20] for $n = 2$ using the trilinear reduction of Lee and Vargas [24]; see also the recent work [16]. In this note, we further extend the multilinear approach to any dimension and any level of linearity in the case $q > p$. We remark that whereas partial results for $q = p$ using this method were discussed in [7], our focus is on sharp results with $q > p$. Rather than working with k-linear estimates, we will work with their *k-broad* variants (see §3), which hold in the full range $p \geq \bar{p}_{n,k}$. The idea of substituting the missing k-linear estimates by k-broad estimates goes back to the work of Guth [12] on Fourier restriction estimates for the paraboloid. Analogous results for conic surfaces have recently been obtained in [28, 7, 30].

We use a by now standard broad-narrow analysis from [13, 5]. To do so, we cannot restrict the attention to the half-wave propagator $e^{it\sqrt{-\Delta}}$ but we are forced to consider a larger class of operators that remains closed under Lorentz rescaling (see §5). Note that, unlike Fourier restriction estimates for the cone, local smoothing estimates for $e^{it\sqrt{-\Delta}}$ are not invariant under Lorentz rescaling as they are not invariant under rotations in $\mathbb{R}^n \times \mathbb{R}$. As a consequence, one needs to use the k-broad estimates for perturbations of the light cone from [7, 30] instead of only those for the light cone from [28].

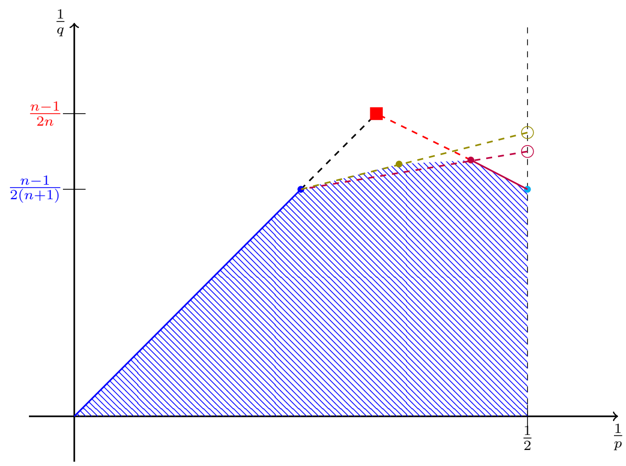

FIGURE 1. $L^p - L^q$ local smoothing estimates for all $\sigma < \sigma_{p,q}$ for $\frac{1}{q} \leq \frac{n-1}{n+1}\frac{1}{p'}$, $p \geq 2$ hold in the shaded region $\mathfrak{P}_n$. The critical point of the local smoothing conjecture is depicted as a red square, and the descending red dashed line is the critical line of the $L^p - L^q$ conjecture. The dark blue point follows from the decoupling theorem [4]. The hollow circles denote k-broad restriction estimates in [28, 7] $k = 3, 4$. The estimates at the purple and olive points are the content of our Theorem 1.3. Higher degrees of multilinearity imply further points, but they have been left out from the picture for clarity.

Our first result is a sharp $L^p - L^q$ local smoothing estimate on the critical line $\frac{1}{q} = \frac{n-1}{n+1}\frac{1}{p'}$. For future applicability, we state the next theorem in terms of the best exponent for which there is sharp regularity results in Conjecture 1.1 when $q = p$. Such a statement requires the aforementioned larger class of operators, which we introduce in what follows. Let Φ^+_{conic} denote the class of functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ smooth away from 0, homogeneous of degree 1 and satisfying that $\partial^2_{\xi\xi}\varphi(\xi)$ has $n-1$ positive eigenvalues on $\mathbb{R}^n\backslash\{0\}$. Given $\varphi \in \Phi^+_{\text{conic}}$, define the wave-like propagator

$$U_\varphi f(x,t) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\varphi(\xi))}\widehat{f}(\xi)\,\mathrm{d}\xi.$$

Note that a standard computation reveals that $\varphi(\xi) = |\xi| \in \Phi^+_{\text{conic}}$ and thus $e^{it\sqrt{-\Delta}}$ is of the form U_φ. It is well-known [26, 6] that if $\varphi \in \Phi^+_{\text{conic}}$, U_φ continues to satisfy (1.1), (1.5) and (1.6), that is,

$$\|U_\varphi f\|_{L^q_{-\bar{s}_{p,q}}(\mathbb{R}^n)} \lesssim_t \|f\|_{L^p(\mathbb{R}^n)} \tag{1.10}$$

for $1 < p \leq q < \infty$. One can formulate Conjecture 1.1 for U_φ, which is also known to hold for $q = p \geq \frac{2(n+1)}{n-1}$ by [4].

Theorem 1.2. *Let $n \geq 2$ and let $\bar{p}_n \geq 2n/(n-1)$ be the smallest number p for which*

$$\|U_\varphi f\|_{L^p_{-\bar{s}_p+\sigma}(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $\sigma < 1/p$ and all $\varphi \in \Phi^+_{\text{conic}}$. Then Conjecture 1.1 holds for all $1 < p \le q < \infty$ satisfying

$$q \ge \frac{2\bar{p}_n\left(n^2+3n-1\right)-4n(n+4)}{(n-1)((n+2)\bar{p}_n-2(n+3))}, \qquad \frac{1}{q}=\frac{n-1}{n+1}\frac{1}{p'},$$

and, more generally,

$$\|U_\varphi f\|_{L^q_{-\bar{s}_{p,q}+\sigma}(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $\sigma < 1/q$, p and q as above and all $\varphi \in \Phi^+_{\text{conic}}$.

In particular, as $\bar{p}_n \le 2(n+1)/(n-1)$ by [4], then Conjecture 1.1 holds for

$$q \ge \frac{2(n^2+6n-1)}{(n-1)(n+5)}, \quad \frac{1}{q}=\frac{n-1}{n+1}\frac{1}{p'}.$$

For $n \ge 3$ our results are an improvement over the estimates obtained by bilinear methods in [22], which implied Conjecture 1.1 for $q \ge \frac{2(n^2+2n-1)}{(n-1)(n+1)}$, $\frac{1}{q}=\frac{n-1}{n+1}\frac{1}{p'}$.

Theorem 1.2 is proved using 3-broad estimates only. The use of higher degrees of multilinearity in the proof would cause the method to become increasingly inefficient on the critical line. However, higher orders of multilinearity can be used away from the critical line. This is the content of our next theorem, from which Theorem 1.2 follows by setting $k = 3$.

Theorem 1.3. *Let $n \ge 2$ and let $\bar{p}_n \ge 2n/(n-1)$ be the smallest number p for which*

$$\|U_\varphi f\|_{L^p_{-\bar{s}_p+\sigma}(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $\sigma < 1/p$ and all $\varphi \in \Phi^+_{\text{conic}}$. Then Conjecture 1.1 holds for all pairs $(\bar{p}(k), \bar{q}(k))$,

$$\bar{p}(k) = \frac{2\bar{p}_n\left(2n^2+k(n+4)-k^2+3n-5\right)-4\left(n+k+1\right)\left(2n-k+3\right)}{\bar{p}_n\left(n+k+1\right)\left(2n-k+1\right)-2\left(2n^2+k(n-2)-k^2+5n+9\right)},$$
$$\bar{q}(k) = \frac{2\bar{p}_n\left(2n^2+k(n+4)-k^2+3n-5\right)-4\left(n+k+1\right)\left(2n-k+3\right)}{\bar{p}_n\left(n+k-1\right)\left(2n-k+1\right)-2\left(2n^2+kn-k^2+n+3\right)}.$$

with $k \in \{2, \dots, n+1\}$.

Furthermore, Conjecture 1.1 holds for all $(1/p, 1/q) \in (\mathfrak{P}_n \cup \mathfrak{T}_n)\backslash(\overline{P_0P_1} \cup \overline{P_1P_2})$ where $\mathfrak{P}_n$ is the convex hull of

$$(1/\bar{p}(k), 1/\bar{q}(k)), \quad (1/\bar{p}_n, 1/\bar{p}_n), \quad P_0=(0,0), \quad P_1=(1,0),$$

see Figure 1. The set $\mathfrak{T}_n$ is the triangle formed by

$$(1/\bar{p}(3), 1/\bar{q}(3)), \quad P_1=(1,0), \quad P_2=(1/2,1/2).$$

More generally,

$$\|U_\varphi f\|_{L^q_{-\bar{s}_{p,q}+\sigma}(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $\sigma < 1/q$, p, q as above and all $\varphi \in \Phi^+_{\text{conic}}$.

In particular, as $\bar{p}_n \le 2(n+1)/(n-1)$ by [4], we have the values

$$\bar{p}(k) = \frac{2(n^2+2nk-k^2+3k-1)}{n^2+2nk-k^2-k+5} \quad \text{and} \quad \bar{q}(k) = \frac{2(n^2+2nk-k^2+3k-1)}{n^2+2nk-k^2+k-2n+1}.$$

We remark that the sharp regularity estimates from Theorem 1.3 can be interpolated against any current non-sharp regularity $L^p - L^p$ local smoothing estimates (such as those in [7]) to obtain partial results in the exterior of $\mathfrak{T}_n$.

We finish the introduction with a contextual remark. One of our original motivations was to investigate how close the state of art in $L^p - L^q$ local smoothing estimates for $e^{it\sqrt{-\Delta}}$ is from solving a problem that was left open in our earlier joint work with Ramos [2]: Let $n = 4$ and let σ be the normalised surface measure of the unit sphere in $\mathbb{R}^n$. Does there exist $p \in (1, \infty)$ and $\alpha \in [1, n-1)$ such that

$$f \mapsto \sup_{t>0} |t^\alpha \sigma_t * f|$$

maps L^p to a first order Sobolev space? The question has a positive answer if sharp $L^p - L^q$ local smoothing estimates hold for $q \geq 3 - 1/6 - \epsilon$. Using the best known estimates in Theorem 1.2, we only get sharp local smoothing for $q \geq 3 - 1/9$ and hence miss the threshold by 1/18.

Structure of the paper. We begin by making some standard reductions in §2, which reduce Theorem 1.3 to the upcoming Theorem 2.3, which is a local estimate for functions with compact Fourier support that is well separated from the origin. In §5 we address the Lorentz rescaling, which is a main ingredient in the proof of Theorem 1.3 and the reason to introduce the class of phase functions Φ^+_{conic}. The concept of k-broad norm is introduced in §3 and in §4 we present a narrow decoupling for the operators U_φ. The proof of Theorem 1.3 is presented in §6.

Notation. Given $R \geq 1$, B^n_R denotes a ball of radius R in $\mathbb{R}^n$ and B_R denotes a ball of radius R in $\mathbb{R}^n \times \mathbb{R}$. Given a measurable set $A \subset \mathbb{R}^{n+1}$, A^c denotes its complementary set. The notation $A \lesssim B$ is used if $A \leq CB$ for some constant $C > 0$. If the constant C depends on a certain list of relevant parameters L, we use the notations $C = C_L$ and $\lesssim_L$. The case of dimension and the integrability parameters p and q may also be suppressed from the notation. The relations $A \gtrsim_L B$ and $A \sim_L B$ are defined similarly.

A weight function adapted to a ball $B_R \subset \mathbb{R}^{n+1}$ of radius R and centre c is defined as

$$w^N_{B_R}(z) = \Big(1 + \Big(\frac{|z - c|}{R}\Big)^2\Big)^{-N/2},$$

where N is a large dimensional constant.

Given $1 < p \leq q < \infty$, $p' \leq q$, $0 < \bar{\sigma} \leq \sigma_{p,q}$ and $\varphi \in \Phi^+_{\text{conic}}$, we say that there is $(p, q, \bar{\sigma})$ *local smoothing* for U_φ or that a $(p, q, \bar{\sigma})$ *local smoothing estimate* for U_φ holds if

$$\|U_\varphi f\|_{L^q_{-\bar{s}_{p,q}+\sigma}(\mathbb{R}^n \times [1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $\sigma < \bar{\sigma}$. If $\bar{\sigma} = \sigma_{p,q}$, we say that there is *sharp regularity* (p, q) *local smoothing* for U_φ.

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2. Initial reductions

Before proceeding with the proof of Theorem 1.3 we perform some standard reductions which are useful to show that there is $(p, q, \bar{\sigma})$ local smoothing for U_φ.

2.1. **Dyadic decomposition.** Given $\varphi \in \Phi^+_{\mathrm{conic}}$, the first step is to break up the operator U_φ into pieces which are Fourier supported on dyadic annuli. Let $\zeta \in C^\infty_c(\mathbb{R})$ with $\operatorname{supp} \zeta \subseteq [1/2, 2]$ be such that $\sum_{k\in\mathbb{Z}} \zeta(2^{-k}r) = 1$ for all $r > 0$. Define $\eta(\xi) = \zeta(|\xi|)$ for $\xi \in \mathbb{R}^n$. Thus,

$$U_\varphi f(x,t) = U_\varphi(\check{\tilde{\eta}} * f)(x,t) + \sum_{k\geq 0} U_\varphi(\check{\eta}_k * f)(x,t)$$

where $\eta_k(\xi) := \eta(2^{-k}\xi)$ and $\tilde{\eta} := \sum_{k<0} \eta_k$. An elementary integration by parts argument quickly reveals that the first term satisfies

$$\|U_\varphi(\check{\tilde{\eta}} * f)\|_{L^q(\mathbb{R}^n\times[1,2])} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 \leq p \leq q \leq \infty$. Thus, there is $(p, q, \bar{\sigma})$ local smoothing for U_φ if

$$\|U_\varphi(\check{\eta}_k * f)\|_{L^q(\mathbb{R}^n\times[1,2])} \lesssim_\varepsilon 2^{k(\bar{s}_{p,q}-\bar{\sigma}+\varepsilon)}\|f\|_{L^p(\mathbb{R}^n)} \tag{2.1}$$

holds for all $\varepsilon > 0$ with the implicit constant uniform in $k \geq 0$. By rescaling and setting $\lambda = 2^k$, (2.1) is equivalent to showing

$$\|U_\varphi(\check{\eta} * f)\|_{L^q(\mathbb{R}^n\times[\lambda,2\lambda])} \lesssim_\varepsilon \lambda^{\beta+\varepsilon}\|f\|_{L^p(\mathbb{R}^n)} \tag{2.2}$$

uniformly in $\lambda \geq 1$, where

$$\beta = \bar{s}_{p,q} - \bar{\sigma} + \frac{n+1}{q} - \frac{n}{p} = (n-1)\Big(\frac{1}{2} - \frac{1}{p}\Big) + \frac{1}{q} - \bar{\sigma}.$$

We further note that the best constant in (2.2) is comparable to the best constant in

$$\|U_\varphi(\check{\eta} * f)\|_{L^q(\mathbb{R}^n\times[-2\lambda,2\lambda])} \lesssim_\varepsilon \lambda^{\beta+\varepsilon}\|f\|_{L^p(\mathbb{R}^n)} \tag{2.3}$$

uniformly in $\lambda \geq 1$. Indeed, set $\lambda = 2^k$ with $k \geq 0$. By rescaling

$$\begin{aligned}
\|U_\varphi(\check{\eta} * f)\|^q_{L^q(\mathbb{R}^n\times[0,2\lambda])} &\leq \sum_{j=0}^{\infty} \|U_\varphi(\check{\eta} * f)\|^q_{L^q(\mathbb{R}^n\times[2^{-j}\lambda,2^{-j+1}\lambda])} \\
&\leq \sum_{j=0}^{k} \|U_\varphi(\check{\eta} * f)\|^q_{L^q(\mathbb{R}^n\times[2^{-j}\lambda,2^{-j+1}\lambda])} \\
&\qquad + \sum_{j=k+1}^{\infty} 2^{-(j-k)(n+1)}\|U_\varphi(\check{\tilde{\eta}} * f_{j-k})\|^q_{L^q(\mathbb{R}^n\times[1,2])}
\end{aligned}$$

where

$$\widehat{f_j}(\xi) = 2^{jn}\eta(2^j\xi)\widehat{f}(2^j\xi).$$

Note that we can add for free the Fourier localisation given by $\tilde{\eta}$. By (2.2) the first term admits the desired bound by geometric summation. By the elementary integration by parts bound $\|U_\varphi(\check{\tilde{\eta}} * f_{j-k})\|_{L^q(\mathbb{R}^n\times[1,2])} \lesssim \|f_{j-k}\|_{L^q(\mathbb{R}^n)}$ and Bernstein's inequality, that is,

$$\|f_{j-k}\|_{L^q(\mathbb{R}^n)} \lesssim 2^{-(j-k)n(1/p-1/q)}\|f_{j-k}\|_{L^p(\mathbb{R}^n)},$$

the second term admits the bound

$$\sum_{j=k+1}^{\infty} 2^{-(j-k)} \|f\|^q_{L^p(\mathbb{R}^n)}$$

which is also acceptable. We conclude that the best constant in (2.3) is controlled by the best constant in (2.2). The other direction is immediate.

2.2. **A quantitative family of wave propagators.** In §5 we will show that the class of operators $\{U_\varphi : \varphi \in \Phi^+_{\mathrm{conic}}\}$ is invariant under Lorentz rescaling. To show this, it will be convenient to work with a quantitative version of the class Φ^+_{conic}.

Fix parameters $D_1, D_2 > 0, \vec{\mu} = (\mu_{\min}, \mu_{\max}) \in \mathbb{R}^2_+, M \geq 100n$ and $\varepsilon_\circ > 0$. Let $b \in C^\infty_c(\mathbb{R}^n)$ be supported in

$$\Xi := \{\xi \in \mathbb{R}^n : 1/2 \leq \xi_1 \leq 2,\ |\xi_j| \leq |\xi_1| \text{ for all } 2 \leq j \leq n\}$$

satisfying

B1) $|\partial^\gamma_\xi b(\xi)| \leq D_1$ for all $\gamma \in \mathbb{N}^n_0$ such that $|\gamma| \leq M$.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth function homogeneous of degree 1 satisfying

H1) $h(1, 0') = \partial_\xi h(1, 0') = 0$;

H2) $|\partial^\gamma_\xi h(\xi)| \leq D_2$ for all $\gamma = (\gamma_1, \gamma') \in \mathbb{N}_0 \times \mathbb{N}^{n-1}_0$ such that $|\gamma| \leq M$ and $|\gamma'| \geq 3$ and all $\xi \in \operatorname{supp} b$;

H3) $|\partial^2_{\xi'\xi'} h(\xi) - \frac{1}{\xi_1}\mathrm{L}| < \varepsilon_\circ$ for some matrix $\mathrm{L} \in \mathrm{GL}(n-1, \mathbb{R})$ with eigenvalues in $[\mu_{\min}, \mu_{\max}]$ and for all $\xi \in \operatorname{supp} b$.

It is noted that the above conditions on the derivatives imply, by homogeneity of h, that the remaining derivatives up to order M are bounded by $C(D_2, \vec{\mu}, M, n, \varepsilon_\circ)$. We denote by $\mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ the family of all phase-amplitude pairs $[h; b]$ satisfying B1), H1), H2) and H3), and define

$$U_{[h;b]} f(x, t) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t h(\xi))} b(\xi) \widehat{f}(\xi)\, \mathrm{d}\xi.$$

Given a phase $\varphi \in \Phi^+_{\mathrm{conic}}$, the operator $U_{[\varphi;\eta]}$ in (2.2) can be written as a sum of $C(\varphi, n)$ operators of the type $U_{[h;b]}$ with $[h; b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$.

Proposition 2.1. *Let $n \geq 2$ and $1 < p \leq q < \infty$, $s \in \mathbb{R}$ and $\varphi \in \Phi^+_{\mathrm{conic}}$. Assume that for any $D_1, D_2 > 0$, $\vec{\mu} \in \mathbb{R}^2_+$, $M > 100n$, $\varepsilon_\circ > 0$, and all $[h; b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$, the inequality*

$$\|U_{[h;b]} f\|_{L^q(\mathbb{R}^n \times [-\lambda, \lambda])} \lesssim C_{\mathbf{H}} \lambda^s \|f\|_{L^p(\mathbb{R}^n)}$$

holds uniformly in $\lambda \geq 1$. Then

$$\|U_{[\varphi;\eta]} f\|_{L^q(\mathbb{R}^n \times [-\lambda, \lambda])} \lesssim_{n,\varphi} \lambda^s \|f\|_{L^p(\mathbb{R}^n)}$$

holds uniformly in $\lambda \geq 1$.

Proof. Let $\varphi \in \Phi^+_{\mathrm{conic}}$. By a finite partition of unity and a rotation in the ξ-space, we may assume that

$$\partial^2_{\xi'\xi'} \varphi(1, 0') \quad \text{has positive eigenvalues} \quad \text{and} \quad \operatorname{supp} \widehat{f} \subseteq [1/2, 2] \times [-c_\circ, c_\circ]^{n-1} \subseteq \Xi$$

for some small constant $0 < c_\circ \ll 1$. This gives rise to an amplitude b, which satisfies the condition B1) for a dimensional constant $D_1 > 0$ depending also on

$\|\eta\|_{C^M}$, $c_\circ$, and the partition of unity. Moreover, by a translation of the x-space, one may add and subtract linear terms to replace the phase φ by

$$h(\xi) := \varphi(\xi) - \varphi(1,0')\xi_1 - \langle \partial_{\xi'}\varphi(1,0'), \xi'\rangle = \int_0^1 (1-r)\langle \partial^2_{\xi'\xi'}\varphi(\xi_1, r\xi')\xi', \xi'\rangle \, \mathrm{d}r.$$

It then suffices to verify that h satisfies H1)-H3) for some choice of $D_2, M, \varepsilon_\circ$, L and $\vec{\mu}$. Note that the dependency on the chosen $c_\circ$ is admissible in any case.

To show H1), we observe that h is homogeneous of degree 1 and satisfies $h(1,0') = \partial_{\xi_1} h(1,0') = \partial_{\xi'} h(1,0') = 0$; note that $\varphi(1,0') = \partial_{\xi_1}\varphi(1,0')$ by homogeneity of φ.

Regarding H2) and H3), note that $\partial_\xi^\gamma h(\xi) = \partial_\xi^\gamma \varphi(\xi)$ for all $\gamma \in \mathbb{N}_0^n$ such that $|\gamma| \geq 2$. For fixed $M > 0$, the choice $D_2 = \|\varphi\|_{C^M}$ clearly verifies H2). Finally, as $\partial^2_{\xi'\xi'} h(\xi) = \partial^2_{\xi'\xi'}\varphi(\xi)$, one can take $\mathrm{L} = \partial^2_{\xi'\xi'}\varphi(1,0')$ and $\mu_{\max}$ and $\mu_{\min}$ be its largest and smallest eigenvalues. By the mean value theorem and the bounds on $|\partial_\xi^\gamma h(\xi)|$ for $\gamma \in \mathbb{N}_0^3$ with $|\gamma| = 3$, it is clear that H3) holds with $\varepsilon_\circ = O_n(c_\circ D_2)$. $\square$

2.3. **Reduction to a local estimate.** For fixed time t, the propagators $U_{[h;b]}$ can be interpreted as Fourier multiplier operators in the x-variable. Thus, one may write $U_{[h;b]}f(x,t) = K_{[h;b]}(\cdot, t) * f(x)$ where

$$K_{[h;b]}(y,t) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(y\cdot\xi + th(\xi))} b(\xi)\, \mathrm{d}\xi$$

and $*$ denotes the convolution in the x-variable. As $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$, there exists $C_{\mathbf{H}} > 1$ such that $|\nabla_\xi (y\cdot\xi + th(\xi))| \geq |y|/2$ for $|y| \geq C_{\mathbf{H}}\lambda$ and $|t| \leq \lambda$. The method of non-stationary phase hence yields

$$|K_{[h;b]}(y,t)| \lesssim_{N,\mathbf{H}} |y|^{-N}, \quad |y| \geq C_{\mathbf{H}}\lambda,\ |t| \leq \lambda,\ N \in \mathbb{N}. \tag{2.4}$$

Denoting $\Psi_\lambda^N := (1 + \lambda^{-2}|\cdot|^2)^{-N/2}$, one obtains

$$|U_{[h;b]}f(x,t)\mathbb{1}_{B^n_\lambda}(x)| \leq \big(U_{[h;b]}(f\mathbb{1}_{B^n_{2C_{\mathbf{H}}\lambda}})(x,t) + c_{N,\mathbf{H}}\lambda^{-N}\Psi_\lambda^N * |f|(x)\big)\mathbb{1}_{B^n_\lambda}(x) \tag{2.5}$$

for $|t| \leq \lambda$ which allows for the following local reduction.

Proposition 2.2. *Let $n \geq 1$, $1 < p \leq q < \infty$ and $s \in \mathbb{R}$. Assume that a phase amplitude pair $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ for some fixed choice of D_1, $D_2 > 0$, $\vec{\mu} \in \mathbb{R}^2_+$, $M > 100n$, $\varepsilon_\circ > 0$ is given. Assume that*

$$\|U_{[h;b]}f\|_{L^q(B^n_\lambda \times [-\lambda,\lambda])} \leq C\lambda^s \|f\|_{L^p(\mathbb{R}^n)} \tag{2.6}$$

holds uniformly in $\lambda \geq 1$ and all balls B^n_λ. Then

$$\|U_{[h;b]}f\|_{L^q(\mathbb{R}^n\times[-\lambda,\lambda])} \lesssim_{p,q,n,s,\mathbf{H}} C\lambda^s\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let $\mathcal{B}^n_\lambda$ be a family of finitely overlapping balls B^n_λ covering $\mathbb{R}^n$. By (2.5) and (2.6) applied to $f\mathbb{1}_{B^n_{2C_{\mathbf{H}}\lambda}}$ one has

$$\begin{aligned}
&\|U_{[h:b]}f\|_{L^q(\mathbb{R}^n\times[-\lambda,\lambda])} \\
&\qquad \leq \Big(\sum_{B^n_\lambda \in \mathcal{B}^n_\lambda} \|U_{[h;b]}f\|^q_{L^q(B^n_\lambda\times[-\lambda,\lambda])}\Big)^{1/q} \\
&\qquad \leq C\lambda^s\Big(\sum_{B^n_\lambda\in\mathcal{B}^n_\lambda} \|f\|^q_{L^p(B^n_{2C_{\mathbf{H}}\lambda})}\Big)^{1/q} + c_{N,\mathbf{H}}\lambda^{-N+1/q}\|\Psi^N_\lambda * f\|_{L^q(\mathbb{R}^n)} \\
&\qquad \lesssim_{\mathbf{H}} \lambda^s\|f\|_{L^p(\mathbb{R}^n)} + c_{N,\mathbf{H}}\lambda^{-N+1/q+n(1/q+1/p')}\|f\|_{L^p(\mathbb{R}^n)} \\
&\qquad \lesssim_{p,q,n,s,\mathbf{H}} \lambda^s\|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

We used the embedding $\ell^p \subseteq \ell^q$ for $1 \leq p \leq q \leq \infty$ and required $N > \max\{1/q + n(1/q + 1/p') - s, n\}$. □

Thus, by §2.1 and Propositions 2.1 and 2.2, Theorem 1.3 follows from the following spatially and frequency localised version for the quantitative class of operators.

Theorem 2.3. *Let $n \geq 2$ and $1 < p \leq q < \infty$ be as in Theorem 1.3. Let D_1, $D_2 > 0$, $\vec{\mu} \in \mathbb{R}^2_+$, $M > 100n$, $\varepsilon_\circ > 0$. Then, for all $[h; b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ and for any $\varepsilon > 0$, the inequality*

$$\|U_{[h;b]} f\|_{L^q(B^n_\lambda \times [-\lambda,\lambda])} \lesssim_{n,p,q,\mathbf{H},\varepsilon} \lambda^{\beta+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)} \tag{2.7}$$

holds uniformly in $\lambda \geq 1$ and over all balls B^n_λ, where $\beta = (n-1)\big(\frac{1}{2} - \frac{1}{p}\big) + \frac{1}{q} - \sigma_{p,q}$.

Note that (2.7) is translation invariant in the x-variables so that the estimate over any ball B^n_λ guarantees estimates over all balls B^n_λ.

3. k-broad estimates

In this section we recall the definition of the k-broad norm introduced in [13] and state the key k-broad estimates needed in the proof of Theorem 1.3.

3.1. **k-broad norm.** Let $K \gg 1$ be a fixed large parameter. Fix a maximally K^{-1}-separated subset of $\{1\} \times B^{n-1}(0, 1)$ and for each ω belonging to this subset define the K^{-1}-*plate*

$$\tau := \big\{(\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1} : 1/2 \leq \xi_1 \leq 2 \ \text{ and } \ |\xi'/\xi_1 - \omega| \leq K^{-1}\big\}$$

and set $\omega_\tau := \omega$. The collection of all K^{-1}-plates forms a partition of Ξ into finitely overlapping subsets. Consider a smooth partition of unity $\{\chi_\tau\}$ adapted to that covering, where $\chi_\tau(\xi) := \chi(K(\xi'/\xi_1 - \omega_\tau))$ for some $\chi \in C^\infty_c(\mathbb{R}^{n-1})$; and set $\widehat{f_\tau} := \widehat{f}\chi_\tau$. It is also useful to consider $\tilde{\chi} \in C^\infty_c(\mathbb{R}^{n-1})$ such that $\tilde{\chi} \cdot \chi = \chi$ and define $\tilde{f}_\tau$ by $\widehat{\tilde{f}}_\tau := \widehat{f}\tilde{\chi}_\tau$, where $\tilde{\chi}_\tau$ is defined analogously to χ_τ.

Let B_{K^2} be a ball in $\mathbb{R}^{n+1}$ of radius K^2, $\varphi \in \Phi^+_{\text{conic}}$ and $b \in C^\infty_c(\mathbb{R}^n)$ supported in Ξ. For a fixed integer $A \geq 1$ and $1 \leq p < \infty$, define

$$\mu_{U_{[\varphi;b]}f}(B_{K^2}) := \min_{V_1,\dots V_A \in \mathrm{Gr}(k-1,n+1)} \Big(\max_{\substack{\tau : \measuredangle(G(\tau), V_a) > K^{-2} \\ \text{for } a=1,\dots,A}} \|U_{[\varphi;b]} f_\tau\|^p_{L^p(B_{K^2})} \Big)$$

where

- $\mathrm{Gr}(k-1, n+1)$ is the Grassmannian of all $(k-1)$-dimensional subspaces in $\mathbb{R}^{n+1}$;
- $G(\tau)$ denotes the set of unit normal vectors
$$G(\tau) := \Big\{ \frac{(-\nabla\varphi(\xi), 1)}{\sqrt{1 + |\nabla\varphi(\xi)|}} : \xi \in \tau \Big\};$$
- $\measuredangle(G(\tau), V_a)$ denotes the smallest angle between non-zero vectors $v \in G(\tau)$ and $v' \in V_a$.

Let $\mathcal{B}_{K^2}$ be a collection of finitely-overlapping balls B_{K^2} of radius K^2 which cover $\mathbb{R}^{n+1}$. Given any open set $W \subseteq \mathbb{R}^{n+1}$, define the *$k$-broad norm* of $U_{[\varphi;b]}f$ over W (or *k-broad part* of $\|U_{[\varphi;b]}f\|_{L^p(W)}$) by

$$\|U_{[\varphi;b]} f\|_{\mathrm{BL}^p_{k,A}(W)} := \Big(\sum_{\substack{B_{K^2} \in \mathcal{B}_{K^2} \\ B_{K^2} \cap W \neq \varnothing}} \mu_{U_{[\varphi;b]}f}(B_{K^2}) \Big)^{1/p}.$$

The quantity $\|U_{[\varphi;b]}f\|_{\mathrm{BL}^p_{k,A}(W)}$ is smaller than the left-hand side of the conjectured multilinear estimate (1.9). We refer to [13] and [14, §6.2] for further discussion regarding its relation with multilinear estimates.

Despite $\|U_{[\varphi;b]}f\|_{\mathrm{BL}^p_{k,A}}$ not being literally a norm, it satisfies versions of the triangle and Hölder's inequalities. The latter will be used in the forthcoming arguments.

Lemma 3.1 ([13, Lemma 4.2])**.** *Let $1 \le p, p_1, p_2 < \infty$ and $0 \le \alpha \le 1$ such that $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$. Suppose that $A = A_1 + A_2$ for integers $A_1, A_2 \ge 1$. Then*

$$\|U_{[\varphi;b]}f\|_{\mathrm{BL}^p_{k,A}(W)} \le \|U_{[\varphi;b]}f\|^{\alpha}_{\mathrm{BL}^{p_1}_{k,A_1}(W)} \|U_{[\varphi;b]}f\|^{1-\alpha}_{\mathrm{BL}^{p_2}_{k,A_2}(W)}.$$

3.2. **k-broad estimates for $U_{[\varphi;b]}$.** The following k-broad estimates for the wave propagators $U_{[\varphi;b]}$ are a key ingredient in the proof of Theorem 1.3.

Theorem 3.2 ([7, Theorem 5.3], [30, Theorem 1.2])**.** *Let D_1, $D_2 > 0$, $\vec{\mu} \in \mathbb{R}^2_+$, $M > 100n$, $\varepsilon_\circ > 0$. For any $2 \le k \le n+1$ and any $\varepsilon > 0$, there is a large integer $1 \ll A \lesssim K^\varepsilon$ and $d_\varepsilon > 1$ so that*

$$\|U_{[h;b]}f\|_{\mathrm{BL}^p_{k,A}(B_\lambda)} \lesssim_{\varepsilon,\mathbf{H}} K^{d_\varepsilon} \lambda^\varepsilon \|f\|_{L^2(\mathbb{R}^n)}$$

holds for any $1 \le K^\varepsilon \lesssim \lambda^{\varepsilon^2}$, any $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ and any $p \ge \bar{p}_{n,k} := \frac{2(n+k+1)}{n+k-1}$ uniformly over all balls B_λ of radius λ.

Note that the parameter A can be chosen independently of the location of the ball B_λ, as a translation of the ball only induces an admissible modulation on $\widehat{f}$.

3.3. **Reverse Hölder inequality: a decomposition lemma.** For functions satisfying a reverse Hölder type inequality, it is possible to interpolate k-broad norms at the expense of increasing the parameter A.

Lemma 3.3. *Let $1 \le p, p_1, p_2, q, q_1, q_2 < \infty$ and $0 \le \alpha \le 1$ such that $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$ and $\frac{1}{q} = \frac{\alpha}{q_1} + \frac{1-\alpha}{q_2}$. Suppose that $A = A_1 + A_2$ for integers $A_1, A_2 \ge 1$. Assume that*

$$\|U_{[\varphi;b]}f\|_{\mathrm{BL}^{q_i}_{k,A_i}(W)} \le C_i \|f\|_{p_i} \qquad \text{for } i = 1, 2. \tag{3.1}$$

Then

$$\|U_{[\varphi;b]}f\|_{\mathrm{BL}^q_{k,A}(W)} \le C C_1^\alpha C_2^{1-\alpha} \|f\|_p$$

for all functions f satisfying the reverse Hölder inequality

$$\|f\|^\alpha_{p_1} \|f\|^{1-\alpha}_{p_2} \le C\|f\|_p, \tag{3.2}$$

Proof. This follows from Lemma 3.1 and the hypotheses (3.1) and (3.2). □

Whilst the hypothesis (3.2) does not hold in general, a function with compact Fourier support can be decomposed, up to an error term, into finitely many pieces satisfying a reverse Hölder inequality. This fact can be found in the informal lecture notes [11]. We recall the proof below for completeness.

Lemma 3.4. *Let $1 \le p < \infty$ and fix $R \ge 1$ and $m > 0$. Suppose that $f \in L^p(\mathbb{R}^n)$ and $\operatorname{supp} \widehat{f} \subseteq B(0, 10)$. Then the function f can be written as*

$$f = \sum_{\nu=0}^{m\lfloor \log R \rfloor} f^\nu + e, \tag{3.3}$$

where

i) $\|e\|_{L^\infty(\mathbb{R}^n)} \lesssim R^{-m}\|f\|_{L^p(\mathbb{R}^n)}$;
ii) $\|f^\nu\|_{L^r(\mathbb{R}^n)} \lesssim \|f\|_{L^r(\mathbb{R}^n)}$ *for any* $1 \le r \le \infty$ *and all* $\nu = 0, \dots, m\lfloor \log R \rfloor$;
iii) if $1 \le r, r_1, r_2 \le \infty$ *satisfy* $\frac{1}{r} = \frac{\alpha}{r_1} + \frac{1-\alpha}{r_2}$ *for some* $0 \le \alpha \le 1$, *then*

$$\|f^\nu\|^{\alpha}_{L^{r_1}(\mathbb{R}^n)} \|f^\nu\|^{1-\alpha}_{L^{r_2}(\mathbb{R}^n)} \le 2\|f^\nu\|_{L^r(\mathbb{R}^n)}$$

for all $\nu = 0, \dots, m\lfloor \log R \rfloor$.

Proof. As $\operatorname{supp} \widehat{f} \subseteq B(0, 10)$, Young's convolution inequality implies $\|f\|_\infty \lesssim \|f\|_p$, the right-hand side being finite. Let $\nu_\circ \in \mathbb{Z}$ such that $2^{\nu_\circ - 1} < \|f\|_\infty \le 2^{\nu_\circ}$. For any $\nu \in \mathbb{Z}$, let $f^\nu := f \mathbb{1}_{\{2^{\nu-1} < |f| \le 2^\nu\}}$ and write $f = \sum_{\nu=-\infty}^{\nu_\circ} f^\nu$. Then ii) follows by definition. Because $1 \le p < \infty$ and

$$2^{\nu-1} |\operatorname{supp} f^\nu|^{1/p} \le \|f^\nu\|_p \le \|f\|_p < \infty,$$

we have that $|\operatorname{supp} f^\nu| < \infty$. This immediately implies iii). Furthermore, writing $f = \sum_{\nu=\nu_\circ - m\lfloor \log R \rfloor}^{\nu_\circ} f^\nu + e$ for any fixed $R \ge 1$ and $m > 0$, one has that $\|e\|_\infty \lesssim R^{-m}\|f\|_\infty \lesssim R^{-m}\|f\|_p$. This implies i) and (3.3) follows by relabelling ν. $\square$

3.4. **Local smoothing estimate for k-broad norms.** Any local smoothing estimate can be written in a k-broad norm formulation.

Proposition 3.5. *Let* $n \ge 1$, $2 \le k \le n+1$, $K \ge 2$ *and* $A \ge 1$. *Let* $\bar{p}_n \ge \frac{2n}{n-1}$ *and assume that the* $(\bar{p}_n, \bar{p}_n, 1/\bar{p}_n)$ *local smoothing estimate holds. Then for any* $\varepsilon > 0$, *the inequality*

$$\|U_{[\varphi;b]} f\|_{\mathrm{BL}^{\bar{p}_n}_{k,A}(B^n_R \times [-R,R])} \lesssim_\varepsilon R^\varepsilon R^{(n-1)(\frac{1}{2} - \frac{1}{\bar{p}_n})} \|f\|_{L^{\bar{p}_n}(\mathbb{R}^n)}$$

also holds for any $R \ge 1$, *with constant independent of* A.

Proof. By definition of the k-broad norm and the embedding $\ell^{\bar{p}_n} \subseteq \ell^\infty$

$$\|U_{[\varphi;b]} f\|_{\mathrm{BL}^{\bar{p}_n}_{k,A}(B^n_R \times [-R,R])} \le \Big(\sum_{B_{K^2} \subset B^n_R \times [-R,R]} \sum_{\tau} \|U_{[\varphi;b]} f_\tau\|^{\bar{p}_n}_{L^{\bar{p}_n}(B_{K^2})} \Big)^{1/\bar{p}_n},$$

where the sum in τ ranges over all K^{-1}–plates. The claim follows by changing the order of summation, applying the hypothetical local smoothing estimate on $\|U_{[\varphi;b]} f_\tau\|_{L^{\bar{p}_n}(B^n_R \times [-R,R])}$ (via §2.1) and using the bound $(\sum_\tau \|f_\tau\|^{\bar{p}_n}_{\bar{p}_n})^{1/\bar{p}_n} \lesssim \|f\|_{\bar{p}_n}$, which follows by interpolation between the cases $p = 2$ and $p = \infty$. $\square$

4. Narrow decoupling and flat phases

If the contribution to $U_{[\varphi;b]} f$ comes from plates whose normal vectors lie close to a $(k-1)$-dimensional subspace, one can essentially use the Bourgain–Demeter decoupling inequality [4] in $\mathbb{R}^{k-1}$. This phenomenon is normally referred to as *narrow* decoupling. The case $\phi(\xi) = |\xi|$ was established by Harris [17, Theorem 2.3] and can be used to show that the same result holds for suitably small perturbations of $|\xi|$.

Definition 4.1. Let $D_1, D_2 > 0$, $\vec{\mu} \in \mathbb{R}^2_+$, $M > 100n$, $\varepsilon_\circ > 0$. Let $L > 0$. Given a phase-amplitude pair $[h; b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$, we say that $[h; b]$ is L-flat if

$$|\partial^\alpha_\xi h(\xi)| \lesssim L^{-1} D_2 \quad \text{for } |\alpha'| \ge 3,\ |\alpha| \le M,\ \xi \in \operatorname{supp} b,$$

where $\alpha = (\alpha_1, \alpha') \in \mathbb{N}_0 \times \mathbb{N}_0^{n-1}$.

If $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ is L-flat, Taylor expansion immediately reveals that

$$h(\xi_1, \xi') = \frac{\langle \partial^2_{\xi'\xi'} h(1, 0')\xi', \xi' \rangle}{2\xi_1} + L^{-1}E(\xi),$$

where E is homogeneous of degree 1 and $|\partial^\alpha E(\xi)| \lesssim 1$ for all $|\alpha| \leq M-3$ on $\operatorname{supp} b$. This concept was introduced in [7] (see also [30]) to deduce the following narrow decoupling inequality.

Theorem 4.2. *Let $n \geq 2$, $3 \leq k \leq n+1$ and $K \geq 2$. Let D_1, $D_2 > 0$, $\vec{\mu} \in \mathbb{R}^2_+$, $M > 100n$, $\varepsilon_\circ > 0$. Let $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ be K^2-flat. Then for any $\varepsilon > 0$ and $N > 0$, the inequality*

$$\|U_{[h;b]}f\|_{L^p(B_{K^2})} \lesssim_{\varepsilon, \mathbf{H}, N} K^\varepsilon \Big(\sum_\tau \|U_{[h;b]} f_\tau\|^2_{L^p(w^N_{B_{K^2}})} \Big)^{1/2}$$

holds for all $2 \leq p \leq \frac{2(k-1)}{k-3}$ whenever $U_{[h;b]}f = \sum_\tau U_{[h;b]} f_\tau$ and τ are K^{-1}-plates such that $\measuredangle(G(\tau), V) \leq K^{-2}$ for some $(k-1)-$dimensional vector space V.

In order to see this, let $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ be K^2-flat and let $\tilde{h}$ denote its second order Taylor polynomial. By a suitable change of variables, the result of Harris for $\phi(\xi) = |\xi|$ can be extended to the phase $\tilde{h}$. For the extension to h, let Γ^K_h denote the K^{-2} neighbourhood of the cone generated by h and $\Gamma^K_{\tilde{h}}$ its analogue for $\tilde{h}$. Because of the K^2-flat hypothesis, the objects Γ^K_h and $\Gamma^K_{\tilde{h}}$ are indistinguishable from one another and, moreover, the normals to Γ^K_h lie in the K^{-2}-neighbourhood of the normals to $\Gamma^K_{\tilde{h}}$. Hence the decoupling inequality extends to the K^2-flat case (via its equivalent formulation in terms of the Fourier support lying on a neighbourhood of a cone).

5. Lorentz rescaling

5.1. **Lorentz rescaling.** We will next prove that the target estimate (2.7) self-improves if the support of $\widehat{f}$ is small. This is achieved by applying a standard Lorentz rescaling argument.

Before turning to the proof, it is instructive to compare the situation with Fourier restriction estimates, which are of the type (2.7) but with the right-hand side replaced by $\|\widehat{f}\|_p$. Such estimates are invariant under rotations in $\mathbb{R}^{n+1}$, and one can then apply the rotation $L(\xi_1, \xi', \tau) = (\xi_1 + \tau, \xi', \tau - \xi_1)$, where $(\xi_1, \xi', \tau) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, which maps the forward light cone $\Gamma := \{(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R} : \tau = |\xi|\}$ into the *tilted* cone $\Gamma_{\mathrm{par}} := \{(\xi_1, \xi', \tau) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : \tau = |\xi'|^2/\xi_1\}$. Thus, Fourier restriction estimates for the phase $\varphi(\xi) = |\xi|$ follow from those for $h_{\mathrm{par}}(\xi) = |\xi'|^2/\xi_1$. The phase function h_{par} satisfies the special property of being invariant under Lorentz rescaling, due to its perfect parabolic structure.

The invariance under Lorentz rescaling is no longer true for local smoothing estimates for $e^{it\sqrt{-\Delta}}$, as they are not rotationally invariant in $\mathbb{R}^{n+1}$. However, the class of phase functions in $\mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ is invariant under rescaling: given a generic h in this class, the rescaled phase $\tilde{h}$ is different from the original h, but still satisfies H1), H2) and H3). This is the underlying reason for introducing the larger family of wave-propagators U_φ when proving estimates for $e^{it\sqrt{-\Delta}}$ via an induction-on-scales argument.

Lemma 5.1. *Let $n \geq 2$ and $1 < p \leq q < \infty$ be as in Theorem 1.3. Let D_1, $D_2 > 0$, $\vec{\mu} \in \mathbb{R}^2_+$, $M > 100n$, $\varepsilon_\circ > 0$ and $L > 0$. Assume (2.7) holds for all $[h; b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ that are L-flat and all $\lambda \geq 1$. Let $K \geq 2$ be sufficiently large, depending on n and M, and τ be a K^{-1}-plate. Then*

$$\|U_{[h;b]} f_\tau\|_{L^q(B^n_R \times [-R,R])} \lesssim_{\mathbf{H},\varepsilon} K^{\frac{n+1}{q} - \frac{n-1}{p}} (R/K^2)^{\beta+\varepsilon} \|\tilde{f}_\tau\|_{L^p(\mathbb{R}^n)},$$

where f_τ and $\tilde{f}_\tau$ are defined as in §3.1.

Proof. Let $(1, \omega) \equiv (1, \omega_\tau)$ be the center of the K^{-1}-plate τ upon which $\widehat{f_\tau}$ is supported. Perform the change of variables $(\xi_1, \xi') = (\eta_1, \eta_1 \omega + K^{-1}\eta')$, so that $h(\xi) = h(\eta_1, \eta_1\omega + K^{-1}\eta')$. By a Taylor expansion around $(\eta_1, \eta_1\omega)$ and the homogeneity of h, $h(\xi)$ equals to

$$\eta_1 h(1, \omega) + K^{-1}\langle \partial_{\xi'} h(1, \omega), \eta' \rangle + K^{-2} \int_0^1 (1 - r) \langle \partial^2_{\xi'\xi'} h(1, \omega + rK^{-1}\frac{\eta'}{\eta_1})\eta', \eta' \rangle \frac{\mathrm{d}r}{\eta_1}. \tag{5.1}$$

Let $\tilde{h}(\eta)$ be the function associated with the integral above,

$$\tilde{h}(\eta) = K^2 h(\eta_1, \eta_1\omega + K^{-1}\eta') - K^2 \eta_1 h(1, \omega) - K \langle \partial_{\xi'} h(1, \omega), \eta' \rangle. \tag{5.2}$$

Let $\mathrm{D}_K, \Upsilon_\omega : \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ be linear functions given by

$$\begin{aligned} \mathrm{D}_K(x_1, x', t) &= (x_1, K^{-1}x', K^{-2}t) \\ \Upsilon_\omega(x_1, x', t) &= (x_1 + \langle x', \omega \rangle + th(1, \omega), x' + t\partial_{\xi'} h(1, \omega), t). \end{aligned}$$

Note that

$$U_{[h;b]} f_\tau(x, t) = U_{[\tilde{h};\tilde{b}]} g(\mathrm{D}_K \circ \Upsilon_\omega(x_1, x', t))$$

where[2]

$$\begin{aligned} \widehat{g}(\eta) &:= \widehat{\tilde{f}}_\tau(\eta_1, \eta_1\omega + K^{-1}\eta') K^{-(n-1)} \\ \tilde{b}(\eta) &:= b(\eta_1, \eta_1\omega + K^{-1}\eta')\chi(\eta'). \end{aligned} \tag{5.3}$$

We then have

$$\|U_{[h;b]} f_\tau\|_{L^q(B^n_R \times [-R,R])} = K^{\frac{n+1}{q}} \|U_{[\tilde{h};\tilde{b}]} g\|_{L^q(\mathrm{D}_K \circ \Upsilon_\omega(B^n_R \times [-R,R]))} \tag{5.4}$$

and

$$\|g\|_{L^p(\mathbb{R}^n)} = K^{-\frac{n-1}{p}} \|\tilde{f}_\tau\|_{L^p(\mathbb{R}^n)}. \tag{5.5}$$

Let $\mathcal{B}_{R/K^2}$ be a finitely overlapping collection of cylinders of the form

$$B_{R/K^2} \equiv B^n_{R/K^2} \times [-R/K^2, R/K^2]$$

such that

$$\mathrm{D}_K \circ \Upsilon_\omega(B^n_R \times [-R, R]) \subseteq \bigcup_{B_{R/K^2} \in \mathcal{B}_{R/K^2}} B_{R/K^2}.$$

Assuming temporarily that $[\tilde{h}; \tilde{b}]$ belongs to $\mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ and is L-flat, we may use the hypothesis (2.7) on each B_{R/K^2} to deduce

$$\|U_{[\tilde{h};\tilde{b}]} g\|_{L^q(B_{R/K^2})} \lesssim_{n,p,q,\mathbf{H},\varepsilon} (R/K^2)^{\beta+\varepsilon} \|g\|_{L^p(\mathbb{R}^n)}.$$

[2]Technically, one should divide $\tilde{b}$ and multiply g by a dimensional constant to ensure that $\tilde{b}$ satisfies B1). This only causes an admissible dimensional constant loss in the resulting inequality.

By Proposition 2.2 (at scale R/K^2), this implies

$$\|U_{[\tilde{h};\tilde{b}]}g\|_{L^q(\mathrm{D}_K\circ\Upsilon_\omega(B^n_R\times[-R,R]))} \lesssim_{n,p,q,\mathbf{H},\varepsilon} (R/K^2)^{\beta+\varepsilon}\|g\|_{L^p(\mathbb{R}^n)}.$$

Combining this with (5.4) and (5.5) allows us to conclude

$$\|U_{[h;b]}f_\tau\|_{L^q(B^n_R\times[-R,R])} \lesssim K^{\frac{n+1}{q}-\frac{n-1}{p}}(R/K^2)^{\beta+\varepsilon}\|f_\tau\|_{L^p(\mathbb{R}^n)}.$$

It remains to verify that $[\tilde{h};\tilde{b}] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ and that is L-flat. It follows from the expression of $\tilde{b}$ in (5.3) that it is supported in Ξ and satisfies B1) (see footnote 2). Regarding the phase $\tilde{h}$, it is clear from its definition in (5.2) that it is homogeneous of degree 1. Moreover, either (5.2) and the homogeneity of h, or simply the integral expression (5.1) quickly reveal that

$$\tilde{h}(1,0') = \partial_{\eta_1}\tilde{h}(1,0') = \partial_{\eta'}\tilde{h}(1,0') = 0.$$

This verifies that H1) holds. Furthermore, note that (5.2) yields

$$\partial^{\gamma'}_{\eta'}\tilde{h}(\eta) = K^{-(|\gamma'|-2)}\partial^{\gamma'}_{\xi'}h(\eta_1, \eta_1\omega + K^{-1}\eta')$$

for any $\gamma' \in \mathbb{N}_0^{n-1}$ such that $|\gamma'| \geq 2$. This and the assumptions on h immediately imply H2) and H3) for $\tilde{h}$, provided that $K \geq 2$ is sufficiently large depending on M and n. Moreover, as $[h;b]$ is L-flat, the above identity also implies that $[\tilde{h};\tilde{b}]$ is L-flat. □

Remark 5.2. We emphasise that if $[h;b]$ is L-flat, then the rescaled pair $[\tilde{h};\tilde{b}]$ is LK-flat, as can be read from the proof above. This fact will be referred to later.

6. Proof of Theorems 1.2 and 1.3

As discussed in Section 2, Theorem 1.3 is a consequence of Theorem 2.3, which can be further reduced to an equivalent statement for flat functions. Indeed, fix[3] $\lambda \gg 1$, $\varepsilon > 0$, and a pair $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$. Let $\tilde{\delta} = \frac{\varepsilon}{2(n-1)} > 0$ and decompose the support of b into $\lambda^{-\tilde{\delta}}$-plates. Applying the Lorentz rescaling Lemma 5.1 to each piece, the rescaled phase-amplitude pairs are in $\mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ and are $\lambda^{\tilde{\delta}}$-flat (see Remark 5.2). As there are $O(\lambda^{\tilde{\delta}(n-1)})$ many plates, (2.7) follows if we can prove that for all $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ that are $\lambda^{\tilde{\delta}}$-flat, the inequality

$$\|U_{[h;b]}f\|_{L^q(B^n_\lambda\times[-\lambda,\lambda])} \lesssim_{n,p,q,\mathbf{H},\varepsilon} \lambda^{\beta+\varepsilon/2}\|f\|_{L^p(\mathbb{R}^n)} \tag{6.1}$$

holds uniformly in $\lambda \geq 1$ and over all balls B^n_λ. Here again

$$\beta = (n-1)\Big(\frac{1}{2} - \frac{1}{p}\Big) + \frac{1}{q} - \sigma_{p,q}.$$

To this end, we introduce the following definition.

Definition 6.1. Given $\varepsilon > 0$, $R \geq 1$, $1 < p \leq q < \infty$ and $\beta = (n-1)\big(\frac{1}{2} - \frac{1}{p}\big) + \frac{1}{q} - \sigma_{p,q}$, let $Q_{\varepsilon,p,q}(R)$ denote the infimum over all constants $C \geq 0$ such that the inequality

$$\|U_{[h;b]}f\|_{L^q(B_R)} \leq CR^{\beta+\varepsilon}\|f\|_{L^p(\mathbb{R}^n)}$$

holds for all cylinders $B_R = B^n_R \times [-R,R]$, all phase/amplitude pairs $[h;b] \in \mathbf{H}(D_1, D_2, \vec{\mu}, M, \varepsilon_\circ)$ that are $\lambda^{\varepsilon/(n-1)}$-flat, all $\lambda \geq R$ and all functions $f \in L^p(\mathbb{R}^n)$.

[3] If $\lambda \lesssim 1$, Theorem 1.3 holds from the kernel estimate (2.4).

Thus, in order to verify (6.1), it suffices to show that for any $\varepsilon > 0$,

$$Q_{\varepsilon,p,q}(R) \le C(\varepsilon) \tag{6.2}$$

for all $R \ge 1$. The constant $C(\varepsilon)$ is allowed to depend on the quantities listed in Definition 6.1, namely p, q, n, D_1, D_2, $\vec{\mu} \in \mathbb{R}^2_+$, M and $\varepsilon_\circ$. We do not track dependencies on them from this point on, and whenever necessary, we refer to them as the data. The proof of (6.2) will proceed via induction on scales.

By the kernel estimate (2.4), the inequality (6.2) holds for small values $R \lesssim_\varepsilon 1$. This allows us to induct on the quantity R, with $R \lesssim_\varepsilon 1$ as a base case. In particular, one can state the following induction hypothesis.

Induction Hypothesis. *There exists a constant $\bar{C}_\varepsilon$ depending only on ε and the data such that*

$$Q_{\varepsilon,p,q}(R') \le \bar{C}_\varepsilon$$

holds for all $1 \le R' \le R/2$.

We shall next show that $Q_{\varepsilon,p,q}(R) \le \bar{C}_\varepsilon$. Let $f \in L^p(\mathbb{R}^n)$. By the support properties of b, we can assume that $\operatorname{supp} \widehat{f} \subseteq B(0,10)$, and by Lemma 3.4 and the triangle inequality one has

$$\|U_{[h;b]} f\|_{L^q(B_R)} \le \sum_{\nu=0}^{m\lfloor \log R \rfloor} \|U_{[h;b]} f^\nu\|_{L^q(B_R)} + \|U_{[h;b]} e\|_{L^q(B_R)} \tag{6.3}$$

for any $m > 0$. By Hölder's inequality and the kernel estimate (2.4), one has

$$\|U_{[h;b]} e\|_{L^q(B_R)} \lesssim R^{\frac{n+1}{q}} \|\Psi_R^{n+1} * |e|\|_{L^\infty(B_R^n)} \lesssim R^{\frac{n+1}{q}+n} \|e\|_{L^\infty(\mathbb{R}^n)}.$$

Using that $\|e\|_{L^\infty(\mathbb{R}^n)} \lesssim R^{-m} \|f\|_{L^p(\mathbb{R}^n)}$, one readily obtains

$$\|U_{[h;b]} e\|_{L^q(B_R)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{6.4}$$

provided $m > \frac{n+1}{q} + n$.

Broad-narrow analysis. We shall next perform a Bourgain–Guth broad–narrow analysis (cf. [5, 13]) on each function f^ν. By Theorem 3.2, there exists an integer $1 \ll A \lesssim_\varepsilon K^{\varepsilon/2}$ independent of the ball B_R such that

$$\|U_{[h;b]} f^\nu\|_{\mathrm{BL}^{\bar{p}_{n,k}}_{k,A}(B_R)} \lesssim_\varepsilon K^{d_\varepsilon} R^{\varepsilon/2} \|f^\nu\|_{L^2(\mathbb{R}^n)}$$

provided $K^{\varepsilon/2} \lesssim R^{\varepsilon^2/4}$. Here $K \ge 2$ is the parameter used to define the k-broad norm and will be specified later. Moreover, by Proposition 3.5

$$\|U_{[h;b]} f^\nu\|_{\mathrm{BL}^{\bar{p}_n}_{k,1}(B_R)} \lesssim_\varepsilon R^{\varepsilon/2} R^{(n-1)\left(\frac{1}{2}-\frac{1}{\bar{p}_n}\right)} \|f^\nu\|_{L^{\bar{p}_n}(\mathbb{R}^n)},$$

where $\bar{p}_n$ is the (hypothetical) smallest exponent for which sharp local smoothing holds.

By iii) of Lemma 3.4, f^ν satisfies the reverse Hölder inequality (3.2). Hence one can interpolate the above inequalities via Lemma 3.3 and use ii) of Lemma 3.4 to obtain

$$\|U_{[h;b]} f^\nu\|_{\mathrm{BL}^q_{k,A+1}(B_R)} \lesssim_\varepsilon K^{d_\varepsilon} R^{\varepsilon/2} R^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^p(\mathbb{R}^n)} \tag{6.5}$$

for

$$\frac{1}{p} - \frac{1}{\bar{p}_n} = (n+k+1)\Big(\frac{1}{2} - \frac{1}{\bar{p}_n}\Big)\Big(\frac{1}{p} - \frac{1}{q}\Big),$$

$2 \le p \le \bar{p}_n$ and $\bar{p}_{n,k} \le q \le \bar{p}_n$.

Consider next the decomposition of the support of b into K^{-1}-plates τ. For each $B_{K^2} \subset B_R$, let $V_1, \dots, V_{A+1}$ be a collection of $(k-1)$-dimensional subspaces in $\mathbb{R}^{n+1}$ attaining the minimum

$$\min_{V_1,\dots V_{A+1}\in \mathrm{Gr}(k-1,n+1)} \Big(\max_{\tau \notin V_a} \|U_{[h;b]} f_\tau^\nu\|^q_{L^q(B_{K^2})} \Big),$$

where $\tau \notin V_a$ stands for $\measuredangle(G(\tau), V_a) > K^{-2}$ for all $a = 1, \dots, A+1$. Then

$$\int_{B_{K^2}} |U_{[h;b]} f^\nu|^q \lesssim K^C \max_{\tau \notin V_a} \int_{B_{K^2}} |U_{[h;b]} f_\tau^\nu|^q + \sum_{a=1}^{A+1} \int_{B_{K^2}} \Big| \sum_{\tau \in V_a} U_{[h;b]} f_\tau^\nu \Big|^q.$$

When summing over $B_{K^2} \subset B_R$, the first term corresponds to the *broad* part $\|U_{[h;b]} f^\nu\|^q_{\mathrm{BL}^q_{k,A+1}(B_R)}$, which satisfies the estimate (6.5). The second term corresponds to the *narrow* part, for which the plates accumulate on a $(k-1)$-dimensional subspace. Provided that $K^2 \le \lambda^{\varepsilon/(n-1)}$, the pair $[h;b]$ is K^2-flat. By Theorem 4.2 and Hölder's inequality, for every $\delta' > 0$ and $N > 0$,

$$\int_{B_{K^2}} \Big| \sum_{\tau \in V_a} U_{[h;b]} f_\tau^\nu \Big|^q \lesssim_{\delta',N} K^{q\delta'} \max\{1, K^{q(k-3)(\frac{1}{2}-\frac{1}{q})}\} \sum_{\tau \in V_a} \int_{\mathbb{R}^{n+1}} |U_{[h;b]} f_\tau^\nu|^q w^N_{B_{K^2}}$$

holds for all $2 \le q \le \frac{2(k-1)}{k-3}$ (with $2 \le q \le \infty$ for $k \in \{2,3\}$), using Hölder's inequality in the sum and noting that there are $O(K^{k-3})$ K^{-1}-plates $\tau \in V_a$. As we have already taken advantage of the reduced number of plates, we can further control the sum over $\tau \in V_a$ by the sum over all K^{-1}-plates τ. Thus, summing over a and the balls $B_{K^2} \subset B_R$,

$$\Big(\sum_{B_{K^2} \subset B_R} \sum_{a=1}^{A+1} \int_{B_{K^2}} \Big| \sum_{\tau \in V_a} U_{[h;b]} f_\tau^\nu \Big|^q \Big)^{1/q}$$
$$\lesssim_{\delta',N} A^{1/q} K^{\delta'} \max\{1, K^{(k-3)(\frac{1}{2}-\frac{1}{q})}\} \Big(\sum_{\tau : K^{-1}-\text{plates}} \|U_{[h;b]} f_\tau^\nu\|^q_{L^q(w^N_{B_R})} \Big)^{1/q},$$

where we used $\sum_{B_{K^2} \subset B_R} w^N_{B_{K^2}} \lesssim w^N_{B_R}$.

Next, note that for any $\delta > 0$ and $\tilde{N} > 0$ one has

$$\|U_{[h;b]} f_\tau^\nu\|_{L^q(w^N_{B_R})} \lesssim_{\delta,\tilde{N}} \|U_{[h;b]} f_\tau^\nu\|_{L^q(R^\delta B_R)} + R^{-\tilde{N}} \|f_\tau^\nu\|_{L^p(\mathbb{R}^n)}.$$

This follows from the kernel estimate (2.4) and the decay of the weight $w^N_{B_R}$ on $\mathbb{R}^{n+1} \backslash R^\delta B_R$ provided N is chosen sufficiently large depending on $\delta, \tilde{N}, p, q$ and n. Applying a trivial decoupling (via the triangle inequality) of K^{-1}-plates τ into $(KR^{\delta/2})^{-1}$-plates τ', we obtain

$$\Big(\sum_{B_{K^2} \subset B_R} \sum_{a=1}^{A+1} \int_{B_{K^2}} \Big| \sum_{\tau \in V_a} U_{[h;b]} f_\tau^\nu \Big|^q \Big)^{1/q}$$
$$\lesssim R^{\delta\frac{(n-1)}{2}} K^{\delta'+\varepsilon} \max\{1, K^{(k-3)(\frac{1}{2}-\frac{1}{q})}\} \Big(\sum_{\tau} \sum_{\tau' \cap \tau \neq \varnothing} \|U_{[h;b]} (f_\tau^\nu)_{\tau'}\|^q_{L^q(R^\delta B_R)} \Big)^{1/q}$$
$$+ R^{-\tilde{N}} \Big(\sum_\tau \|f_\tau^\nu\|^q_{L^p(\mathbb{R}^n)} \Big)^{\frac{1}{q}} \quad (6.6)$$

where we have used $A \lesssim K^{\varepsilon/2}$ and the constant depends on $\delta, \delta', \tilde{N}$. Using the Lorentz rescaling in Lemma 5.1 and the induction hypothesis $Q_{\varepsilon,p,q}(R') \leq \bar{C}_\varepsilon$ with

$$R' = R^{1+\delta}/(R^\delta K^2) = R/K^2 \leq R/2,$$

one has

$$\Big(\sum_\tau \sum_{\tau' \cap \tau \neq \varnothing} \|U_{[h;b]}(f_\tau^\nu)_{\tau'}\|^q_{L^q(R^\delta B_R)}\Big)^{1/q} \lesssim \bar{C}_\varepsilon K^{\frac{n+1}{q} - \frac{n-1}{p}} (R/K^2)^{\beta+\varepsilon} \Big(\sum_\tau \sum_{\tau' \cap \tau \neq \varnothing} \|(\widetilde{f}_\tau^\nu)_{\tau'}\|^q_{L^p(\mathbb{R}^n)}\Big)^{1/q}.$$

By the embedding $\ell^p \subseteq \ell^q$ for $p \leq q$; the bounds

$$(\sum_\tau \|f_\tau^\nu\|_p^p)^{1/p} \lesssim \|f^\nu\|_p \quad \text{and} \quad \Big(\sum_\tau \sum_{\tau' \cap \tau \neq \varnothing} \|(\widetilde{f}_\tau^\nu)_{\tau'}\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p} \lesssim \|f^\nu\|_p,$$

which follow by interpolation between $p = 2$ and $p = \infty$; ii) of Lemma 3.4, that is, $\|f^\nu\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$; and choosing $\tilde{N}$ sufficiently large, one has that the right-hand side of (6.6) is controlled by

$$C\bar{C}_\varepsilon \max\{1, K^{(k-3)(\frac{1}{2}-\frac{1}{q})}\} K^{\frac{n+1}{q} - \frac{n-1}{p} - 2\beta - \varepsilon + \delta'} R^{\delta\frac{(n-1)}{2}} R^{\beta+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}. \tag{6.7}$$

Closing the induction. By (6.3), the estimates (6.5) and (6.7) for each $\nu = 0, \ldots, m\lfloor \log R \rfloor$, where $m > \frac{n+1}{q} + n$, and the error estimate (6.4), one obtains

$$\|U_{[h;b]} f\|_{L^q(B_R)} \leq \log R \cdot (\mathrm{I} + \mathrm{II})\, \|f\|_{L^p(\mathbb{R}^n)}, \tag{6.8}$$

where

$$\mathrm{I} = D(\varepsilon) K^{D(\varepsilon)} R^{\varepsilon/2} R^{(n-1)(\frac{1}{2}-\frac{1}{p})},$$
$$\mathrm{II} = D(\delta, \delta')\bar{C}_\varepsilon \max\{1, K^{(k-3)(\frac{1}{2}-\frac{1}{q})}\} K^{\frac{n+1}{q} - \frac{n-1}{p} - 2\beta - \varepsilon + \delta'} R^{\delta\frac{(n-1)}{2}} R^{\beta+\varepsilon}$$

and

$$\frac{1}{p} - \frac{1}{\bar{p}_n} = (n+k+1)\Big(\frac{1}{2} - \frac{1}{\bar{p}_n}\Big)\Big(\frac{1}{p} - \frac{1}{q}\Big),$$

$2 \leq p \leq \bar{p}_n$, $\bar{p}_{n,k} \leq q \leq \bar{p}_n$. Here $D(\cdot)$ is a constant depending on the data as described after Definition 6.1 and the arguments in the parenthesis but not on K and not on R. It is allowed to change from line to line.

We need to show that $\log R \cdot (\mathrm{I} + \mathrm{II}) \leq \bar{C}_\varepsilon R^{\beta+\varepsilon}$. This will require the exponent of K in the second term of the right-hand side above to be negative. It is useful to note that

$$\frac{n+1}{q} - \frac{n-1}{p} - 2\beta = \frac{n+1}{q} - \frac{n-1}{p'} - 2\Big(\frac{1}{q} - \bar{\sigma}\Big), \tag{6.9}$$

as

$$\beta = (n-1)\Big(\frac{1}{2} - \frac{1}{p}\Big) + \Big(\frac{1}{q} - \bar{\sigma}\Big).$$

We next analyse what choices of p, q and k allow to close the induction and lead to sharp estimates on the critical line and off the critical line respectively.

Sharp regularity estimates on the critical line. If $\bar\sigma = \sigma_{p,q} = 1/q$ and $(1/p, 1/q)$ is on the critical line

$$\frac{1}{q} = \frac{n-1}{n+1}\frac{1}{p'}, \tag{6.10}$$

the expression in (6.9) equals to 0. Thus, the exponent of K in the second term of the right-hand side of (6.8) can only be negative if $k = 2$ or $k = 3$. Note that the critical line (6.10) meets the interpolation line

$$\frac{1}{p} - \frac{1}{\bar p_n} = (n+k+1)\Big(\frac{1}{2} - \frac{1}{\bar p_n}\Big)\Big(\frac{1}{p} - \frac{1}{q}\Big) \tag{6.11}$$

at

$$p(k) = \frac{2\bar p_n\left(n^2+kn-1\right) - 4n(n+k+1)}{(n-1)\bar p_n(n+k+1) - 2(k(n-1)+n(n+1))},$$
$$q(k) = \frac{2\bar p_n\left(n^2+kn-1\right) - 4n(n+k+1)}{(n-1)\bar p_n(n+k-1) - 2(n-1)(k+n)},$$

which satisfy $2 \le p(k) \le \bar p_n$, $\bar p_{n,k} \le q(k) \le \bar p_n$ for $2 \le k \le n+1$. As $q(k)$ decreases with k, the best estimates are obtained using $k = 3$, which is the highest possible k that is still admissible. It is then our goal to show that

$$\|U_{[h;b]}f\|_{L^{q(3)}(B_R)} \le \bar C_\varepsilon R^{\beta+\varepsilon}\|f\|_{L^{p(3)}(\mathbb{R}^n)}. \tag{6.12}$$

To this end, consider (6.8) and use the bounds $\log R \lesssim_\varepsilon R^{\varepsilon/4}$ for the first term I and $\log R \lesssim_\delta R^{\delta(n-1)/2}$ for the second term II; recall that $R \gtrsim_\varepsilon 1$. As $\beta = (n-1)(\frac{1}{2} - \frac{1}{p})$, the inequality (6.8) reads now

$$\|U_{[h;b]}f\|_{L^{q(3)}(B_R)} \le \Big(D(\varepsilon)K^{D(\varepsilon)}R^{\beta+3\varepsilon/4} + D(\delta,\delta')\bar C_\varepsilon K^{\delta'-\varepsilon}R^{\delta(n-1)}R^{\beta+\varepsilon}\Big)\|f\|_{L^{p(3)}(\mathbb{R}^n)}.$$

Choose $\delta' = \varepsilon/2$ so that $K^{\delta'-\varepsilon} = K^{-\varepsilon/2}$. Then choose $K = K_0R^{2\delta(n-1)/\varepsilon}$ for sufficiently large $K_0 \ge 1$, depending on δ, ε and the data so that $D(\delta,\delta')K_0^{-\varepsilon/2} \le 1/2$. Then

$$\|U_{[h;b]}f\|_{L^{q(3)}(B_R)} \le \Big(D(\varepsilon)K^{D(\varepsilon)}R^{\beta-\varepsilon/4} + \bar C_\varepsilon/2\Big)R^{\beta+\varepsilon}\|f\|_{L^{p(3)}(\mathbb{R}^n)}$$

with $K = K_0R^{2\delta(n-1)/\varepsilon}$. We choose

$$2\delta = \min\Big\{\frac{\varepsilon^2}{4D(\varepsilon)(n-1)}, \frac{\varepsilon^2}{4(n-1)^2}\Big\}.$$

Finally, choose $\bar C_\varepsilon$ large enough so that

$$D(\varepsilon)K_0^{D(\varepsilon)} \le \bar C_\varepsilon/2,$$

which is admissible as the parameter K_0 only depends on ε and the data. One then concludes that

$$\|U_{[h;b]}f\|_{L^{q(3)}(B_R)} \le \bar C_\varepsilon R^{\beta+\varepsilon}\|f\|_{L^{p(3)}(\mathbb{R}^n)}$$

using the first value in the definition of δ, which is the desired estimate (6.12). This closes the induction provided we can verify the flatness condition $K^2 \le \lambda^{\varepsilon/(n-1)}$ and the condition $K^{\varepsilon/2} \lesssim R^{\varepsilon^2/4}$ required by the broad estimate. Note that by the second entry in the definition of δ, and using $R \le \lambda$,

$$K^2 = K_0^2R^{4\delta(n-1)/\varepsilon} \le K_0^2R^{\varepsilon/2(n-1)} \le (K_0^2\lambda^{-\varepsilon/2(n-1)})\lambda^{\varepsilon/(n-1)} \le \lambda^{\varepsilon/(n-1)}$$

as $\lambda \gg 1$ and the parameter K_0 only depends on ε and the data. Similarly, by the first entry in the definition of δ,

$$K^{\varepsilon/2} = K_0^{\varepsilon/2} R^{\delta(n-1)} \leq K_0^{\varepsilon/2} R^{\varepsilon^2/8} \lesssim R^{\varepsilon^2/4}$$

as we are only concerned with $R \gtrsim_\varepsilon 1$.

Therefore a sharp $(p(3), q(3), \sigma_{p(3),q(3)})$ local smoothing estimate holds, and a further interpolation with the elementary $(1, \infty, 0)$ estimate yields the estimates $(p, q, \sigma_{p,q})$ on the critical line (6.10) for all $q \geq q(3)$. This proves Theorem 1.2.

Sharp regularity estimates away from the critical line. Consider first the case

$$\frac{1}{q} > \frac{n-1}{n+1}\frac{1}{p'}, \quad 2 \leq p \leq \frac{2n}{n-1}, \quad \bar{\sigma} = \sigma_{p,q} = \frac{(n-1)}{2}\Big(\frac{1}{p'} - \frac{1}{q}\Big),$$

with $p \leq q$, $p' < q$. For this data, the expression (6.9) is identically zero for any pair (p, q). Thus, the only possibilities for the exponent of K to be negative in (6.8) are again $k = 2$ and $k = 3$. The estimates arising repeating the above analysis are implied by the interpolation of the sharp estimates $(p, q, \sigma_{p,q})$ for $q \geq q(3)$ in the critical line (6.10) with the fixed-time estimate $p = q = 2$. This proves the sharp bounds in the region $\mathfrak{T}_n \backslash \overline{P_1 P_2}$ in Theorem 1.3.

Consider next the case

$$\frac{1}{q} < \frac{n-1}{n+1}\frac{1}{p'}, \quad \frac{2n}{n-1} < q, \quad \bar{\sigma} = \sigma_{p,q} = \frac{1}{q}, \quad p \leq q.$$

Using the relation (6.9), one has that the exponent of K in the second term in (6.8) is negative provided that

$$(k-3)\Big(\frac{1}{2} - \frac{1}{q}\Big) + \frac{n+1}{q} - \frac{n-1}{p'} \leq 0. \tag{6.13}$$

We are thus allowed to use higher values for $k \geq 3$. For (p, q) in the interpolation line (6.11), the condition (6.13) is saturated at $(p, q) = (\bar{p}(k), \bar{q}(k))$, where

$$\bar{p}(k) = \frac{2\bar{p}_n\left(2n^2 + k(n+4) - k^2 + 3n - 5\right) - 4\,(n+k+1)\,(2n-k+3)}{\bar{p}_n\,(n+k+1)\,(2n-k+1) - 2\,(2n^2 + k(n-2) - k^2 + 5n + 9)},$$

$$\bar{q}(k) = \frac{2\bar{p}_n\left(2n^2 + k(n+4) - k^2 + 3n - 5\right) - 4\,(n+k+1)\,(2n-k+3)}{\bar{p}_n\,(n+k-1)\,(2n-k+1) - 2\,(2n^2 + kn - k^2 + n + 3)}.$$

The pair of exponents $(\bar{p}(k), \bar{q}(k))$ satisfies the constraints

$$2 \leq \bar{p}(k) \leq \bar{p}_n, \quad \bar{p}_{n,k} \leq \bar{q}(k) \leq \bar{p}_n, \quad \frac{2n}{n-1} < \bar{q}(k) < \infty$$

for any integer $2 \leq k \leq n+1$. Arguing as for the critical line, one can close the induction and obtain the sharp estimate

$$\|U_{[h;b]} f\|_{L^{\bar{q}(k)}(B_R)} \leq \bar{C}_\varepsilon R^{\beta+\varepsilon} \|f\|_{L^{\bar{p}(k)}(\mathbb{R}^n)}.$$

This yields a set of $(p, q, \sigma_{p,q})$ local smoothing estimates for each $2 \leq k \leq n+1$, which can all be interpolated together with $(1/\bar{p}_n, 1/\bar{p}_n)$ and the fixed time estimates at $P_0 = (0, 0)$, $P_1 = (1, 0)$ from (1.10) to yield sharp estimates for $(1/p, 1/q) \in \mathfrak{P}_n \backslash \overline{P_0 P_1}$. This proves Theorem 1.3.

Remark. Despite this paper focuses on sharp regularity local smoothing estimates, it is natural to explore what non-sharp regularity estimates would higher degrees of linearity imply on the critical line (6.10). In order to close the induction in the proof of Theorem 1.2, one requires the exponent of K in the second term in the right-hand side of (6.8) to be negative. By (6.9), this requires

$$\bar{\sigma} \le \sigma(k) := \frac{k-1}{2q(k)} - \frac{k-3}{4}.$$

Note that $0 < \sigma(k) \le 1/q(k)$ if $2 \le q(k) < \frac{2(k-1)}{k-3}$. Controlling $R^{(n-1)(\frac{1}{2}-\frac{1}{p})} \le R^{\beta}$, one can then argue as above to obtain, for each fixed k, a $(p(k), q(k), \sigma(k))$ local smoothing estimate. However, with the input $\bar{p}_n = \frac{2(n+1)}{n-1}$, such estimates are worse than those obtained by interpolation of the case $k = 3$ and the known local smoothing estimates for all $\sigma < 1/(2p)$ at $p = q = \frac{2n}{n-1}$, which themselves follow by interpolation from the sharp estimates at $\bar{p}_n = \frac{2(n+1)}{n-1}$ and L^2. Indeed, that interpolation yields (p, q, σ^*) estimates in the critical line (6.10) with

$$\sigma^* = \frac{n+5}{4} - \frac{n^2+4n+1}{2q(n-1)}, \quad \frac{2n}{n-1} \le q \le \frac{2(n^2+6n-1)}{(n-1)(n+5)}.$$

One can hence verify that for $q = q(k)$, one has $\sigma^* > \sigma(k)$ if $k > 3$. Better non-sharp regularity results could be obtained by interpolation with the most recent results at $p = q = \frac{2n}{n-1}$ from [7]. The computation is left to the interested reader.

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