A STABILITY RESULT FOR PARABOLIC MEASURES OF OPERATORS WITH SINGULAR DRIFTS

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ABSTRACT. We study the operator

$$\partial_t - \operatorname{div} A\nabla + B \cdot \nabla$$

in parabolic upper-half-space, where A is an elliptic matrix satisfying an oscillation condition and B is a singular drift with a Carleson control. Our main result establishes quantitative A_∞-estimates for the parabolic measure in terms of oscillation of A and smallness of B. The proof relies on new estimates for parabolic Green functions that quantify their deviations from linear functions of the normal variable and on a novel, quantitative Carleson measure criterion for anisotropic A_∞-weights.

CONTENTS

1. Introduction

We work in the parabolic upper-half-space

$$\mathbb{R}^{n+1}_+ := \{(t, x, \lambda) : t \in \mathbb{R}, x \in \mathbb{R}^{n-1}, \lambda > 0\}$$

and study differential operators of the form

$$\mathcal{L} := \partial_t - \operatorname{div} A\nabla + B \cdot \nabla, \tag{1.1}$$

where $A : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n\times n}$ is an elliptic matrix-valued function (with ellipticity constant M_0) and $B : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n$ is a singular drift term with a uniform bound $|\lambda B(t, x, \lambda)| \leq \varepsilon_0$. The matrices A satisfy a (weak) Dahlberg–Kenig–Pipher oscillation condition in all variables and λB satisfies a smallness condition, both being measured by certain coefficients $\alpha_{A,B}(t, x, \lambda)$ and quantified through the requirement that

$$d\nu_{A,B}(t, x, \lambda) := \alpha_{A,B}(t, x, \lambda)^2 \frac{dt\, dx\, d\lambda}{\lambda}$$

should be a (parabolic) Carleson measure on $\mathbb{R}^{1+n}_+$, see Section 2 for precise definitions and further notation. This class of operators arises as pull-backs when studying parabolic equations in parabolic Lipschitz domains [HL01].

Our main theorem is a stability result for the parabolic measure, analogous to the culmination of the sequence of the three papers [DLM22b], [BTZ23a] and [BES21] from the elliptic setting. Moreover, it can be viewed as an analytic version of a small constant, variable coefficients variant of [HLN04]. It quantitatively demonstrates that small oscillations of A and small drift B lead to small oscillations of the parabolic measure when the pole is sufficiently far from the boundary.

Theorem 1.2. *Let M_0 be given. There are $\varepsilon_0 > 0$, $\delta_0 > 0$ and $C \geq 1$ such that the following hold whenever $\mathcal{L}$ is (M_0, ε_0)-parabolic and $(t_0, x_0, \lambda_0) \in \mathbb{R}^{1+n}_+$.*

(1) For every $\delta \in (0, \delta_0)$ there is $\kappa \geq 40$ such that if $\|\nu_{A,B}\|_C \leq \delta$, then the $\mathcal{L}$-parabolic measure ω with pole at the forward corkscrew point $a^+(t_0, x_0, \kappa\lambda_0)$ is absolutely continuous with respect to Lebesgue measure on $Q^n(t_0, x_0, 2\lambda_0)$.

(2) Denoting $k := \frac{d\omega}{dt\,dx}$ in the setting of (1), the quantitative A_∞-estimate

$$\log\left(\fint\!\!\fint_Q k(t, x)\, dt\, dx\right) - \fint\!\!\fint_Q \log k(t, x)\, dt\, dx \leq C\sqrt{\delta}$$

holds for all parabolic cubes $Q \subset Q^n(t_0, x_0, \lambda_0)$.

The supremum of the left-hand side in part (2) of the theorem can be taken as a definition of the (local) A_∞-constant of ω. It implies the following quantitative statement for L^p-solvability of the Dirichlet problem for $\mathcal{L}$, see Section 5.

Corollary 1.3. *Let M_0 be given. There is $\varepsilon_0 > 0$ such that if $\mathcal{L}$ is (M_0, ε_0)-parabolic, then for every $q \in (1, \infty)$ there exists $\delta > 0$ such that $\|\nu\|_C \leq \delta$ implies that all parabolic measures in Theorem 1.2 are locally reverse Hölder weights with exponent q and uniform constants. In particular, the L^p-Dirichlet problem for $\mathcal{L}$ in the upper-half-space is solvable, where $1/p + 1/q = 1$.*

Remark 1.4. The methods here also seem to apply for $\mathcal{L}$ with no drift but without the smallness assumption on the measure ν. In that case, one expects to obtain that the associated parabolic measure belongs to A_∞ without quantitative control on the constant. The only modification required is to replace the proof of Theorem 3.21 with its "large constant" analogue closely following [DLM22b, Section 4]. This result, in itself, would be new because of the **weak**-DKP condition, but is not the focus of the current work (small constant results).

1.1. Strategy.

The three main steps of the proof are

(a) an estimate on deviation of the Green function from a linear function,
(b) an estimate on approximations to distributional derivatives of the parabolic measures,
(c) and a local and quantitative criterion for anisotropic A_∞-weights in terms of Carleson conditions.

This general idea corresponds to the sequence of papers [DLM22b], [BTZ23a] and [BES21] from the elliptic setting; but the first and the third part of the proof exhibit significant differences due to the evolutionary nature of the problem.

Part (a) is carried out in Sections 3.1 - 3.3. The main difficulty is the time-directedness of the Harnack inequality in the context of general solutions. Already the proof for non-degeneracy of the Dirichlet energy for constant coefficient operators in Lemma 3.1 requires a completely novel argument and the perturbation argument for variable coefficients in Lemma 3.10 would have not been possible, had we been not able to sweep away one of the essential technical difficulties pointed out in [DLM22b]. Our proof could hence be used to improve even the elliptic estimate (compare to Lemma 3.19 in [DLM22b]).

Part (b) consists in using the Riesz formula for solutions in order to bound approximants to distributional derivatives of the parabolic measure by quantities related to the Green function. This is carried out in Section 3.4 and follows the lines of thought of [BTZ23a] with few modifications.

Part (c) constitutes our second main result. It is presented in Section 4 and should be useful in different contexts. Our Carleson condition is quantitative in the spirit of [BES21] but without global doubling as a background hypothesis on the weight. Providing a 'Carleson test' that is amenable for weights that are doubling only locally is important, as parabolic measures fall precisely into this class. To this end, we renounce the algebraic properties of the heat kernel in [FKP91, BES21] to be able to directly work with approximations of identity based on compactly supported test functions. Even though the results in [FKP91] also include a characterization based on compactly supported test functions, their argument still uses representation through heat kernels and thus global doubling seems to be necessary.

1.2. Background on parabolic measures.

Let us provide some additional context from the extensive literature related to parabolic measure.

Based on Dahlberg's result [Dah77] that harmonic measure above a Lipschitz graph satisfies a reverse Hölder inequality with exponent $p = 2$, R. Hunt conjectured (see [KW80, p. 2]) that caloric measure above a parabolic Lipschitz graph is locally an A_∞-weight and therefore absolutely continuous with respect to surface measure. This turned out to be untrue [KW88]. Only very recently, it was shown that caloric measure above a parabolic Lipschitz graph is in A_∞ if and only if the function defining the graph has a half-order time derivative in the parabolic BMO-space [BHMN21]. One direction was known for some time and is due to Lewis and Murray [LM95] and the same condition on the half-order time derivative had already been identified as a natural one from the context of parabolic singular integrals, boundary value problems and Layer potentials in [Hof95, Hof97, HL96, HL05, Mur85]. Moreover, the true analog of Dahlberg's result (density with reverse Hölder exponent $p = 2$) requires smallness of the half-order time derivative [HL96].

Contrarily to the elliptic setting [JK81], the most obvious flattening pull-back from a parabolic graph domain to the half-space cannot be used because it fails to be Lipschitz in the t-variable, and the more elaborate smoothed pullback of Dahlberg–Kenig–Stein [Dah86b] introduces a singular drift term, even when starting from the heat equation [HL96, HL01, HL05]. Moreover, if in the situation of the Dahlberg–Kenig–Stein pullback the half-order time derivative for the graph function is in parabolic BMO and the original coefficients A and B in the graph domain satisfy the (strong) Dahlberg–Kenig–Pipher condition (supremum of $|\partial_t A|^2\delta^3$, $|\nabla A|^2\delta$ and $|B|^2\delta$ on Whitney regions are all Carleson measures [HL05]), then so do the pulled-back coefficients. In particular, the pull-backs are operators of type (1.1) with a stronger oscillation condition than in the present paper.

For operators of type $\mathcal{L}$ in the upper-half-space, Hofmann and Lewis [HL01] proved that the (strong) Dahlberg–Kenig–Pipher condition implies that the associated parabolic measure is an A_∞-weight. Subsequently, L^p-solvability of the Dirichlet problem for p close or equal to 1 in parabolic graph domains (with small half order time derivative measured in BMO) were shown by Dindoš–Petermichl–Pipher [DPP07] and for p close to 1 in rougher domains by Dindoš–Dyer–Hwang [DDH20].

In this regard, the main contribution of the present work is the fist qualitative small constant A_∞-results and the use of the (weak a.k.a L^2) Dahlberg–Kenig–Pipher condition on the coefficients. We conjecture that a direct application of our results here (along with tracking constants) will give small constant A_∞-results in the graph case and that — less directly and following the work of [DLM22a] — it may be possible to obtain small constant A_∞-results in rougher settings for operators satisfying a weak Dahlberg-Kenig-Pipher condition.

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2. Preliminaries

2.1. **General notation.** Throughout, we work in space time $\mathbb{R}^{1+n}$ with $n \in \mathbb{N}$ satisfying $n \geq 2$. For notational convenience, given any two integers $n_1, n_2 \geq 1$, we identify $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $\mathbb{R}^{n_1+n_2}$ so that

$$\mathbb{R}^{1+n} := \{(t, x, \lambda) : t \in \mathbb{R},\ x \in \mathbb{R}^{n-1},\ \lambda \in \mathbb{R}\}.$$

We define the parabolic distances by setting for $z, w \in \mathbb{R}^m$ with $m \geq 2$,

$$d_m(z, w) := \max\left(|z_1 - w_1|^{1/2}, \max_{1<i\leq m} |z_i - w_i|\right).$$

This definition will be used with $m \in \{n, n+1\}$. Given $z \in \mathbb{R}^{1+n}$ and $r > 0$, we define the parabolic cylinder with radius r and center x as

$$Q(z, r) := \{w \in \mathbb{R}^{1+n} : d_{n+1}(z, w) < r\}.$$

The parabolic boundary of $Q(z, r)$ is

$$\partial_{par} Q(z, r) := \{w \in Q(z, r) : d_{1+n}(z, w) = r, w_1 \neq z_1 + r^2\}$$

and likewise the adjoint parabolic boundary $\partial_{par^*} Q(z, r)$ is obtained from the topological boundary by excluding the initial boundary $w_1 = z_1 - r^2$. If E is a measurable set with Lebesgue measure $|E| \in (0, \infty)$ and $f \in L^1_{loc}(\mathbb{R}^{1+n}_+)$, we write

$$\langle f \rangle_E := \fint_E f(z)\, dz := \frac{1}{|E|} \int_E f(z)\ dz.$$

Further, given a time parameter τ and a set $E \subset \mathbb{R}^{1+n}$, we denote the time slice by

$$E^\tau := \{(t, x, \lambda) \in E : t = \tau\}.$$

2.2. **Point-set correspondence on the upper-half-space.** Our focus will be on solutions to parabolic equations in subsets of the upper-half-space $\mathbb{R}^{1+n}_+$. The points $(t, x, \lambda) \in \mathbb{R} \times \mathbb{R}^{n-1} \times (0, \infty) = \mathbb{R}^{1+n}_+$ are set in a one-to-one correspondence with the following geometric objects.

(1) The n-dimensional parabolic cylinder or boundary cylinder is defined as

$$Q^n(t, x, \lambda) := \{(s, y, 0) \in \mathbb{R}^{1+n} : d_n((t, x), (s, y)) < \lambda\}.$$

(2) The parabolic Carleson region is defined as

$$R(t, x, \lambda) := Q^n(t, x, \lambda) \times (0, 2\lambda).$$

(3) The parabolic Whitney region is defined as

$$W(t, x, \lambda) := Q^n(t, x, \lambda) \times (\lambda, 2\lambda).$$

(4) The forward and backward (in time) corkscrew points are defined as

$$a^\pm(t, x, \lambda) := (t \pm 2\lambda^2, x, 2\lambda).$$

Given $(t, x, \lambda) \in \mathbb{R}^{1+n}_+$ and symbols Y, Z, we sometimes write

$$Y(Z(t, x, \lambda)) = Y(t, x, \lambda).$$

to ease notation. Useful examples for this notation are the parabolic Carleson box $R(Q^n(t, x, \lambda)) = R(t, x, \lambda)$ above the boundary cylinder $Q^n(t, x, \lambda)$ and the corkscrew points $a^\pm(R) = a^\pm(R(t, x, \lambda)) = a(t, x, \lambda)$. For regions $Z = Z(t, x, \lambda)$ we also write $\theta B := Z(t, x, \theta\lambda)$, $\theta > 0$, for re-scaled regions of the same type.

2.3. **Structure of the equation.** We denote by ∇ and div the gradient and divergence with respect to all variables except for the time variable.We denote the derivative with respect to the time variable by ∂_t.

Let $M_0 \geq 1$. A measurable function $A : \mathbb{R}^{1+n}_+ \to \mathbb{R}^{n\times n}$ is said to be M_0-elliptic if for all $z \in \mathbb{R}^{1+n}_+$ and $\xi \in \mathbb{R}^n$,

$$A(z)\xi \cdot \xi \geq M_0^{-1}|\xi|^2, \quad |A(z)\xi| \leq M_0|\xi|.$$

Let $\varepsilon_0 > 0$. A function $B : \mathbb{R}^{1+n}_+ \to \mathbb{R}^n$ is said to be an ε_0-small drift if for all $z \in \mathbb{R}^{1+n}_+$,

$$z_{n+1}|B(z)| \leq \varepsilon_0,$$

We call

$$\mathcal{L} = \partial_t - \operatorname{div}(A\nabla\,\cdot) + B\cdot\nabla$$

an (M_0, ε_0)-parabolic operator if A is M_0-elliptic and B is ε_0-small. The adjoint of an (M_0, ε_0)-parabolic operator is formally written as

$$\mathcal{L}^* = -\partial_t - \operatorname{div}(A^*\nabla\,\cdot) - \operatorname{div}(B\,\cdot).$$

Once the parameters M_0 and ε_0 are specified, we call $\mathcal{L}$ and $\mathcal{L}^*$ briefly parabolic and adjoint parabolic operators. If $\varepsilon_0 = 0$, we call the (M_0, ε_0)-parabolic operator drift free.

2.4. **Weak solutions.** Let $\mathcal{L}$ be an (M_0, ε_0)-parabolic operator. Given an open cube $Q \subset \mathbb{R}^{1+n}_+$ and source terms $f \in L^2_{loc}(Q)$ and $F \in L^2_{loc}(Q;\mathbb{R}^n)$, we say that u is a weak solution to $\mathcal{L}u = f + \operatorname{div} F$ in Q if $u \in L^2_{loc}(I; W^{1,2}_{loc}(\Sigma)) \cap C_{loc}(I; L^2_{loc}(\Sigma))$, where $Q = I \times \Sigma$ with I corresponding to the time variable, and

$$\int_Q -u\partial_t\varphi + A\nabla u\cdot\nabla\varphi + (B\cdot\nabla u)\varphi\,dz = 0 \qquad (\varphi \in C^\infty_c(Q)).$$

The adjoint equation $\mathcal{L}^*u = f + \operatorname{div} F$ has an analogous interpretation. We say that u is a global weak solution in Q if the 'loc-subscripts' above can be dropped.

The definitions of weak solutions follow the classical references [Aro68, HL01]. Since B is ε_0-small, we can also write $Bu, B\cdot\nabla u \in L^2_{loc}(Q)$ as source terms on the right-hand side and conclude from [ABES19b, Theorem 4.2] that for local solutions the a priori requirement $C_{loc}(I; L^2_{loc}(\Sigma))$ is not needed but follows from the equation.

If $Q = R(t, x, \lambda)$ is a Carleson region, we usually work with global weak solutions to $\mathcal{L}u = 0$ or $\mathcal{L}^*u = 0$ that satisfy '$u = 0$ on $Q^n(t, x, \lambda)$'. This should be understood as $u(s,\cdot) = 0$ in the Sobolev sense for a.e. $s \in I$. In this scenario, for adjoint solutions, we have $Bu \in L^2(Q)$, since the one-dimensional Hardy inequality yields

$$\int_Q |Bu|^2\,dz \leq \varepsilon_0 \iint_{Q^n(t,x,\lambda)} \int_0^{\lambda^2} \frac{|u(s,y,\mu)|^2}{\mu^2}\,d\mu\,ds\,dy \leq 4\varepsilon_0 \int_Q |\partial_\lambda u|^2\,dz.$$

This observation will be used frequently in the following. Sometimes we impose the stronger condition that u is continuous up to $Q^n(t, x, \lambda)$ and vanishes thereon; 'u vanishes continuously on $Q^n(t, x, \lambda)$' for short.

2.5. **Carleson measures and energies.** We impose additional smoothness on the coefficients of the differential operators in terms of a Carleson measure condition on the oscillation of A and the size of B.

Definition 2.1 (Carleson Measure)**.** Given a measure μ on $\mathbb{R}^{1+n}_+$ and a set $E \subset \mathbb{R}^{1+n}_+$, we write

$$\|\mu\|_{C(E)} := \sup_{R' \subset E} \frac{\mu(R')}{|Q^n(R')|},$$

where the supremum is taken over all parabolic Carleson regions $R' \subset E$. If $\|\mu\|_{C(E)} < \infty$, we call μ a parabolic Carleson measure in E and $\|\mu\|_{C(E)}$ its parabolic Carleson constant or norm. If $E = \mathbb{R}^{1+n}_+$, then we plainly call μ a parabolic Carleson measure.

Definition 2.2 (Weak-DKP condition)**.** Let $A : \mathbb{R}^{1+n}_+ \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^{1+n}_+ \to \mathbb{R}^n$ be locally integrable functions. Let $C \in \{W, R\}$. Define for $z \in \mathbb{R}^{1+n}_+$ the quantities

$$\begin{aligned}
\alpha^C_A(z) &:= \left(\fint_{C(z)} |A(w) - \langle A \rangle_{C(z)}|^2 \, dw \right)^{1/2}, \\
\alpha^C_B(z) &:= \left(\fint_{C(z)} |B(w)|^2 w_{n+1}^2 \, dw \right)^{1/2}, \\
\alpha^C_{A,B}(z) &:= \alpha^C_A(z) + \alpha^C_B(z),
\end{aligned}$$

and the associated measures

$$\nu^C_{A,B} := \left(\alpha^C_{A,B}(z)\right)^2 \frac{dz}{z_{n+1}}, \quad \nu^C_A := \left(\alpha^C_A(z)\right)^2 \frac{dz}{z_{n+1}} \quad \text{and} \quad \nu^C_B := \left(\alpha^C_B(z)\right)^2 \frac{dz}{z_{n+1}}.$$

We say that (A, B) satisfies a weak DKP-condition in a set $E \subset \mathbb{R}^{1+n}_+$ if

$$\|\nu^W_{A,B}\|_{C(E)} < \infty.$$

We say that (A, B) satisfies a weak DKP-condition if it satisfies a weak DKP condition with $E = \mathbb{R}^{1+n}_+$.

The following proposition shows that the Carleson condition above improves from Whitney to Carleson regions. Hence, the two conditions are in fact equivalent and we will drop the index C from the notation.

Proposition 2.3. *Let M_0 and ε_0 be given. Then there exists a constant C such that the following holds. Let A be M_0-elliptic, B be an ε_0-small drift and let Q^n be a boundary cylinder. Then*

$$\|\alpha^R_{A,B}\|^2_{L^\infty(R(Q^n))} + \|\nu^R_{A,B}\|_{C(Q^n)} \le C \|\nu^W_{A,B}\|_{C(9Q^n)}.$$

Proof. The estimate for the α_A follows by the very same argument that proves [DLM22b, Lemma 3.16 & Remark 3.22] in the elliptic setting and is a general fact about Carleson measures. The estimate for α_B is even easier, since this quantity is an average of a function that is independent of the external variable z. Indeed, introducing additional averages over Whitney regions $\widetilde{W}(t, x, \lambda) := Q^n(t, x, \lambda) \times$

$(\frac{\lambda}{2}, \lambda)$, we conclude by Fubini's theorem that

$$\begin{aligned} \fint_{R(2Q^n)} |B(w)|^2 w_{n+1}^2 \, dw &= \fint_{R(2Q^n)} |B(w)|^2 w_{n+1}^2 \left(\fint_{\widetilde{W}(y)} dz \right) dw \\ &\leq C \fint_{R(6Q^n)} \left(\fint_{W(z)} |B(w)|^2 w_{n+1}^2 \, dw \right) dz. \end{aligned} \tag{2.4}$$

The right-hand side is bounded from above by $C\|\nu_B^W\|_{C(6Q^n)}$ and the left-hand side is bounded from below by $C\alpha_B^R(z)^2$ for every $z \in R(Q^n)$. Thus, the L^∞-bound for α_B^R follows. By a similar use of Fubini's theorem we obtain

$$\begin{aligned} \int_{R(Q^n)} \left(\alpha_B^R(z)\right)^2 \frac{dz}{z_{n+1}} &\leq C \int_{R(3Q^n)} |B(w)|^2 w_{n+1}^2 \left(\int_{w_{n+1}/2}^{\infty} \frac{dz_{n+1}}{z_{n+1}^2} \right) dw \\ &= C \int_{R(3Q^n)} |B(w)|^2 w_{n+1} \, dw \end{aligned}$$

and repeating the argument in (2.4) for this integral, we find

$$\begin{aligned} &\leq C \int_{R(9Q^n)} \left(\fint_{W(z)} |B(w)|^2 w_{n+1} \, dw \right) dz \\ &\leq C \int_{R(9Q^n)} \left(\alpha_B^W(z)\right)^2 \frac{dz}{z_{n+1}}. \end{aligned}$$

This estimate is valid for every boundary cube, so $\|\nu_B^R\|_{C(Q^n)} \leq C\|\nu_B^W\|_{C(9Q^n)}$. □

Finally, we reserve the following notation for the L^2-norms of gradients, their linear approximations and their ratio.

Definition 2.5 (Energies J and E)**.** For measurable functions u on $\mathbb{R}_+^{1+n}$ with locally integrable gradient we define

$$\begin{aligned} J_u(t, x, \lambda) &:= \fint_{R(t,x,\lambda)} \left| \nabla(u - z_{n+1}\langle \partial_{n+1} u \rangle_{R(t,x,\lambda)}) \right|^2 dz, \\ E_u(t, x, \lambda) &:= \fint_{R(t,x,\lambda)} \left| \nabla u(z) \right|^2 dz, \\ \beta_u(t, x, \lambda) &:= \left(\frac{J_u(t, x, r)}{E_u(t, x, r)} \right)^{1/2}. \end{aligned}$$

2.6. **Basic estimates for weak solutions.** Next, we recall some basic estimates for solutions to parabolic equations. These estimates assume varying simplifications in the structure of the equation, and will be needed in respective generality.

The following Caccioppoli inequality is a special case of Lemma 1 on p. 623 in [Aro68]. Indeed, by the observation in Section 2.4, the singular drift can first be included in the source term F for [Aro68] and then be absorbed to the left-hand side if ε_0 is small enough.

Lemma 2.6 (Caccioppoli inequality)**.** *Let M_0 be given. Then there exists $\varepsilon_0, C > 0$ such that for any $c \in (1, 2]$ the following holds. Let $Q = Q(z, r) \subset \mathbb{R}_+^{1+n}$ be* *a parabolic cylinder, let $\mathcal{L}$ be (M_0, ε)-parabolic, $F \in L^2_{loc}(\overline{\mathbb{R}^{1+n}_+}; \mathbb{R}^n)$ and u be a solution to*

$$\mathcal{L}u = \operatorname{div} F \quad or \quad \mathcal{L}^* u = \operatorname{div} F \quad in\ c^2 Q \cap \mathbb{R}^{1+n}_+.$$

Then

$$\sup_\tau \fint_{Q^\tau} \frac{|u(w)|^2}{r^2}\, dw + \fint_Q |\nabla u(w)|^2\, dw \\ \leq \frac{C}{(c-1)^2} \fint_{cQ \cap \mathbb{R}^{1+n}_+} \frac{|u(w)|^2}{r^2}\, dw + C \fint_{cQ \cap \mathbb{R}^{1+n}_+} |F(w)|^2\, dw.$$

in either of the following two scenarios:

(1) Interior case & no drift: $cQ \subset \mathbb{R}^{1+n}_+$ and $\varepsilon = 0$.
(2) Boundary case & Dirichlet conditions: $z \in \partial\mathbb{R}^{1+n}_+$ and $u = 0$ on $cQ \cap \partial\mathbb{R}^{1+n}_+$.

As a consequence, we get:

Lemma 2.7 (Boundary Reverse Hölder inequality)**.** *Let M_0 be given. There exists $\varepsilon_0, C > 0$ such that for any $c \in (1, 2]$ the following holds. Whenever $\mathcal{L}$ is (M_0, ε_0)-parabolic, $R = R(t, x, \lambda) \subset \mathbb{R}^{1+n}_+$ is a Carleson region, $F \in L^2_{loc}(\overline{\mathbb{R}^{1+n}_+}; \mathbb{R}^n)$ and u is a solution to*

$$\mathcal{L}u = \operatorname{div} F \quad or \quad \mathcal{L}^* u = \operatorname{div} F$$

in $c^2 R$ such that $u = 0$ on $Q^n(cR)$, then for $q = \frac{2(n+2)}{n}$ it follows that

$$\left(\fint_R |u(z)|^q\, dz \right)^{1/q} \leq \frac{C}{(c-1)} \left(\fint_{cR} |u(z)|^2\, dz \right)^{1/2} + C\lambda \left(\fint_{cR} |F(z)|^2\, dz \right)^{1/2}.$$

Proof. Pick any $p \in (q, 2n/(n-2))$ and fix $\theta \in (0,1)$ with $1/q = (1-\theta)/2 + \theta/p$. Interpolating the slice-wise L^q-norm, we obtain

$$\left(\fint_R |u(z)|^q\, dz \right)^{1/q} \leq \left(\sup_\tau \fint_{R^\tau} |u(z)|^2\, dz \right)^{(1-\theta)/2} \\ \times \left(\frac{1}{\lambda^2} \int_0^{\lambda^2} \left(\fint_{R^\tau} |u(z)|^p\, dz \right)^{\theta q/p} dt \right)^{1/q}. \tag{2.8}$$

A slice-wise Sobolev–Poincaré inequality yields,

$$\int_0^{\lambda^2} \left(\fint_{R^\tau} |u(z)|^p\, dz \right)^{\theta q/p} dt \leq C \int_0^{\lambda^2} \left(\fint_{R^\tau} |\lambda \nabla u(z)|^2\, dz \right)^{\theta q/2} dt \\ \leq C \lambda^{2+\theta q/2} \left(\fint_R |\nabla u(z)|^2\, dz \right)^{\theta q/2},$$

where in the second step we have used $\theta q/2 = (2/n) \cdot (1 - 2/p)^{-1} \geq 1$ and Jensen's inequality. Now, both factors on the left of (2.8) can be controlled by the Caccioppoli inequality (Lemma 2.6) and the claim follows. □

The following boundary estimates on derivatives to drift free constant coefficient equations are folklore but difficult to trace down in the literature. We include a proof for convenience.

Lemma 2.9. *Let M_0 and $c > 1$ be given. For any $k \in \mathbb{N}_0, \gamma \in \mathbb{N}_0^n$ there exists a constant C such that the following holds. If $\mathcal{L}_0$ is a constant coefficient $(M_0, 0)$-parabolic operator, $R = R(t, x, \lambda) \subset \mathbb{R}^{1+n}_+$ is a Carleson region and u is a global solution to either $\mathcal{L}_0 u = 0$ or $\mathcal{L}_0^* u = 0$ in cR with $u = 0$ on $Q^n(cR)$, then*

$$\sup_{z \in R} |\partial_t^k \partial^\gamma u(z)| \leq \frac{C}{\lambda^{2k+|\gamma|-1}} \left(\fint_{cR} |\nabla u(z)|^2 \, dz \right)^{1/2}.$$

Moreover, u vanishes continuously on $Q^n(cR)$.

Proof. We can assume $(t, x, \lambda) = (0, 0, 1)$ since the general case can be obtained by re-scaling. As the coefficients of $\mathcal{L}_0$ are constant, it follows from the method of difference quotients and the Caccioppoli inequality on interior cubes that all derivatives $\partial_t^k \partial^\gamma u$ are locally square-integrable functions that solve the same equation.

In order to prove the boundary estimates, we first claim that

$$\|\nabla \partial_{x_j} u\|_{L^2(R)} + \|\partial_t u\|_{L^2(R)} + \|\nabla \partial_t u\|_{L^2(R)} \leq C \|\nabla u\|_{L^2(cR)} \tag{2.10}$$

for all $j = 1, \ldots, n-1$. The estimate for $\nabla \partial_{x_j} u$ follows by applying the boundary Caccioppoli inequality to (difference quotients approximating) $\partial_{x_j} u$. In order to estimate $\partial_t u$, we let χ be a smooth function with $\chi = 1$ on R, $\chi = 0$ on $\mathbb{R}^{1+n}_+ \setminus cR$ and $\|\partial_t \chi\|_\infty + \|\nabla \chi\|_\infty \leq C$ for some constant $C = C(n, c)$. The localized solution $\tilde{u} := u\chi$ vanishes on the entire parabolic boundary of cR and satisfies $\mathcal{L}_0 \tilde{u} = f + \operatorname{div} F$ with

$$\begin{aligned} f &:= u\chi + (\partial_t \chi) u - A\nabla u \cdot \nabla \chi, \\ F &:= -A(u \nabla \chi). \end{aligned} \tag{2.11}$$

Since A is constant, we have

$$\|f + \operatorname{div} F\|_{L^2(cR)} \leq C \big(\|u\|_{L^2(cR)} + \|\nabla u\|_{L^2(cR)} \big) \leq C \|\nabla u\|_{L^2(cR)},$$

where we have used the Poincaré inequality in the second step. Classical higher regularity for second-order parabolic equations as in [Eva98, Chapter 7, Theorem 5] yields $\|\partial_t \tilde{u}\|_{L^2(R)} \leq C \|\nabla u\|_{L^2(cR)}$ and hence the bound for $\partial_t u$ in (2.10). The bound for $\nabla \partial_t u$ then follows by the Caccioppoli inequality applied to $\partial_t u$.

Now, let $k \in \mathbb{N}_0$ and $\gamma \in \mathbb{N}_0^n$. By the equation, we may write $\partial_t^k \partial^\gamma u$ as a linear combination of terms $\partial_t^{k'} \partial^{\gamma'} u$ such that the order of differentiation is one to the vertical λ-direction. Since (2.10) applies to the x- and t-derivatives of u, we find $\|\partial_t^k \partial^\gamma u\|_{L^2(R)} \leq C \|\nabla u\|_{L^2(cR)}$. Since this bound holds for all k, γ, the analogous bound with L^∞-norm on the left follows by Sobolev embeddings and so does the claim that u vanishes continuously on $Q^n(cR)$. □

Finally, we need a reverse Hölder estimate for the gradient of adjoint solutions near the boundary. A Gehring-type argument is not applicable here, as we do not anticipate such an estimate in the interior. Instead, we utilize an analytic perturbation argument similar to [ABES19b] in the Banach spaces

$$\begin{aligned} E_p &:= L^p(\mathbb{R}; W_0^{1,p}(\mathbb{R}^n_+)) \cap H^{1/2,p}(\mathbb{R}; L^p(\mathbb{R}^n_+)), \\ \|\cdot\|_{E_p}^p &:= \|\cdot\|_p^p + \|\nabla \cdot\|_p^p + \|D_t^{1/2} \cdot\|_p^p, \end{aligned} \tag{2.12}$$

where $p \in (1, \infty)$, $t \in \mathbb{R}$ is the distinguished variable and the half-order time derivative $D_t^{1/2}$ is defined via the Fourier multiplier $\tau \mapsto |\tau|^{1/2}$ in the t-variable.

Further background on the role of these spaces in parabolic PDEs can be found in [ABES19b, Sect. 7]. For completeness, we provide a proof of the following complex interpolation result at the end of the section.

Lemma 2.13. *The spaces E_p in (2.12) interpolate by the complex method according to the rule*

$$[E_{p_0}, E_{p_1}]_\theta = E_p, \qquad \theta \in (0,1), \quad (1-\theta)/p_0 + \theta/p_1 = 1/p$$

and the same result holds for the (anti-)duals E_p^*.

Lemma 2.14 (Boundary Reverse Hölder inequality for the gradient)**.** *Let M_0 and $c > 1$ be given. Then there exist $\varepsilon_0, C > 0$ and $p > 2$ such that the following holds. Let $R = R(t, x, \lambda) \subset \mathbb{R}^{1+n}_+$ be a Carleson box and let $\mathcal{L}$ be an (M_0, ε_0)-parabolic operator. Whenever u is a solution to $\mathcal{L}^* u = 0$ on cR with $u = 0$ on $Q^n(cR)$, then*

$$\left(\fint_R |\nabla u(z)|^p \, dz \right)^{1/p} \leq C \left(\fint_{cR} |\nabla u(z)|^2 \, dz \right)^{1/2}.$$

Proof. By scaling, we may assume $\lambda = 1$. Let χ be a smooth function with $\chi = 1$ on R, $\chi = 0$ on $\mathbb{R}^{1+n}_+ \setminus c^{1/2} R$ and $\|\partial_t \chi\|_\infty + \|\nabla \chi\|_\infty \leq C$ for some dimensional constant C, and set

$$L_1 u := \partial_t u + \operatorname{div} A \nabla u, \quad L_2 u := \operatorname{div}(Bu), \quad Lu := u + L_1 u + L_2 u.$$

Then $\tilde{u} := u\chi \in L^2(\mathbb{R}; W^{1,2}_0(\mathbb{R}^n_+))$ satisfies a global equation

$$L\tilde{u} = f + \operatorname{div} F$$

on the upper-half-space with right-hand side

$$\begin{aligned} f &:= u\chi + (\partial_t \chi) u - A \nabla u \cdot \nabla \chi - Bu \cdot \nabla \chi, \\ F &:= -A(u \nabla \chi). \end{aligned}$$

By the observation in Section 2.4 we have $Bu \in L^2_{loc}(\overline{\mathbb{R}^{1+n}_+})$ and therefore the equation can be rewritten as $\partial_t \tilde{u} = \operatorname{div}(G) + h$ with right-hand sides $G, h \in L^2(\mathbb{R}^{n+1}_+)$. This means that $\partial_t \tilde{u} \in L^2(\mathbb{R}; W^{1,2}_0(\mathbb{R}^n_+)^*)$ and, using Plancherel's theorem for the Fourier transform in the t-variable, we conclude $u \in E_2$.

Step 1: Variational Estimates. We are going to use the full scale of spaces E_p and define the operator $L : E_p \to E^*_{p'}$ variationally by

$$\langle Lu, v \rangle = \int_{\mathbb{R}^{1+n}_+} u \cdot \overline{v} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v} + A \nabla u \cdot \overline{\nabla v} - Bu \cdot \overline{\nabla v} \, dz,$$

where H_t is the Hilbert transform in the t-variable. In order to see that L is well-defined, we use Hölder's inequality to bound

$$\begin{aligned} |\langle L_1 v, w \rangle| &\leq \int_{\mathbb{R}^{1+n}_+} |H_t D_t^{1/2} v| |D_t^{1/2} w| + M_0 |\nabla v| |\nabla w| \, dz \\ &\leq \|H_t D_t^{1/2} v\|_p \|D_t^{1/2} w\|_{p'} + M_0 \|\nabla v\|_p \|\nabla w\|_{p'}, \end{aligned}$$

and also invoke the one-dimensional Hardy inequality to get

$$|\langle L_2 v, w \rangle| \leq \int_{\mathbb{R}^{1+n}_+} \varepsilon_0 \frac{|v|}{z_{n+1}} |\nabla w| \, dz \leq \frac{\varepsilon_0 p}{p-1} \|\partial_{n+1} v\|_p \|\nabla w\|_{p'}. \tag{2.15}$$

Altogether, $L : E_p \to E^*_{p'}$ with norm depending on M_0, ε_0, p.

If $p = 2$, then we have the 'hidden coercivity bound'

$$\mathrm{Re}\langle (1 + L_1)v, (1 + \delta H_t)v\rangle \geq (M_0^{-1} - \delta M_0)\|\nabla v\|_2^2 + \delta \|D_t^{1/2} v\|_2^2 + \|v\|_2^2,$$

compare [ABES19b, Lemma 2.3]. We take $\delta := M_0^{-1}(M_0 + 1)^{-1} < 1$ to have equal constants for the first terms on the right-hand side and then require $\varepsilon_0 < \delta/4$ in order to obtain from (2.15) a similar estimate for the full operator,

$$\mathrm{Re}\langle Lv, (1 + \delta H_t)v\rangle \geq \frac{\delta}{2}\|v\|_{E_2}^2.$$

This implies that $L : E_2 \to E_2^*$ is invertible, see again [ABES19b, Lemma 2.3].

Since the spaces E_p and their duals interpolate by the complex method, we can apply Sneiberg's lemma ([ABES19a, Theorem A1], [Shn74]) to conclude that L remains invertible as an operator $E_p \to E^*_{p'}$ and that the inverses agree on common subspaces, whenever $2 \leq p < p_0$, where p_0 and the norm of the inverse depend only on the dimension and M_0.

Step 2:Conclusion. If necessary, we lower p_0 to achieve $p_0 < \min(2(n+2)/n, n)$ and lower ε_0 to have Lemma 2.7 at our disposal.

Let $2 < p < p_0$. By Hölder's inequality, $\|\operatorname{div} F\|_{E^*_{p'}} \leq \|F\|_p$ and the Sobolev embedding for $W_0^{1,p}(\mathbb{R}^n_+)$, also $\|f\|_{E^*_{p'}} \leq \|f\|_{p_*}$ with exponent $1/p_* = 1/p - 1/n$. Thus,

$$\|\operatorname{div} F\|_{E^*_{p'}} \leq C\|u\|_{L^p(c^{1/2}R)} \leq C\|u\|_{L^2(cR)} \leq C\|\nabla u\|_{L^2(cR)},$$

where we use Lemma 2.7 in the third step and the Poincaré inequality with zero boundary values on one side of a cube in the fourth step. Likewise, in estimating

$$\|f\|_{E^*_{p'}} \leq C\big(\|u\|_{L^{p_*}(c^{1/2}R)} + C\|\nabla u\|_{L^{p_*}(c^{1/2}R)}\big) \leq C\|\nabla u\|_{L^{p_*}(c^{1/2}R)} \leq C\|\nabla u\|_{L^2(c^{1/2}R)},$$

we use the one-dimensional Hardy inequality, the same Poincaré inequality and finally that $p_* \leq 2$. Altogether, $u \in E_2$ satisfies $L\tilde{u} = \operatorname{div} F + f \in E^*_{p'}$. Invertibility shown in Step 1 yields the claim

$$\|\nabla u\|_{L^p(R)} \leq \|\tilde{u}\|_{E_p} \leq \|L\tilde{u}\|_{E^*_{p'}} \leq C\|\nabla u\|_{L^2(cR)}. \qquad \square$$

Proof of Lemma 2.13. Let us call

$$E_p^{full} := L^p(\mathbb{R}; W^{1,p}(\mathbb{R}^n)) \cap H^{1/2,p}(\mathbb{R}; L^p(\mathbb{R}^n))$$

the corresponding spaces on the full space $\mathbb{R}^{1+n}$. By [AEN20, Lemma 6.1] they are isomorphic to $L^p(\mathbb{R}; L^p(\mathbb{R}^n))$ and follow the interpolation rules in question.

Extending by zero to the lower half-space, the space E_p is hence isometrically identified as a closed subspace of E_p^{full} that we also call E_p. Denoting by π_1 the restriction $L^p(\mathbb{R}; W^{1,p}(\mathbb{R}^{n+1})) \to L^p(\mathbb{R}; W^{1,p}(\mathbb{R}^{n+1}_-))$ and by $\pi_2 : L^p(\mathbb{R}; W^{1,p}(\mathbb{R}^{n+1}_-)) \to L^p(\mathbb{R}; W^{1,p}(\mathbb{R}^{n+1}))$ the extension by even reflection, we obtain a bounded projection $\pi := \pi_2 \circ \pi_1$ of E_p^{full} with kernel E_p. The interpolation principle for complemented subspaces [Tri78, Sect. 1.17.1] tells us that E_p and E_p^{full} interpolate according to the same rules.

The construction above reveals that E_p-spaces are isomorphic to closed subspaces of reflexive spaces, hence reflexive. Moreover, $C_c^\infty(\mathbb{R}^{1+n}_+)$ is dense in all E_p

by a standard smoothing procedure. Consequently, the interpolation rules for E_p^* follow by a duality principle for the complex method [BL76, Cor. 4.5.2]. □

2.7. **Pointwise estimates for weak solutions.** Local boundedness and (Hölder-) continuity of solutions to general (M_0, ε_0)-parabolic equations and their adjoints is known since the work of Aronson [Aro68]. We need the following pointwise estimates.

Lemma 2.16 (Parabolic Harnack's inequality [Mos64, Theorem 2], [Mos71])**.** *Let M_0 be given. There is a constant C such that the following holds. Let Q be a parabolic cylinder with $2Q \subset \mathbb{R}^{1+n}_+$ and let $\mathcal{L}$ be $(M_0, 0)$-parabolic. Then*

$$u(t, x, \lambda) \leq u(s, y, \mu) \exp\left(\frac{C|(y, \mu) - (x, \lambda)|^2}{|t - s|} + 1\right)$$

for all $(t, x, \lambda), (s, y, \mu) \in Q$ in either of the following scenarios:

(1) $s > t$ and $u \geq 0$ solves $\mathcal{L}u = 0$ in $2Q$.
(2) $t > s$ and $u \geq 0$ solves $\mathcal{L}^* u = 0$ in $2Q$.

Lemma 2.17 (Boundary Harnack's inequality [FS97] [FGS86, Theorem 1.6])**.** *Let M_0 be given. There is a constant C such that whenever $R = R(t, x, \lambda)$ is a parabolic Carleson region, $\mathcal{L}$ is $(M_0, 0)$-parabolic and $u, v \geq 0$ are weak solutions to $\mathcal{L}u = 0$ on $2R$ that vanish continuously on the boundary cylinder $Q^n(2R)$, then*

$$\frac{u(z)}{v(z)} \leq C \frac{u(a^+(t, x, \lambda))}{v(a^-(t, x, \lambda))} \qquad (z \in R).$$

Lemma 2.18 (Carleson Estimate [Sal81, Theorem 3.1])**.** *Let M_0 be given. There is a constant C such that whenever $R = R(t, x, \lambda)$ is a parabolic Carleson region, $\mathcal{L}$ is $(M_0, 0)$-parabolic and $u \geq 0$ is a weak solution to $\mathcal{L}u = 0$ on $2R$ that vanishes continuously on the boundary cylinder $Q^n(2R)$, then*

$$u(z) \leq C u(a^+(R)) \qquad (z \in \tfrac{1}{2}R).$$

By a simple reversal of time, we see that in Lemmas 2.16 and 2.17 for solutions to the adjoint equation $\mathcal{L}^* u = 0$, the roles of forward and backward time-lag are interchanged. Combining [Lie96, Theorem 6.32], the Carleson Estimate and the Parabolic Harnack's inequality also yields the following useful bound.

Lemma 2.19 (Boundary Hölder Continuity)**.** *Let M_0 be given. There are constants C and $\alpha \in (0, 1)$ such that whenever $R = R(t, x, \lambda)$ is a parabolic Carleson region, $\mathcal{L}$ is $(M_0, 0)$-parabolic and $u \geq 0$ is a weak solution to $\mathcal{L}u = 0$ or $\mathcal{L}^* u = 0$ on $2R$ that vanishes continuously on the boundary cylinder $Q^n(2R)$, then*

$$u(z) \leq C \left(\frac{z_{n+1}}{\lambda}\right)^{\alpha} \left(\fint_{2R} |u(w)|^2 \, dw\right)^{1/2} \qquad (z \in \tfrac{1}{2}R).$$

2.8. **Green functions for equations with drift.** In this section, we provide the definition and estimates for the Green function and the parabolic measure of an (M_0, ε_0)-parabolic operator $\mathcal{L}$.

The continuous Dirichlet problem for $\mathcal{L}$ consists in finding for given $f \in C_c(\mathbb{R}^n)$ a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+$ that is continuous up to the boundary and attains the boundary value f. If this problem is uniquely solvable for all f, then the parabolic measure ω^z at $z \in \mathbb{R}^{1+n}_+$ is defined as the measure for which

$$u(z) = \int_{\partial \mathbb{R}^{1+n}_+} f(w)\, d\omega^z(w),$$

whenever u and f are related as above. By a Green's function $G : \mathbb{R}^{1+n}_+ \times \mathbb{R}^{1+n}_+ \to [0, \infty)$ we mean a function satisfying the following properties.

(1) $G(\cdot, z)$ with $z \in \mathbb{R}^{1+n}_+$ fixed is a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{1+n}_+ \setminus \{z\}$.
(2) $G(z, \cdot)$ with $z \in \mathbb{R}^{1+n}_+$ fixed is a weak solution to $\mathcal{L}^* u = 0$ on $\mathbb{R}^{1+n}_+ \setminus \{z\}$.
(3) If $z, w \in \mathbb{R}^{1+n}_+$ and $w_1 > z_1$, then $G(z, w) = 0$.
(4) If $z \in \mathbb{R}^{1+n}_+$, then both $G(\cdot, z)$ and $G(z, \cdot)$ extend continuously to $\overline{\mathbb{R}^{1+n}_+}$ and vanish at $\partial \mathbb{R}^{1+n}_+$.
(5) If $\Psi \in C^\infty_c(\mathbb{R}^{1+n})$, then

$$\Psi(z) = \langle \mathcal{L}^* G(z, \cdot), \Psi \rangle + \int_{\partial \mathbb{R}^{1+n}_+} \Psi(w)\, d\omega^z(w) \qquad (z \in \mathbb{R}^{1+n}_+). \tag{2.20}$$

The identity (2.20) is called the Riesz formula (for $\mathcal{L}$).

For the equations we consider here, these objects exist. For readers that are perhaps worried about smoothness assumptions (on the drift) in [HL01], we remark that [HL01] constructs these objects in two settings: drift-free equations and a perturbative regime, where the 'unperturbed' operator has parabolic measure in (weak-)A_∞ [HL01, Chapter III]. Therefore one can run our arguments here first without the drift and prove that the associated parabolic measure is in A_∞. Then the perturbative regime covers our ε_0-small drifts with a Carleson condition. For instance, in Chapter III of [HL01], Theorem 1.7 implies (1.5)(b) for the operator $\mathcal{L}$ (with drift) and therefore, by Lemma 2.2 and Lemma 2.6, the Green function and parabolic measure for $\mathcal{L}$ exist. We state this observation explicitly as

Proposition 2.21. *Given M_0, there exists $\varepsilon_0 > 0$ such that if $\mathcal{L}$ is an (M_0, ε_0)-parabolic operator, then the continuous Dirichlet problem for $\mathcal{L}$ is uniquely solvable and the parabolic measure and the parabolic Green function on $\mathbb{R}^{1+n}_+$ exist.*

We recall the following set of useful estimates on parabolic measures and Green functions.

Lemma 2.22 (Estimates for parabolic measure and Green function)**.** *Given M_0, there exists $\varepsilon_0 > 0$, $\kappa_0 \geq 40$ and $C \geq 1$ such that the following statement is valid. If $\kappa \geq \kappa_0$, $(t_0, x_0, \lambda_0) \in \mathbb{R}^{1+n}_+$ and $p := a^+(t_0, x_0, \kappa\lambda_0)$, then for an (M_0, ε_0)-parabolic operator the Green function G and the parabolic measure ω^p at p satisfy:*

(1) Strong Harnack inequality*:*

$$\sup_{w \in W(z)} G(p, w) \leq C \inf_{w \in W(z)} G(p, w) \qquad (z \in R(t_0, x_0, 4\lambda_0)).$$

(2) Local doubling for parabolic measure*:*

$$\omega^p(Q^n(t, x, 2\lambda)) \leq C \omega^p(Q^n(t, x, \lambda)) \qquad ((t, x, \lambda) \in R(t_0, x_0, 4\lambda_0)).$$

(3) CFMS-estimate*:*

$$\frac{\omega^p(Q(t, x, \lambda))}{\lambda^{n+1}} \approx_C \frac{G(p, (t, x, \lambda))}{\lambda} \qquad ((t, x, \lambda) \in R(t_0, x_0, 4\lambda_0)).$$

(4) (Backward) Carleson estimate*: If* $(t,x,\lambda) \in R(t_0,x_0,4\lambda_0)$*, then*

$$G(p,(s,y,\mu)) \leq CG(p,a^+(t,x,\lambda)) \qquad ((s,y,\mu) \in R(t,x,\tfrac{3}{4}\lambda)).$$

(5) Gradient estimate*:*

$$\frac{G(p,(t,x,\lambda))}{\lambda} \approx_C \left(\fint_{R(t,x,\lambda)} |\nabla_z G(p,z)|^2 \, dz \right)^{1/2} \qquad ((t,x,\lambda) \in R(t_0,x_0,4\lambda_0)),$$

where ∇_z *is the spatial gradient in the second set of variables for* G*.*

Proof. The first four items are straightforward reformulations of Lemmata 3.9 to 3.14 in [HL01]. The uniformity of the estimates with respect to κ and the possibility to take an explicit ambient box $R(t_0,x_0,4\lambda_0)$ also follows from [HL01], which states results for all (t_0,x_0,λ_0), some κ_0 and some ambient box $R(t_0,x_0,c\lambda_0)$: indeed, $a^+(t_0,x_0,\kappa\lambda_0) = a^+(t_0,x_0,\kappa_0\lambda_1)$ for $\lambda_1 := \lambda_0\kappa/\kappa_0$ and under this change this change of reference point, $Q(t_0,x_0,(c\kappa/\kappa_0)\lambda_0) = Q(t_0,x_0,c\lambda_1)$.

In one direction, we use the strong Harnack inequality to control the left-hand side in (5) by the Whitney average $(\fint_{W(t,x,\lambda)} |\frac{G(p,z)}{\lambda}|^2 \, dz)^{1/2}$, enlarge to a Carleson average and then apply the Poincaré inequality with vanishing boundary values at $z_{n+1} = 0$. Conversely, we use the boundary Caccioppoli inequality and the backward Carleson estimate to control the right-hand side in (5) by $\frac{G(p,a^+(t,x,2\lambda))}{\lambda}$ and then conclude by the strong Harnack inequality. □

3. Just estimates

Here, we prove our key estimates on the energies introduced in Definition 2.5. We split this part into four: First constant coefficient operators, then error estimates for differences of solutions and finally we combine the two points of view to conclude oscillation estimates for Green functions and parabolic measure.

3.1. Estimates for energies of constant coefficient operators.

We start with estimates for the energies from Definition 2.5 in case of constant coefficient equations.

The decay estimate is a standard estimate on regularity of solutions. An interior version can be found in many textbooks and the boundary version for elliptic equations is provided in [DLM22b, Lemma 3.4]. The parabolic version follows with no additional complications. The non-degeneracy estimate is very different: it relies on the strong Harnack inequality and we do not expect a full analogy of the elliptic theory. For our applications, however, it suffices to work with solutions that share many values with a parabolic Green function.

Lemma 3.1. *Let* M_0 *be given. There exists* $\varepsilon_0 > 0$*,* $\kappa_0 \geq 40$ *and* $C \geq 1$ *such that whenever* $\mathcal{L}_0$ *is a constant coefficient* $(M_0,0)$*-parabolic operator,* $\kappa \geq \kappa_0$ *and* $(t_0,x_0,\lambda_0),(t,x,\lambda) \in \mathbb{R}^{1+n}_+$*, then the following holds.*

(1) Decay of J*: Whenever* u_0 *solves* $\mathcal{L}_0^* u = 0$ *in* $R(t,x,\lambda)$ *and satisfies* $u_0 = 0$ *on* $Q^n(t,x,\lambda)$*, then*

$$J_{u_0}(t,x,\varrho) \leq C \left(\frac{\varrho}{\lambda}\right)^2 J_{u_0}(t,x,\lambda) \qquad (0 < \varrho \leq \lambda).$$

(2) Non-degeneracy of E: *Let G be the Green function for an (M_0, ε_0)-parabolic operator and set $u := G(a^+(t_0, x_0, \kappa\lambda_0), \cdot)$. If $(t, x, \lambda) \in R(t_0, x_0, \lambda_0)$ and u_0 solves $\mathcal{L}_0^* u_0 = 0$ in $R(t, x, 2\lambda)$ with $u_0 = u$ on $\partial_{par^*} R(t, x, 2\lambda)$ (continuously on $Q^n(t, x, 2\lambda)$), then*

$$E_{u_0}(t, x, \lambda) \leq C E_{u_0}(t, x, \varrho) \qquad (0 < \varrho \leq \lambda).$$

Proof. We start with the first item. The estimate follows by the triangle inequality if $\varrho' > \lambda/2$, so we may assume $\varrho \leq \lambda/2$ from now on. In this case we bound

$$J_{u_0}(t, x, \varrho) \leq \sup_{(s,y,\mu) \in R(t,x,\varrho)} |\nabla_y u_0(s, y, \mu)|^2 + \fint_{R(t,x,\varrho)} |\partial_\mu u_0 - \langle \partial_\mu u_0 \rangle_{R(t,x,\varrho)}|^2 \, dz := \mathrm{I} + \mathrm{II}\,.$$

As u_0 and its lateral derivatives vanish on $Q^n(t, x, \lambda)$, the mean-value theorem yields

$$\begin{aligned} \mathrm{I} &\leq \sup_{(s,y,\mu) \in R(t,x,\varrho)} |\mu \partial_\mu \nabla_y u_0(s, y, \mu)|^2 \leq C \left(\frac{\varrho}{\lambda}\right)^2 \fint_{R(t,x,\lambda/2)} |\lambda \nabla(\nabla_y u_0)|^2 \, dz \\ &\leq C \left(\frac{\varrho}{\lambda}\right)^2 J_{u_0}(t, x, \lambda), \end{aligned}$$

where in the second and third step we have used Lemma 2.9 and the boundary Caccioppoli inequality for the y-derivatives of u_0. Next, we introduce $v := u_0 - \mu \langle \partial_\mu u_0 \rangle_{R(t,x,\varrho)}$ and note that this function has the same properties as u_0 since $\mathcal{L}_0$ has constant coefficients. Poincaré's inequality yields

$$\begin{aligned} \mathrm{II} \leq C \fint_{R(t,x,\varrho)} |\varrho \nabla \partial_\mu u_0|^2 \, dz &= C \fint_{R(t,x,\varrho)} |\varrho \nabla \partial_\mu v|^2 \, dz \\ &\leq C \sup_{(s,y,\mu) \in R(t,x,\lambda/2)} |\varrho \nabla (\partial_\mu v)|^2 \\ &\leq C \left(\frac{\varrho}{\lambda}\right)^2 \fint_{R(t,x,\lambda)} |\nabla v|^2 \, dz = C \left(\frac{\varrho}{\lambda}\right)^2 J_{u_0}(t, x, \lambda), \end{aligned}$$

where we have used Lemma 2.9 in the penultimate step. This completes the proof.

We turn to the second item. By the boundary Caccioppoli inequality (Lemma 2.6) and the maximum principle (e.g. [Lie96, Corollary 6.26]),

$$E_{u_0}(t, x, \lambda) \leq \frac{C}{\lambda^2} \fint_{R(t,x,2\lambda)} |u_0|^2 \, dz \leq \frac{C}{\lambda^2} \sup_{z \in \partial_{par^*} R(t,x,2\lambda)} |u_0(z)|^2.$$

By assumption, u_0 and u coincide on the adjoint parabolic boundary of $R(t, x, 2\lambda)$. Thus, the Carleson estimate and the strong Harnack inequality from Lemma 2.22 yield

$$\sup_{z \in \partial_{par^*} R(t,x,2\lambda)} |u_0(z)| \leq C u(a^-(t, x, 4\lambda)) \leq \inf_{z \in \Gamma} |u_0(z)|$$

where $\Gamma := \{(s, y, 4\lambda) \in \partial_{par^*} R(t, x, 2\lambda) : s \geq t + 2\lambda^2\}$ denotes the 'forward top part' of the parabolic boundary. Altogether,

$$E_{u_0}(t, x, \lambda) \leq C \frac{\inf_{z \in \Gamma} |u_0(z)|^2}{\lambda^2}. \tag{3.2}$$

Next, we will replace the infimum in a quantitative way by the value of u_0 at an interior point of $R(t, x, 2\lambda)$. To this end, let $\Delta \subset \Gamma$ be the parabolic boundary cylinder of radius λ around $(t+3\lambda^2, x, 4\lambda) \in \Gamma$ and let $\varphi \in C_c^\infty(\mathbb{R}^{1+n}_+)$ be such that $1_{\frac{1}{2}\Delta} \le \varphi \le 1_\Delta$ on Γ. Let h be the weak solution to

$$\begin{aligned} \mathcal{L}_0^* h &= 0 \quad \text{in } R(t,x,2\lambda), \\ h &= \varphi \quad \text{on } \partial_{par^*} R(t,x,2\lambda) \end{aligned}$$

that exists by constant-coefficient theory (e.g. [Eva98, Chapter 7, Theorem 3]). By the maximum principle, $0 \le h \le 1$. Thus, the boundary Hölder estimate (Lemma 2.19) applied to $1-h$ yields $h(t+3\lambda^2, x, (4-\varepsilon)\lambda) \ge 1/2$ for some $\varepsilon \in (0,2)$ that depends only on M_0 and dimension. Finally, the Harnack inequality implies $h(a^+(t,x,\lambda)) \ge C > 0$. On the other hand, $(\inf_{z\in\Gamma} |u_0(z)|)h \le u$ on $\partial_{par^*} R(t,x,2\lambda)$ and by the maximum principle the inequality persists at $a^+(t,x,\lambda)$ in the interior. Going back to (3.2), we obtain

$$E_{u_0}(t,x,\lambda) \le C \frac{u_0(a^+(t,x,\lambda))^2}{\lambda^2 h(a^+(t,x,\lambda))^2} \le C \frac{u_0(a^+(t,x,\lambda))^2}{\lambda^2}.$$

We conclude by an upper bound for $u_0(a^+(t,x,\lambda))/\lambda$: Applying the boundary Harnack inequality (Lemma 2.17) to the solutions u_0 and λ of $\mathcal{L}_0^*$, we find

$$\frac{\varrho/2}{u_0(t,x,\varrho/2)} \le C \frac{\lambda}{u_0(a^+(t,x,\lambda))}$$

and by the mean value theorem and Lemma 2.9,

$$\frac{u_0(a^+(t,x,\lambda))^2}{\lambda^2} \le C \Big(\sup_{\mu \in (0,\varrho/2)} \partial_\mu u_0(t,x,\mu) \Big)^{1/2} \le C E_{u_0}(t,x,\varrho). \qquad \square$$

3.2. **Estimates for differences of adjoint solutions.** In order to extend Lemma 3.1 to certain solutions to (adjoint) equations with variable coefficients, we are going to control the error terms that come from the difference of two solutions. Throughout this subsection, $R = R(t,x,\lambda) \subset \mathbb{R}^{1+n}_+$ is a Carleson region, A is an M_0-elliptic, B is an ε_0-small drift, A_0 is an M_0-elliptic constant matrix and we work with global weak solutions to the equations

$$\begin{aligned} -\partial_t u_0 - \operatorname{div} A_0^* \nabla u_0 &= 0 \quad \text{in } R, \\ u_0 &= 0 \quad \text{on } Q^n(R) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} -\partial_t u - \operatorname{div} A^* \nabla u + \operatorname{div}(Bu) &= 0 \quad \text{in } R, \\ u - u_0 &= 0 \quad \text{on } \partial_{par^*} R, \end{aligned} \tag{3.4}$$

where the boundary values are understood in the Sobolev sense and both solutions vanish continuously on $Q^n(R)$. We start with the following simple and classical perturbation estimate, see for instance [Dah86a] and [DLM22b] for related estimates in the elliptic setting.

Lemma 3.5. *Let M_0 be given. There exist $\varepsilon_0, C > 0$, depending only on M_0 and dimension, such that in the setting above*

$$\begin{aligned}&\int_R |\nabla(u-u_0)|^2 \, dz \\ &\leq C \min\left(\int_R |A-A_0|^2|\nabla u|^2 + |B|^2|u|^2 \, dz, \int_R |A-A_0|^2|\nabla u_0|^2 + |B|^2|u_0|^2 \, dz\right)\end{aligned} \tag{3.6}$$

and

$$C^{-1}\int_R |\nabla u_0|^2 \, dz \leq \int_R |\nabla u|^2 \, dz \leq C\int_R |\nabla u_0|^2 \, dz. \tag{3.7}$$

Proof. The estimate (3.7) follows from (3.6) by the triangle inequality, boundedness of $A - A_0$ and the Hardy estimate for the drift term discussed in Section 2.4.

As for (3.6), we formally use $w := u_0 - u$ as a test function. To make this precise, we write $R = I \times \Sigma$, where $I = (a, b)$ is a time interval and Σ a spatial cube, and set

$$w_h(t, x, \lambda) := \frac{1}{2h}\int_{t-h}^{t+h} 1_{(a+2h,b-2h)}(s)\, w(s, x, \lambda)\, ds.$$

In the limit as $h \to 0$, we have $w_h \to w$ in $L^2((a, b); W_0^{1,2}(\Sigma))$ by the maximal function theorem. Moreover, w_h vanishes for $t = a$ and $t = b$. Since in addition $w_h \in W^{1,2}((a, b); L^2(\Sigma))$, this function is a valid test function for (3.3), (3.4). (This approximation argument relies again on $Bu, Bu_0 \in L^2(R)$.)

Now, by ellipticity of A,

$$\begin{aligned}M_0^{-1}&\int_R |\nabla w|^2 \, dz \\ &\leq \int_R A^*\nabla w \cdot \nabla w \, dz \\ &= \lim_{h\to 0}\int_R A^*\nabla(u_0 - u)\cdot \nabla w_h \, dz \\ &= \lim_{h\to 0}\int_R A_0^*\nabla u_0 \cdot \nabla w_h + (A^* - A_0^*)\nabla u_0 \cdot \nabla w_h - A^*\nabla u \cdot \nabla w_h \, dz \\ &= -\int_R Bu\nabla w \, dz + \int_R (A^* - A_0^*)\nabla u_0 \cdot \nabla w - \lim_{h\to 0}\int_R w\partial_t w_h \, dz \\ &=: -\mathrm{I} + \mathrm{II} - \lim_{h\to 0}\mathrm{III}_h,\end{aligned} \tag{3.8}$$

where the penultimate step follows by (3.3), (3.4) and the convergence properties of w_h. We are going to estimate all terms in a way that produces the term with u_0 on the right-hand side of (3.6). By Cauchy's inequality and Hardy's inequality,

$$\begin{aligned}|\,\mathrm{I}\,| &\leq \frac{1}{2M_0}\int_R |\nabla w|^2 \, dz + \frac{M_0}{2}\int_R |Bu|^2 \, dz \\ &\leq \left(\frac{1}{2M_0} + 4\varepsilon_0 M_0\right)\int_R |\nabla w|^2 \, dz + M_0\int_R |Bu_0|^2 \, dz\end{aligned}$$

and the first term can be absorbed to the left in (3.8) for $\varepsilon_0 := (16M_0^2)^{-1}$. The bound for II follows directly by Cauchy's inequality. To estimate III_h, we write

$w_{\pm h}(t,x,\lambda) := w(t \pm h, x, \lambda)$ and note that by definition and a change of variables,

$$\begin{aligned}\mathrm{III}_h &= \frac{1}{2h}\bigg(\int_{a+3h}^{b-3h}\iint_\Sigma w(w_h - w_{-h})\,dx\,d\lambda\,dt - \int_{b-3h}^{b-h}\iint_\Sigma ww_{-h}\,dx\,d\lambda\,dt \\ &\quad + \int_{a+h}^{a+3h}\iint_\Sigma ww_h\,dx\,d\lambda\,dt\bigg) \\ &= \frac{1}{2h}\bigg(\int_{a+h}^{a+2h}\iint_\Sigma ww_h\,dx\,d\lambda\,dt - \int_{b-3h}^{b-2h}\iint_\Sigma ww_h\,dx\,d\lambda\,dt\bigg).\end{aligned}$$

Since $w \in C([a,b];L^2(Q))$ vanishes for $t=b$, we conclude that

$$\lim_{h\to 0}\mathrm{III}_h = \frac{1}{2}\iint_\Sigma |w(a,x,\lambda)|^2 - |w(b,x,\lambda)|^2\,dx\,d\lambda \geq 0.$$

Going back to (3.8), this completes the proof of (3.6) with u_0 on the right.

In order to end up with u, we repeat the argument but use ellipticity of A_0^* in (3.8). □

The next two lemmas provide estimates for integrals as on the right-hand side of (3.6) but on slightly smaller boxes. They come with additional decay from the weak-DKP condition.

The case of a solution to a constant coefficient equation is particularly simple. The main work horse here is Lemma 2.9.

Lemma 3.9. *Let M_0 and $\theta \in (0,1)$ be given and specialize to $A_0 := \langle A\rangle_R$. There exist $\varepsilon_0, C > 0$ depending only on M_0, θ, n such that*

$$\fint_{\theta R} |A-A_0|^2|\nabla u_0|^2 + |B|^2|u_0|^2\,dz \leq C\alpha_{A,B}(R)^2\fint_R |\nabla u|^2\,dz.$$

Proof. By Lemma 2.9 followed by (3.7),

$$\sup_{z\in\theta R}|\nabla u_0(z)|^2 \leq C\fint_R|\nabla u_0|^2\,dz \leq C\fint_R|\nabla u|^2\,dz$$

and thus

$$\fint_{\theta R}|A-A_0|^2|\nabla u_0|^2\,dz \leq C\bigg(\fint_R|A-A_0|^2\,dz\bigg)\bigg(\fint_R|\nabla u|^2\,dz\bigg),$$

which is half of the claim. By the mean value theorem, Lemma 2.9 and (3.7),

$$\bigg(\sup_{z\in\theta R}\frac{u_0(z)}{z_{n+1}}\bigg)^2 \leq \sup_{z\in\theta R}|\partial_{n+1}u_0(z)|^2 \leq C\fint_R|\nabla u_0|^2\,dz \leq C\fint_R|\nabla u|^2\,dz$$

and we obtain the second half of the claim in the same fashion:

$$\fint_{\theta R}|B|^2|u_0|^2\,dz \leq C\bigg(\fint_R|B|^2z_{n+1}^2\,dz\bigg)\fint_R|\nabla u|^2\,dz. \qquad \square$$

We also need estimates analogous to Lemma 3.9 but with u instead of u_0 on the left. Since the average on the left in Lemma 3.9 is bounded by that of $|\nabla u_0|^2$, controlling the difference of gradients is enough. The use of Lemma 2.9 is now replaced by a variant of the Calderón–Zygmund estimate and the boundary reverse

Hölder inequality for the gradient of solutions. When specialized to the elliptic setting, this argument gives a notable improvement over [DLM22b, Lemma 3.19].

Lemma 3.10. *Let M_0 and $\theta \in (0,1)$ be given, specialize to $A_0 := \langle A\rangle_R$, and assume that u_0, u solve (3.3), (3.4) also on $\theta^{-1}R$. Then there exist $\varepsilon_0, C > 0$ depending only on M_0, θ and dimension, such that*

$$\fint_{\theta^2 R} |\nabla(u-u_0)|^2\,dz \le C\alpha_{A,B}(R)^2 \fint_{\theta^{-1}R} |\nabla u|^2\,dz.$$

Proof. Let $w := u - u_0$. We shall first prove the bound

$$\left(\fint_{\theta^2 R} |\nabla w|^2\,dz\right)^{1/2} \le \frac{C}{\lambda}\fint_{\theta R} |w|\,dz + C\alpha_{A,B}(R)\left(\fint_R |\nabla u|^2\,dz\right)^{1/2} \tag{3.11}$$

To this end, we view w as a global solution to the equation with variable coefficients

$$\begin{aligned} -\partial_t w - \operatorname{div} A^*\nabla w &= \operatorname{div} F \quad &&\text{in } R,\\ w &= 0 &&\text{on } \partial_{par^*}R, \end{aligned} \tag{3.12}$$

where

$$F := F_1 + F_2 + F_3, \quad F_1 := -Bw, \quad F_2 := -Bu_0, \quad F_3 := (A^* - A_0^*)\nabla u_0.$$

If $Q = Q(z,r)$ is an interior parabolic cylinder with $4Q = Q(z,4r) \subset R$, then for $q := 2(n+2)/n$ and $p := 2$ the reverse Hölder inequality for local solutions (e.g. [ABES19b, Proposition 4.4]) yields

$$\left(\fint_Q |w|^q\,dz\right)^{1/q} \le C\left(\fint_{2Q} |w|^p\,dz\right)^{1/p} + r\left(\fint_{2Q} |F_2|^2 + |F_3|^2\,dz\right)^{1/2}. \tag{3.13}$$

The term F_1 does not appear explicitly since $4Q \subset R$ implies $|F_1| \le C\varepsilon_0 r|w|$ on $2Q$. If $Q = R(t,x,r)$ is a Carleson box with $4Q = R(t,x,4r) \subset R$, then (3.13) also holds for the same exponents q, p — in this case, it is precisely the statement of the boundary reverse Hölder inequality (Lemma 2.7) if we write $\operatorname{div} F_1$ on the left-hand side of the equation. The collection $\mathcal{Q}$ of boxes, for which we know (3.13) with these q, p so far has a covering property: every $Q \in \mathcal{Q}$ can be covered by $Q_1, \dots, Q_N \in \mathcal{Q}$ where N only depends on dimension, Q_i is comparable to Q in size and $4Q_i \subset 2Q$. Thus, the classical argument of [BCF16, Theorem B.1] shows that in (3.13) the right-hand exponent can be lowered to $p = 1$ and of course we can lower q to $q = 2$ by Hölder's inequality. Finally, we cover $\theta^{3/2}R$ by sufficiently small boxes from $\mathcal{Q}$ and sum up (3.13) with $q = 2$, $p = 1$ to conclude

$$\left(\fint_{\theta^{3/2}R} |w|^2\,dz\right)^{1/2} \le C\fint_{\theta R} |w|\,dz + \lambda\left(\fint_{\theta R} |F_2|^2 + |F_3|^2\,dz\right)^{1/2}.$$

The claim (3.11) follows by the boundary Caccioppoli inequality (Lemma 2.6) to the left and Lemma 3.9 to control $|F_2|^2 + |F_3|^2$ on the right.

With (3.11) at hand, we shall complete the argument by proving

$$\frac{1}{\lambda}\fint_R |w|\,dz \le C\alpha_{A,B}(R)\left(\fint_{\theta^{-1}R} |\nabla u|^2\,dz\right)^{1/2}. \tag{3.14}$$

To this end, we rearrange (3.12) and view w as a global solution to the equation

$$\begin{aligned} -\partial_t w - \operatorname{div} A_0^* \nabla w &= -\operatorname{div} Bu + \operatorname{div}(A^* - A_0^*)\nabla u && \text{in } R, \\ w &= 0 && \text{on } \partial^*_{par} R \end{aligned} \tag{3.15}$$

with constant coefficients. By Hölder's inequality, Poincaré's inequality and the Calderón–Zygmund estimates (see [LSU68, Chapter IV] or [BBS19, Theorem 4.1]),

$$\begin{aligned} \frac{1}{\lambda} \fint_R |w| \, dz &\le \frac{1}{\lambda} \left(\fint_R |w|^q \, dz \right)^{1/q} \\ &\le C \left(\fint_R |\nabla w|^q \, dz \right)^{1/q} \\ &\le C \left(\fint_R |B|^q |u|^q \, dz \right)^{1/q} + C \left(\fint_R |A - A_0|^q |\nabla u|^q \, dz \right)^{1/q} =: \mathrm{I} + \mathrm{II} \end{aligned}$$

holds for any $q > 1$ and a constant C depending also on q. We fix $p > 2$ to be the exponent provided by Lemma 2.14, take q such that $1/q = 1/2 + 1/p$ and estimate I and II by the right-hand side of (3.14) as follows. First, by Hölder's and Hardy's inequalities,

$$\mathrm{I} \le C \left(\fint_R |B|^2 z_{n+1}^2 \, dz \right)^{1/2} \left(\fint_R \frac{|u|^p}{z_{n+1}^p} \, dz \right)^{1/p} \le C \alpha_B(R)^{1/2} \left(\fint_R |\nabla u|^p \, dz \right)^{1/p}$$

and the correct bound follows from by Lemma 2.14. Similarly,

$$\mathrm{II} \le C \left(\fint_R |A - A_0|^2 \, dz \right)^{1/2} \left(\fint_R |\nabla u|^p \, dz \right)^{1/p} \le C \alpha_A(R)^{1/2} \left(\fint_{\theta^{-1} R} |\nabla u|^2 \, dz \right)^{1/2} . \qquad \square$$

3.3. **Estimates for Green functions.** The following estimates for Green functions are a perturbed version of Lemma 3.1.

Lemma 3.16. *Let M_0 be given. There exists $\varepsilon_0 > 0$, $\kappa_0 \ge 40$ and $C \ge 1$ such that whenever G is the Green function for an (M_0, ε_0)-parabolic operator and $u := G(a^+(t_0, x_0, \kappa\lambda_0), \cdot)$ for some $(t_0, x_0, \lambda_0) \in \mathbb{R}^{1+n}_+$ and $\kappa \ge \kappa_0$, then the following estimates hold for all $(t, x, \lambda) \in R(t_0, x_0, \lambda_0)$ and all $\varrho \in (0, \lambda]$:*

(1) Decay of J:

$$J_u(t, x, \varrho) \le C \left(\frac{\varrho}{\lambda} \right)^2 J_u(t, x, \lambda) + C \left(\frac{\lambda}{\varrho} \right)^{n+2} \alpha_{A,B}(t, x, \lambda)^2 E_u(t, x, \lambda). \tag{3.17}$$

(2) Non-degeneracy of E:

$$E_u(t, x, \lambda) \le C E_u(t, x, \varrho) + C \left(\frac{\lambda}{\varrho} \right)^{n+2} \alpha_{A,B}(t, x, \lambda)^2 E_u(t, x, \lambda). \tag{3.18}$$

Proof. We begin with the first item. As in the proof of Lemma 3.1 we can assume $\varrho \le \lambda/4$. Let $A_0 := \langle A \rangle_{R(t,x,\lambda/2)}$ and let u_0 solve

$$\begin{aligned} -\partial_t u_0 - \operatorname{div} A_0^* \nabla u_0 &= 0 && \text{in } R(t, x, \lambda/2), \\ u_0 &= u && \text{in } \partial_{par^*} R(t, x, \lambda/2). \end{aligned}$$

Such a solution exists — it is given by $u_0 := u - w$, where w is the solution to the problem (3.15) with constant coefficients on $R := R(t, x, \lambda/2)$ and the latter is solvable (e.g. [Eva98, Chapter 7, Theorem 3]). Now, we split

$$J_u(t, x, \varrho) \leq 2J_{u_0}(t, x, \varrho) + 2J_{u-u_0}(t, x, \varrho) \tag{3.19}$$

and estimate both energies separately. By Lemma 3.1, the triangle inequality and Lemma 3.10,

$$\begin{aligned} J_{u_0}(t, x, \varrho) &\leq C\left(\frac{\varrho}{\lambda}\right)^2 J_{u_0}(t, x, \lambda/4) \\ &\leq C\left(\frac{\varrho}{\lambda}\right)^2 \big(J_u(t, x, \lambda/4) + E_{u-u_0}(t, x, \lambda/4)\big) \\ &\leq C\left(\frac{\varrho}{\lambda}\right)^2 \big(J_u(t, x, \lambda/4) + \alpha_{A,B}(t, x, \lambda)^2 E_u(t, x, \lambda)\big). \end{aligned}$$

To bound the remaining term in (3.19), we estimate

$$\begin{aligned} J_{u-u_0}(t, x, \varrho) &\leq E_{u-u_0}(t, x, \varrho) \\ &\leq \left(\frac{\lambda}{\varrho}\right)^{n+2} E_{u-u_0}(t, x, \lambda/4) \\ &\leq \left(\frac{\lambda}{\varrho}\right)^{n+2} \alpha_{A,B}(t, x, \lambda)^2 E_u(t, x, \lambda), \end{aligned}$$

where the last estimate follows from Lemma 3.10 as before. This yields the first item.

As for the second item, Lemma 2.22 and the triangle inequality yield

$$E_u(t, x, \lambda) \leq CE_u(t, x, \lambda/4) \leq C\big(E_{u_0}(t, x, \lambda/4) + E_{u-u_0}(t, x, \lambda/4)\big). \tag{3.20}$$

In the proof of the first assertion, we have already seen that

$$E_{u-u_0}(t, x, \lambda/4) \leq C\alpha_{A,B}(t, x, \lambda)^2 E_u(t, x, \lambda),$$

which is of the desired form. To estimate the first term on the right of (3.20), we use Lemma 3.1, the triangle inequality and once again Lemma 3.10:

$$\begin{aligned} E_{u_0}(t, x, \lambda/4) &\leq CE_{u_0}(t, x, \varrho) \\ &\leq C\big(E_u(t, x, \varrho) + E_{u-u_0}(t, x, \varrho)\big) \\ &\leq CE_u(t, x, \varrho) + C\left(\frac{\lambda}{\varrho}\right)^{n+2} E_{u-u_0}(t, x, \lambda/4) \\ &\leq CE_u(t, x, \varrho) + C\left(\frac{\lambda}{\varrho}\right)^{n+2} \alpha_{A,B}(t, x, \lambda)^2 E_u(t, x, \lambda). \qquad \square \end{aligned}$$

We are in position to state and prove the first central result that was largely inspired by the elliptic counterpart in [DLM22b]. Under an a priori smallness on the coefficients in terms of the $\alpha_{A,B}$-numbers, we obtain a quantitative Carleson measure estimate for the energy ratios

$$\beta_u(t, x, \lambda) = \left(\frac{J_u(t, x, \lambda)}{E_u(t, x, \lambda)}\right)^{1/2}$$

from Definition 2.5. In this context, we also use the measure

$$\mu_{\beta_u} := \beta_u(t,x,\lambda)^2 \frac{dt\,dx\,d\lambda}{\lambda}.$$

Theorem 3.21. *Given M_0, let $\varepsilon_0 > 0$ and $\kappa_0 \geq 40$ be as in Lemma 3.16. There exists constants $\delta_0 > 0$, $\theta_0 \in (0,1/4]$, $\gamma > 0$ and C such that whenever G is the Green function for an (M_0,ε_0)-parabolic operator and $u := G(a^+(t_0,x_0,\kappa\lambda_0),\cdot)$ for some $(t_0,x_0,\lambda_0) \in \mathbb{R}^{1+n}_+$ and $\kappa \geq \kappa_0$, then the uniform smallness*

$$\sup_{(s,y,\mu)\in R(t,x,\lambda)} \alpha_{A,B}(s,y,\mu)^2 \leq \delta_0$$

for fixed $(t,x,\lambda) \in R(t_0,x_0,\lambda_0)$ implies for all $\theta \leq \theta_0$ the

(1) pointwise Carleson bound

$$\frac{1}{(\theta\lambda)^{n+1}} \iiint_{R(t,x,\theta\lambda)} \beta(s,y,\mu)^2 \frac{ds\,dy\,d\mu}{\mu} \leq C\left(\theta^\gamma + \|\nu_{A,B}\|_{C(R(t,x,9\lambda))}\right)$$

(2) and the local Carleson measure estimate

$$\|\mu_{\beta_u}\|_{C(R(t,x,\theta^2\lambda))} \leq C\left(\theta^\gamma + \|\nu_{A,B}\|_{C(R(t,x,\lambda))}\right)$$

Proof. Assertion (2) follows from (1) applied to subregions: Indeed, we only have to take $\theta_0 < 1/36$ in order to guarantee that $R(t',x',\theta\lambda') \subset R(t,x,\theta^2\lambda)$ implies $R(t',x',9\lambda') \subset R(t,x,\lambda)$. Hence, it suffices to prove (1).

To this end, let $\sigma \in (0,1/4]$ be a scaling parameter that we shall fix momentarily. The other constants in question will be taken as $\delta_0 := \sigma^{n+2}/2$, $\theta_0 := \sigma$ and $\gamma := \log(1/2)/\log(\sigma)$. Throughout the proof let $(s,y,\mu) \in R(t,x,\lambda) =: R$. With the estimates above we closely follow [DLM22b, Section 4].

From (3.17), (3.18) and the assumption, we obtain

$$\begin{aligned} J_u(s,y,\sigma\mu) &\leq C_1\sigma^2 J_u(s,y,\mu) + C_2\sigma^{-n-2}\alpha_{A,B}(s,y,\mu)^2 E_u(s,y,\mu), \\ E_u(s,y,\sigma\mu) &\geq C_3(1-\delta_0\sigma^{-n-2})E_u(s,y,\mu). \end{aligned}$$

We decide on $\sigma^2 \leq C_3/(4C_1)$, so that when dividing the two inequalities, we find that β shrinks according to the rule

$$\beta(s,y,\sigma\mu)^2 \leq \frac{1}{2}\beta(s,y,\mu)^2 + C\alpha_{A,B}(s,y,\mu)^2. \tag{3.22}$$

For brevity, let us now write

$$\begin{aligned} b(\mu) &:= \int_{Q^n(\theta R)} \beta(z+(0,0,\mu))^2\,dz, \\ a(\mu) &:= \int_{Q^n(\theta R)} \alpha_{A,B}(z+(0,0,\mu))^2\,dz. \end{aligned}$$

We need to prove that

$$\mathrm{I} := \int_0^{2\theta\lambda} b(\mu)\frac{d\mu}{\mu} \leq C(\theta\lambda)^{n+1}(\theta^\gamma + \|\nu_{A,B}\|_{C(R)}). \tag{3.23}$$

To this end, we split the integral I at height $\mu = 2\sigma\theta\lambda$ and estimate both terms separately. By a change of variables and (3.22),

$$\int_0^{2\sigma\theta\lambda} b(\mu)\,\frac{d\mu}{\mu} = \int_0^{2\theta\lambda} b(\sigma\mu)\,\frac{d\mu}{\mu} \leq \frac{1}{2}\,\mathrm{I} + C\int_0^{2\theta\lambda} a(\mu)\,\frac{d\mu}{\mu}$$
$$\leq \frac{1}{2}\,\mathrm{I} + C(\theta\lambda)^{n+1}\|\nu_{A,B}\|_{C(R)}.$$

For the other piece, we let $k \geq 1$ be the integer with $\sigma^{k+1} \leq \theta < \sigma^k$, so that by definition of γ we have $2^{-(k+1)} \leq \theta^\gamma$. A change of variables and k-fold iteration of (3.22) yield,

$$\int_{2\sigma\theta\lambda}^{2\theta\lambda} b(\mu)\,\frac{d\mu}{\mu} \leq \int_{2\sigma^2\lambda}^{2\lambda} b(\sigma^k\mu)\,\frac{d\mu}{\mu} \leq \int_{2\sigma^2\lambda}^{2\lambda} \Big(2^{-k}b(\mu) + C\sum_{j=0}^{k} 2^{-j}a(\sigma^{k-j}\mu)\Big)\,\frac{d\mu}{\mu}$$
$$\leq C(\theta\lambda)^{n+1}\big(\theta^\gamma + \|\nu_{A,B}\|_{C(9R)}\big),$$

where the last step follows from the uniform pointwise bounds $\beta(s,y,\mu) \leq 2$ (triangle inequality) and $\alpha_{A,B}(s,y,\mu) \leq C\|\nu_{A,B}\|_{C(9R)}$ (Proposition 2.3). The previous two estimates lead to (3.23) under the a priori assumption that I is finite.

The a priori assumption can be removed by repeating the above with $1_{(\varepsilon,\infty)}(\mu)b(\mu)$ in place of $b(\mu)$ for some fixed $\varepsilon > 0$ and passing to the limit as $\varepsilon \to 0$. □

3.4. **Estimates for the parabolic measure.** We conclude this section by giving a lower bound for the energy ratio β in terms of the parabolic measure. This is a straightforward generalization of an analogous result for elliptic measures in [BTZ23b]. For functions f defined in the variables $(t,x) \in \mathbb{R}^n$ we use the parabolic dilations

$$f_\lambda(t,x) := \frac{1}{\lambda^{n+1}} f\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \qquad (\lambda > 0). \tag{3.24}$$

Lemma 3.25. *Given M_0, let $\varepsilon_0 > 0$ and $\kappa_0 \geq 40$ be as in Lemma 3.16 and let $\phi, \psi : \mathbb{R}^n \to \mathbb{R}$ be smooth functions with*

$$\phi \geq 1 \text{ on } Q^n(0,0,\tfrac{1}{2}) \quad \text{and} \quad \phi, \partial_i\psi \in C_c^\infty(Q^n(0,0,\tfrac{3}{4})) \text{ for } i = 1,\ldots,n.$$

There exists a constant C such that the following holds. Let G be the Green function for an (M_0,ε_0)-parabolic operator and set $u := G(p,\cdot)$, where $p := a^+(t_0,x_0,\kappa\lambda_0)$ for some $(t_0,x_0,\lambda_0) \in \mathbb{R}^{1+n}_+$, $\kappa \geq \kappa_0$. Then the associated parabolic measure $\omega := \omega^p$ satisfies for all $(t,x,\lambda) \in R(t_0,x_0,\lambda_0)$ the estimates

$$|(\partial_t\psi)_\lambda * \omega)(t,x)| + |(\nabla_x\psi)_\lambda * \omega)(t,x)| \leq C\sqrt{J_u(t,x,\lambda) + E_u(t,x,\lambda)\alpha_{A,B}(t,x,\lambda)^2}, \tag{3.26}$$

$$\frac{|(\partial_t\psi)_\lambda * \omega)(t,x)|}{(\phi_\lambda * \omega)(t,x)} + \frac{|(\nabla_x\psi)_\lambda * \omega)(t,x)|}{(\phi_\lambda * \omega)(t,x)} \leq C\big(\beta_u(t,x,\lambda) + \alpha_{A,B}(t,x,\lambda)\big) \tag{3.27}$$

Moreover, (t,x,λ) on the right-hand sides can be replaced by any (t',x',λ') that satisfies $d_{n+1}((t',x',\lambda'),(t,x,\lambda)) < \lambda/4$.

Proof. We directly prove the general claim with (t', x', λ') on the right. Once (3.26) has been proved, (3.27) will follow immediately since

$$
\begin{aligned}
(\phi_\lambda * \omega)(t,x) \geq \frac{\omega(Q^n(t,x,\lambda/2))}{\lambda^{n+1}} &\geq C\frac{\omega(Q^n(t',x',\lambda'/5))}{(\lambda')^{n+1}} \\
&\geq C\frac{\omega(Q^n(t',x',\lambda'))}{(\lambda')^{n+1}} \geq C\sqrt{E_u(t',x',\lambda')},
\end{aligned}
$$

where we have used the lower bound for ϕ, local doubling, the CFMS-estimate and the gradient estimate from Lemma 2.22. Hence, it suffices to prove (3.26). To this end, fix a smooth function h of one variable with $h(0) = 1$ and support in $(-1/4, 1/4)$ and define the extension

$$
\Psi(s,y,\mu) = \frac{1}{\lambda^{n+1}}\psi\left(\frac{t-s}{\lambda^2}, \frac{x-y}{\lambda}\right) h\left(\frac{\mu}{\lambda}\right).
$$

We will frequently use that the non-zero values of the derivatives of this function in the upper-half-space are contained in $R(t', x', \lambda')$ and that these derivatives are uniformly bounded by certain powers of $\lambda \approx \lambda'$. The Riesz formula (2.20) applied to $\partial_i\Psi$ for $i = 1, \ldots, n$ yields

$$
\begin{aligned}
\partial_i(\omega * \psi_\lambda)(t,x) &= -\int_{\mathbb{R}^{1+n}_+} A^*\nabla u \cdot \nabla\partial_i\Psi \, dz - \int_{\mathbb{R}^{1+n}_+} Bu \cdot \nabla\partial_i\Psi \, dz - \int_{\mathbb{R}^{1+n}_+} u\partial_t\partial_i\Psi \, dz \\
&=: -\mathrm{I}(i) - \mathrm{II}(i) - \mathrm{III}(i).
\end{aligned}
$$

We estimate these three terms separately. Note carefully that here the left-hand side differs from the left-hand side in (3.26) by a factor of λ^2 when $i = 1$ and a factor λ when $i = 2, \ldots, n$.

As the derivative of a compactly supported function has integral zero, we can write

$$
\mathrm{III}(i) = \int_{\mathbb{R}^{1+n}_+} \big(u - z_{n+1}\langle\partial_{n+1}u\rangle_{R(t',x',\lambda')}\big)\partial_t\partial_i\Psi \, dz.
$$

and a desirable bound in terms of $J_u(t', x', \lambda')^{1/2}$ follows from Hölder's and Poincaré's inequality.

By Hölder's inequality and the support properties of Ψ, we also obtain

$$
|\mathrm{II}(i)| \leq C\lambda^{n+2}\|\nabla\partial_i\Psi\|_\infty \left(\fint_{R(t',x',\lambda')} \frac{|u|^2}{z_{n+1}^2} \, dz\right)^{1/2} \times \left(\fint_{R(t',x',\lambda')} |B|^2 z_{n+1}^2 \, dz\right)^{1/2}
$$

and a desirable bound in terms of $E_u(t', x', \lambda')^{1/2}\alpha_B(t', x', \lambda')$ follows from the usual Hardy's inequality.

In order to control the most substantial contribution $\mathrm{I}(i)$, we introduce $\ell(z) := z_{n+1}\langle\partial_{n+1}u\rangle_{R(t',x',\lambda')}$ and find

$$
\begin{aligned}
\mathrm{I}(i) = &\int_{\mathbb{R}^{1+n}_+} A^*\nabla(u-\ell) \cdot \nabla\partial_i\Psi \, dz + \int_{\mathbb{R}^{1+n}_+} (A^* - \langle A^*\rangle_{R(t',x',\lambda')})\nabla\ell \cdot \nabla\partial_i\Psi \, dz \\
&+ \int_{\mathbb{R}^{1+n}_+} \langle A^*\rangle_{R(t',x',\lambda')}\nabla\ell \cdot \nabla\partial_i\Psi \, dz.
\end{aligned}
$$

By Hölder's inequality and the size and support properties of Ψ, the first term integral admits a desirable bound in terms of $J_u(t', x', \lambda')^{1/2}$. Likewise, a desirable bound for the second integral in terms of $\alpha_A(t', x', \lambda')E_u(t', x', \lambda')^{1/2}$ follows. The third term is zero, because the integrand

$$\langle A^*\rangle_{R(t',x',\lambda')}\nabla\ell \cdot \nabla\partial_i\Psi = \partial_i\big(\langle A^*\rangle_{R(t',x',\lambda')}\nabla\ell \cdot \nabla\Psi\big),$$

is the (tangential) derivative of a smooth and compactly supported function. □

4. A criterion for anisotropic A_∞-weights

In this section, we provide a criterion for anisotropic (local) A^∞-weights, inspired by [FKP91, BES21] and tailored towards the estimates for parabolic measure in Lemma 3.25. Unlike the references cited, we do not assume the weight function to be globally doubling but proceed under the following background assumption.

Definition 4.1 (Local doubling)**.** A locally finite Borel measure ω on a metric space (X, d) is locally doubling in an open set U if there exists a constant C_ω such that for all $x \in U$ and $r > 0$ such that $B(x, 2r) \subset U$ it follows $\omega(B(x, 2r)) \leq C_\omega\omega(B(x, r))$.

In the following, we fix an n-tuple $(p_1, \dots, p_n) \in [1, \infty)^n$ and work in $\mathbb{R}^n$ with the metric

$$d(z, w) \coloneqq \max_{1\leq j\leq n} |z_j - w_j|^{1/p_j}.$$

For cubes in $\mathbb{R}^n$ and Carleson boxes in $\mathbb{R}^{1+n}$ we use the same notation as in Section 2.1. For functions f on $\mathbb{R}^n$ and $\lambda > 0$ we introduce the anisotropic dilations

$$f_\lambda(z) \coloneqq \frac{1}{\lambda^p}\left(\frac{z_1}{\lambda^{p_1}}, \dots, \frac{z_n}{\lambda^{p_n}}\right),$$

where $p \coloneqq \sum_{j=1}^n p_j$. Earlier on in (3.24) we used $(p_1, p_2, \dots, p_n) = (2, 1, \dots, 1)$.

The global doubling hypothesis in [FKP91, BES21] is related to the use of heat extensions and infinite speed of propagation. Replacing heat extensions by more locally behaved averages is the motivation of the following definition. Therein, the heat kernel ϕ would correspond to the profile $g(s) = e^{-s^2/2}$.

Definition 4.2 (Split heat kernel)**.** An $(n + 1)$-tuple $(\phi, \psi_1, \dots, \psi_n)$ of smooth functions on $\mathbb{R}^n$ is called a split heat kernel if there is a smooth profile $g : \mathbb{R} \to \mathbb{R}$ with $1_{[-1/2,1/2]} \leq g \leq 1_{[-1,1]}$ and integral 1, such that

(1) $\phi(z) = \prod_{i=1}^n g(z_i)$,

(2) $\psi_j(z) = \left(\int_{-\infty}^{z_j} g(s)\, s\, ds\right) \prod_{i\neq j} g(z_i)$.

The split heat kernel satisfies $\partial_j\psi_j(z) = z_j\phi(z)$ for $j = 1, \dots, n$ and therefore its (anisotropic) dilations satisfy the following identity, which is reminiscent to the heat equation:

$$\begin{aligned}\lambda\partial_\lambda(\phi_\lambda) &= -\sum_{j=1}^n p_j\big(\phi_\lambda + (z_j\partial_j\phi)_\lambda\big)\\ &= -\sum_{j=1}^n p_j(\partial_j^2\psi_j)_\lambda = -\sum_{j=1}^n p_j\lambda^{2p_j}\partial_j^2\big((\psi_j)_\lambda\big).\end{aligned} \tag{4.3}$$

The following is our main result on anisotropic weights.

Theorem 4.4. *Let $Q_0 \subset \mathbb{R}^n$ be a metric ball and ω be a locally doubling Radon measure on $4Q_0$ that is not identically zero on $2Q_0$. Suppose that there is a split heat kernel $(\phi, \psi_1, \ldots, \psi_n)$ and a constant $\varepsilon > 0$ such that for $j = 1, \ldots, n$ and $k = 1, 2$,*

$$\left\| \frac{|(\partial_j\psi_j)_\lambda * \omega|\,|(\partial_j\phi)_\lambda * \omega|}{|\phi_\lambda * \omega|^2}\,\frac{dz\,d\lambda}{\lambda}\right\|_{C(2Q_0)} \le \varepsilon, \tag{4.5}$$

$$\left\|\frac{(\partial_j^k\phi)_\lambda * \omega}{\phi_\lambda * \omega}\right\|_{L^\infty(R(2Q_0))} + \left\|\frac{(\partial_j^k\psi_j)_\lambda * \omega}{\phi_\lambda * \omega}\right\|_{L^\infty(R(2Q_0))} \le \sqrt{\varepsilon}. \tag{4.6}$$

Then ω is absolutely continuous with respect to Lebesgue measure on $2Q_0$ and there is a constant C depending on $n, p_1, \ldots, p_n$ and the local doubling of ω such that

$$0 \le \log\left(\fint_Q \omega(z)\,dz\right) - \fint_Q \log\omega(z)\,dz \le C\big(\sqrt{\varepsilon} + \varepsilon\big) \tag{4.7}$$

for all metric balls $Q \subset Q_0$.

Remark 4.8. The additional assumption (4.6) compensates the lack of algebraic properties of the heat kernel compared to [FKP91, BES21]. Indeed, in these references $-\psi_j = \phi$ is the heat kernel and therefore (4.6) follows from (4.5) via regularity estimates for solutions to the heat equation [BES21, Lemma 3.3].

Proof. We extend ω to $\mathbb{R}^{1+n}_+$ via the split heat kernel as

$$u(z,\lambda) := (\phi_\lambda * \omega)(z), \quad v_j(z,\lambda) := ((\psi_j)_\lambda * \omega)(z) \qquad (\lambda > 0).$$

In a first step, we control the integral in (4.7) for $u(\cdot, \delta)$ replacing ω. All further assertions will follow upon carefully passing to the limit as $\delta \to 0$.

Step 1: The quantitative estimate. We write $Q_0 = Q(z_0, r_0)$. Let $Q = Q(w, r) \subset 2Q_0$ and $\delta \in (0, r_0)$. Here, we are going to prove

$$0 \le \log\left(\fint_Q u(z,\delta)\,dz\right) - \fint_Q \log u(z,\delta)\,dz \le \begin{cases} C & \text{if } \delta > r \\ C(\sqrt{\varepsilon} + \varepsilon) & \text{if } \delta \le r \end{cases}. \tag{4.9}$$

The lower bound is due to Jensen's inequality. The upper bound for $\delta > r$ follows from doubling of ω since

$$C_\omega^{-1}\frac{\omega(Q(w,\delta))}{\delta^p} \le u(z,\delta) \le C_\omega\frac{\omega(Q(w,\delta))}{\delta^p} \qquad (z \in Q),$$

so that (4.9) follows with $C := 3\log C_\omega$.

We turn to the case $\delta \le r$. Since assumption and claim are translation and scaling invariant, we can assume without loss of generality that $Q = Q(0,1)$. Local doubling implies $\inf_{Q\times(\delta,r)} u > 0$. This will guarantee that all integrals below converge absolutely. We write the left-hand side in (4.9) as

$$\left[\log\left(\fint_Q u(z,\delta)\,dz\right) - \fint_Q \log u(z,1)\,dz\right] + \left[\fint_Q \log u(z,1)\,dz - \fint_Q \log u(z,\delta)\,dz\right] =: \mathrm{I} + \mathrm{II}$$

and start the estimation from the term II.

By the fundamental theorem of calculus and (4.3),

$$\begin{aligned}
\mathrm{II} &= \fint_Q \int_\delta^1 \partial_\lambda(\log u)\,d\lambda\,dz \\
&= \fint_Q \int_\delta^1 \frac{\partial_\lambda u}{u}\,d\lambda\,dz \\
&= \fint_Q \int_\delta^1 -\sum_{j=1}^n \frac{p_j \partial_j^2(v_j)}{u}\,\frac{d\lambda}{\lambda^{1-2p_j}}\,dz \\
&= -\sum_{j=1}^n p_j \int_\delta^1 \fint_Q \partial_j\left(\frac{\partial_j v_j}{u}\right) + \frac{\partial_j v_j \cdot \partial_j u}{|u|^2}\,dz\,\frac{d\lambda}{\lambda^{1-2p_j}} \\
&=: -\sum_{j=1}^n p_j\,\mathrm{II}_{1,j} + p_j\,\mathrm{II}_{2,j}\,.
\end{aligned}$$

Assumption (4.5) implies $|\mathrm{II}_{2,j}| \le \varepsilon$ for all j. For the integrals $\mathrm{II}_{j,1}$ we apply the fundamental theorem of calculus and then use (4.6) to conclude

$$\mathrm{II}_{j,1} \le \int_0^1 \int_{\partial Q\cap\{z_j=\pm 1\}} \left|\frac{\partial_j v_j}{u}\right|\,dz\,\frac{d\lambda}{\lambda^{1-2p_j}} \le C\sqrt{\varepsilon}.$$

Altogether, we have shown that $\mathrm{II} \le C(\varepsilon + \sqrt{\varepsilon})$.

Next, we decompose

$$\mathrm{I} = \left[\log\left(\fint_Q u(z,\delta)\,dz\right) - \log\sup_{z\in Q} u(z,1)\right] + \left[\sup_{z\in Q}\log u(z,1) - \fint_Q \log u(z,1)\,dz\right] =: \mathrm{I}_1 + \mathrm{I}_2\,.$$

By the mean value theorem and (4.6) we have $\mathrm{I}_2 \le C\sqrt{\varepsilon}$. Local doubling of ω also implies $\mathrm{I}_1 \le C$, so that it remains to control I_1 for small ε, say $\sqrt{\varepsilon} \le \min_j \frac{p_j}{2p}$.

Writing I_1 as the quotient of logarithms, we find

$$\mathrm{I}_1 \le \log_+\left(1 + \inf_{z\in\overline{Q}} \frac{1}{u(z,1)} \fint_Q u(z,\delta) - u(z,1)\,dz\right), \tag{4.10}$$

where $\log_+$ is the positive part of the logarithm function. By the fundamental theorem of calculus and (4.3), the average on the right is bounded by

$$\fint_Q u(z,\delta) - u(z,1)\,dz = -\fint_Q \int_\delta^1 \partial_\lambda u\,d\lambda\,dz$$

$$
\begin{aligned}
&= \int_\delta^1 \fint_Q \sum_{j=1}^n p_j \partial_j^2 v_j \, dz \, \frac{d\lambda}{\lambda^{1-2p_j}} \\
&\le C \sum_{j=1}^n p_j \int_\delta^1 \int_{\partial Q \cap \{z_j = \pm 1\}} \partial_j v_j \, dz \, \frac{d\lambda}{\lambda^{1-2p_j}} \\
&\le \sum_{j=1}^n C\sqrt{\varepsilon} \int_0^1 \int_{\partial Q \cap \{z_j = \pm 1\}} |u(z,\lambda)| \, dz \, \frac{d\lambda}{\lambda^{1-p_j}},
\end{aligned}
$$

where the final step follows from (4.6). By the same arguments, we find for all $z \in \overline{Q}$ and all $\lambda \in (0,1)$ that

$$
\begin{aligned}
\frac{u(z,\lambda)}{u(z,1)} &= \exp\left(- \int_\lambda^1 \partial_r \log u(z,r) \, dr \right) \\
&= \exp\left(\sum_{j=1}^n p_j \int_\lambda^1 \frac{\partial_j^2 v_j(z,r)}{u(z,r)} \frac{dr}{r^{1-2p_j}} \right) \\
&\le \exp\left(\sum_{j=1}^n p_j \int_\lambda^1 \sqrt{\varepsilon} \frac{dr}{r} \right) = \lambda^{-\sqrt{\varepsilon} p}.
\end{aligned}
$$

Combining the previous two estimates gives

$$
\inf_{z \in \overline{Q}} \frac{1}{u(z,1)} \fint_Q u(z,\delta) - u(z,1) \, dz \le C\sqrt{\varepsilon} \left(\sum_{j=1}^n \int_0^1 \lambda^{-\sqrt{\varepsilon} p} \frac{d\lambda}{\lambda^{1-p_j}} \right).
$$

The integrals are constants depending on $p_1, \ldots, p_n$ since we assume $\sqrt{\varepsilon} \le \min_j \frac{p_j}{2p}$. Going back to (4.10), we get the desired bound

$$
\mathrm{I}_1 \le \log_+(1 + C\sqrt{\varepsilon}) \le C\sqrt{\varepsilon}
$$

and the proof of (4.9) is complete.

Step 2: Absolute continuity of ω. For clarity, let us write $\omega_\delta := u(\cdot,\delta)\,dz$ for the measures with weight function $u(\cdot,\delta)$. In the limit as $\delta \to 0$, they converge weakly to ω on $2Q_0$. By classical theory of weights, (4.9) implies $\omega_\delta \in A_\infty(2Q_0)$, uniformly in δ, that is, there are constants $c, \gamma > 0$ such that for all balls $Q \subset (3/2)Q_0$, all Borel sets $E \subset Q$ and all $\delta \in (0, r_0)$ we have

$$
\frac{\omega_\delta(E)}{\omega_\delta(Q)} \le c \left(\frac{|E|}{|Q|} \right)^\gamma, \tag{4.11}
$$

see for instance [GCRdF85, Theorem IV.2.15]. By local doubling $\omega_\delta(Q) \le \omega(2Q) \le C_\omega \omega(Q)$ for δ small enough and if E is open, then also $\liminf_{\delta \to 0} \omega_\delta(E) \ge \omega(E)$ by weak convergence. Thus,

$$
\frac{\omega(E)}{\omega(Q)} \le C_\omega c \left(\frac{|E|}{|Q|} \right)^\gamma.
$$

This estimate remains valid for all Borel sets $E \subset Q$ by outer regularity of ω. In particular, ω is absolutely continuous with respect to Lebesgue measure on $(3/2)Q_0$.

Step 3: Conclusion. Let $Q \subset Q_0$. We complete the proof of (4.7) by passing to the limit in (4.9) along a sequence $\delta \to 0$. Since $\omega \in L^1((3/2)Q_0)$, we have $u(\cdot, \delta) \to \omega$ in $L^1(Q)$ and therefore

$$\lim_{\delta \to 0} \log\Big(\fint_Q u(z, \delta)\, dz \Big) = \log\Big(\fint_Q \omega(z)\, dz \Big).$$

The limit with logarithm inside the integral is more delicate. Upon passing to a subsequence, we can arrange $u(\cdot, \delta) \to \omega$ pointwise almost everywhere and hence the same for $\log u(\cdot, \delta) \to \log \omega$. Moreover, (4.7) implies that $(\fint_Q \log u(z, \delta)\, dz)_\delta$ is bounded and (4.11) implies that $(\log u(\cdot, \delta))_\delta$ is bounded in $BMO((4/3)Q)$, see for instance [GCRdF85, Cor. IV.2.19]. Due to the John–Nirenberg inequality, $(\log u(\cdot, \delta))_\delta$ is bounded in $L^2(Q)$. Since boundedness and pointwise convergence imply weak convergence in $L^2(Q)$, we can conclude

$$\lim_{\delta \to 0} \fint_Q \log u(z, \delta)\, dz = \fint_Q \log \omega(z)\, dz.$$

Now, (4.9) follows. □

5. Proofs of the main results

Eventually, we assemble all estimates proved so far in order to obtain Theorem 1.2. We start with a slightly more precise local variant from which the rest will follow effortlessly.

Theorem 5.1. *Let M_0 be given. There are constants $\varepsilon_0 > 0$, $\kappa_0 \geq 40$, $\delta_0 > 0$, $\theta_0 \in (0, 1/4]$, $\gamma > 0$ and $C \geq 1$ such that the following hold whenever $\mathcal{L}$ is (M_0, ε)-parabolic and $\omega := \omega^p$ is the $\mathcal{L}$-parabolic measure with pole at $p := a^+(t_0, x_0, \kappa\lambda_0)$ for some $(t_0, x_0, \lambda_0) \in \mathbb{R}^{1+n}_+$, $\kappa \geq \kappa_0$.*

1. *If $\|\nu_{A,B}\|_C \leq \delta_0$ and $\theta \leq \theta_0$, then ω is absolutely continuous with respect to Lebesgue measure on $Q^n(t_0, x_0, 2\theta^2\lambda_0)$.*
2. *Denoting $k = \frac{d\omega}{dt\, dx}$ in the setting of (1), it follows for all $(t, x, \lambda) \in R(t_0, x_0, \lambda_0)$ that*

$$\log\Big(\fint\!\!\fint_{Q^n(t,x,\theta^2\lambda)} k(t, x)\, dt\, dx \Big) - \fint\!\!\fint_{Q^n(t,x,\theta^2\lambda)} \log k(t, x)\, dt\, dx \leq C\big(\theta^\gamma + \sqrt{\|\nu_{A,B}\|_{C(R(t,x,2\lambda))}}\big).$$

Proof. Let us first sort out the constants. We take ε_0, κ_0 as in Lemma 3.16 and θ_0, γ as in Theorem 3.21. By virtue of Proposition 2.3, we can take the threshold δ_0 such that the assumption $\|\nu_{A,B}\|_C \leq \delta_0$ allows us to use Theorem 3.21 for every $(t, x, \lambda) \in R(t_0, x_0, \lambda_0)$ and in particular for (t_0, x_0, λ_0).

Both assertions will now follow from Theorem 4.4 applied to the parabolic measure ω on $Q_0 := Q^n(t, x, \theta^2\lambda)$.

The local doubling has been discussed in Lemma 2.22 and we fix a split heat kernel with anisotropy $(p_1, \ldots, p_n) = (2, 1, \ldots, 1)$ and profile supported in $[-3/4, 3/4]$. As in Lemma 3.25, we set $u := G(p, \cdot)$ and we also let μ_{β_u} be the same measure as in Theorem 3.21. Then (3.27) yields (4.5) with

$$\varepsilon := C\big(\|\mu_{\beta_u}\|_{C(R(t,x,2\theta^2\lambda))} + \|\nu_{A,B}\|_{C(R(t,x,2\theta^2\lambda))}\big)$$

and by averaging the general version of (3.27) with respect to (t', x', λ') on a suitable Whitney-type region, we get (4.6) with the same choice of ε. Theorem 3.21 guarantees that

$$\varepsilon \leq C\left(\theta^{2\gamma} + \|\nu_{A,B}\|_{C(R(t,x,2\lambda))}\right)$$

and the proof is complete. □

Proof of Theorem 1.2. This is mainly a question of re-labeling objects given that Theorem 5.1 holds uniformly in $\theta \leq \theta_0$ and $\kappa \geq \kappa_0$.

Indeed, given δ, let us take $\theta < \theta_0$ so small that $\theta^\gamma < \sqrt{\delta}$. Setting $\kappa := \kappa_0/\theta^2$ and $\lambda_1 := \lambda_0/\theta^2$, we see that $a^+(t_0, x_0, \kappa\lambda_0) = a^+(t_0, x_0, \kappa_0\lambda_1)$ and that every parabolic cube $Q \subset Q^n(t_0, x_0, \lambda_0)$ is of the form $Q = Q^n(t, x, \theta^2\lambda)$ with $(t, x, \lambda) \in R(t_0, x_0, \lambda_1)$. Hence, Theorem 5.1 yields absolute continuity of ω on $Q^n(t_0, x_0, 2\lambda_0)$ along with the estimate

$$\log\left(\fint\!\!\fint_Q k(t,x)\,dt\,dx\right) - \fint\!\!\fint_Q \log k(t,x)\,dt\,dx \leq C\sqrt{\delta}. \qquad \square$$

Proof of Corollary 1.3. The claim on reverse Hölder classes follows by Theorem 1.2 and [Kor98, Theorem 9], taking into account that the John–Nirenberg theorem remains valid in metric spaces with a locally doubling measure. The claim on solvability of the Dirichlet problem then follows from a standard argument that is similar to the elliptic case, see e.g. [Nys97] or [HL01, Lemma 4.19, Section 2]. □

References

[ABES19a] Pascal Auscher, Simon Bortz, Moritz Egert, and Olli Saari. Nonlocal self-improving properties: a functional analytic approach. *Tunis. J. Math.*, 1(2):151–183, 2019.

[ABES19b] Pascal Auscher, Simon Bortz, Moritz Egert, and Olli Saari. On regularity of weak solutions to linear parabolic systems with measurable coefficients. *J. Math. Pures Appl. (9)*, 121:216–243, 2019.

[AEN20] Pascal Auscher, Moritz Egert, and Kaj Nyström. L^2 well-posedness of boundary value problems for parabolic systems with measurable coefficients. *J. Eur. Math. Soc. (JEMS)*, 22(9):2943–3058, 2020.

[Aro68] Donald G. Aronson. Non-negative solutions of linear parabolic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 22:607–694, 1968.

[BBS19] Miroslav Bulíček, Jan Burczak, and Sebastian Schwarzacher. Well posedness of nonlinear parabolic systems beyond duality. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 36(5):1467–1500, 2019.

[BCF16] Frédéric Bernicot, Thierry Coulhon, and Dorothee Frey. Gaussian heat kernel bounds through elliptic Moser iteration. *J. Math. Pures Appl. (9)*, 106(6):995–1037, 2016.

[BES21] Simon Bortz, Moritz Egert, and Olli Saari. Carleson conditions for weights: The quantitative small constant case. Preprint. arXiv:2107.14217, 2021.

[BHMN21] Simon Bortz, Steve Hofmann, J.M. Martell, and Kaj Nyström. Solvability of the l^p Dirichlet problem for the heat equation is equivalent to parabolic uniform rectifiability in the case of a parabolic Lipschitz graph. Preprint. arXiv:2306.17291, 2021.

[BL76] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*, volume No. 223 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1976.

[BTZ23a] Simon Bortz, Tatiana Toro, and Zihui Zhao. Elliptic measures for Dahlberg-Kenig-Pipher operators: asymptotically optimal estimates. *Math. Ann.*, 385(1-2):881–919, 2023.

[BTZ23b] Simon Bortz, Tatiana Toro, and Zihui Zhao. Optimal Poisson kernel regularity for elliptic operators with Hölder continuous coefficients in vanishing chord-arc domains. *J. Funct. Anal.*, 285(5):Paper No. 110025, 64, 2023.

[Dah77] Björn E. J. Dahlberg. Estimates of harmonic measure. *Arch. Rational Mech. Anal.*, 65(3):275–288, 1977.

[Dah86a] Björn E. J. Dahlberg. On the absolute continuity of elliptic measures. *Amer. J. Math.*, 108(5):1119–1138, 1986.

[Dah86b] Björn E. J. Dahlberg. Poisson semigroups and singular integrals. *Proc. Amer. Math. Soc.*, 97(1):41–48, 1986.

[DDH20] Martin Dindoš, Luke Dyer, and Sukjung Hwang. Parabolic L^p Dirichlet boundary value problem and VMO-type time-varying domains. *Anal. PDE*, 13(4):1221–1268, 2020.

[DLM22a] Guy David, Linhan Li, and Svitlana Mayboroda. Carleson estimates for the Green function on domains with lower dimensional boundaries. *J. Funct. Anal.*, 283(5):Paper No. 109553, 42, 2022.

[DLM22b] Guy David, Linhan Li, and Svitlana Mayboroda. Carleson measure estimates for the Green function. *Arch. Ration. Mech. Anal.*, 243(3):1525–1563, 2022.

[DPP07] Martin Dindos, Stefanie Petermichl, and Jill Pipher. The L^p Dirichlet problem for second order elliptic operators and a p-adapted square function. *J. Funct. Anal.*, 249(2):372–392, 2007.

[Eva98] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[FGS86] Eugene B. Fabes, Nicola Garofalo, and Sandro Salsa. A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations. *Illinois J. Math.*, 30(4):536–565, 1986.

[FKP91] Robert A. Fefferman, Carlos E. Kenig, and Jill Pipher. The theory of weights and the Dirichlet problem for elliptic equations. *Ann. of Math. (2)*, 134(1):65–124, 1991.

[FS97] Eugene B. Fabes and Mikhail V. Safonov. Behavior near the boundary of positive solutions of second order parabolic equations. In *Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996)*, volume 3, pages 871–882, 1997.

[GCRdF85] José García-Cuerva and José L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[HL96] Steve Hofmann and John L. Lewis. L^2 solvability and representation by caloric layer potentials in time-varying domains. *Ann. of Math. (2)*, 144(2):349–420, 1996.

[HL01] Steve Hofmann and John L. Lewis. The Dirichlet problem for parabolic operators with singular drift terms. *Mem. Amer. Math. Soc.*, 151(719):viii+113, 2001.

[HL05] Steve Hofmann and John L. Lewis. The L^p Neumann problem for the heat equation in non-cylindrical domains. *J. Funct. Anal.*, 220(1):1–54, 2005.

[HLN04] Steve Hofmann, John L. Lewis, and Kaj Nyström. Caloric measure in parabolic flat domains. *Duke Math. J.*, 122(2):281–346, 2004.

[Hof95] Steve Hofmann. A characterization of commutators of parabolic singular integrals. In *Fourier analysis and partial differential equations (Miraflores de la Sierra, 1992)*, Stud. Adv. Math., pages 195–210. CRC, Boca Raton, FL, 1995.

[Hof97] Steve Hofmann. Parabolic singular integrals of Calderón-type, rough operators, and caloric layer potentials. *Duke Math. J.*, 90(2):209–259, 1997.

[JK81] David S. Jerison and Carlos E. Kenig. The Dirichlet problem in nonsmooth domains. *Ann. of Math. (2)*, 113(2):367–382, 1981.

[Kor98] Michael Brian Korey. Ideal weights: asymptotically optimal versions of doubling, absolute continuity, and bounded mean oscillation. *J. Fourier Anal. Appl.*, 4(4-5):491–519, 1998.

[KW80] Robert Kaufman and Jang-Mei Wu. Singularity of parabolic measures. *Compositio Math.*, 40(2):243–250, 1980.

[KW88] Robert Kaufman and Jang-Mei Wu. Parabolic measure on domains of class Lip $\frac{1}{2}$. *Compositio Math.*, 65(2):201–207, 1988.

[Lie96] Gary M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[LM95] John L. Lewis and Margaret A. M. Murray. The method of layer potentials for the heat equation in time-varying domains. *Mem. Amer. Math. Soc.*, 114(545):viii+157, 1995.

[LSU68] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*, volume Vol. 23 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1968. Translated from the Russian by S. Smith.

[Mos64] Jürgen Moser. A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.*, 17:101–134, 1964.

[Mos71] J. Moser. On a pointwise estimate for parabolic differential equations. *Comm. Pure Appl. Math.*, 24:727–740, 1971.

[Mur85] Margaret A. M. Murray. Commutators with fractional differentiation and BMO Sobolev spaces. *Indiana Univ. Math. J.*, 34(1):205–215, 1985.

[Nys97] Kaj Nyström. The Dirichlet problem for second order parabolic operators. *Indiana Univ. Math. J.*, 46(1):183–245, 1997.

[Sal81] Sandro Salsa. Some properties of nonnegative solutions of parabolic differential operators. *Ann. Mat. Pura Appl. (4)*, 128:193–206, 1981.

[Shn74] I. Ja. Shneiberg. Spectral properties of linear operators in interpolation families of Banach spaces. *Mat. Issled.*, 9(2(32)):214–229, 254–255, 1974.

[Tri78] Hans Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.

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