Hamiltonian systems. Exercises

Marcel Guardia and Tere M.–Seara

Chapter 1: Introduction to Hamiltonian systems

1. Make the phase portrait of the Hamiltonian system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - \frac{x^3}{3} \end{aligned}$$

   and compute its Hamiltonian.

2. Make the phase portrait of the Hamiltonian system

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y + x^2 \end{aligned}$$

   and compute its Hamiltonian.

3. (Meyer-Hall-Offin) Let x, y, z be the usual coordinates in $\mathbb{R}^3$, $r = xi + yj + zk$,$X = \dot{x}$, $Y = \dot{y}$, $Z = \dot{z}$, $R = \dot{r} = Xi + Yj + Zk$.
   (a) Compute the three components of angular momentum $mr \times R$.
   (b) Compute the Poisson bracket of any two of the components of angular momentum and show that it is $\pm m$ times the third component of angular momentum.
   (c) Show that if a system admits two components of angular momentum as integrals, then the system admits all three components of angular momentum as integrals.

4. (Meyer-Hall-Offin) A Lie algebra A is a vector space with a product $\ : A \times A \to A$ that satisfies
   - $ab = ba$ (anticommutative),
   - $a(b + c) = ab + ac$ (distributive),
   - $(\alpha a)b = \alpha(ab)$ (scalar associative),
   - $a(bc) + b(ca) + c(ab) = 0$ (Jacobi identity), where $a, b, c \in A$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$.

   (a) Show that vectors in $\mathbb{R}^3$ form a Lie algebra where the product $*$ is the cross product.
   (b) Show that smooth functions on an open set in $\mathbb{R}^{2n}$ form a Lie algebra, where $fg = \{f, g\}$, the Poisson bracket.
   (c) Show that the set of all $n \times n$ matrices, $gl(n, \mathbb{R})$, is a Lie algebra, where $AB = ABBA$, the Lie product.

5. (Meyer-Hall-Offin) The pendulum equation is $\ddot{\theta} + sin\theta = 0$.
   (a) Show that $2I = \frac{1}{2}\dot{\theta}^2 + (1 \cos\theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2)$ is an integral.
   (b) Sketch the phase portrait.
   (c) Make the substitution $y = \sin(\theta/2)$ to get $\dot{y}^2 = (1 - y^2)(I - y^2)$. Show that when $0 < I < 1$, $y = ksn(t, k)$ solves this equation when $k^2 = I$ (Look at the definition of elliptic sine function of Section 1.6 of Meyer-Hall-Offin).
6. (Meyer-Hall-Offin) Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a globally defined conservative Hamiltonian, and assume that $H(z) \to +\infty$ as $z \to +\infty$. Show that all solutions of $\dot{z} = J\nabla H(z)$ are bounded. (Hint: Think like Dirichlet.)
7. Consider a $\mathcal{C}^2$ Hamiltonian $H = H(q, p, t) : U \subset \mathbb{R}^{2n+1} \to \mathbb{R}$ such that $\det(\partial_p^2 H) \neq 0$ on U. Define $v = \partial_p H(q, p, t)$. Prove
   (a) $\partial_{q_i} L(q, v, t) = -\partial_{q_i} H(q, p, t)$, $\partial_{v_i}(q, v, t) = p_i$, $\partial_t L(q, v, t) = -\partial_t H(q, p, t)$.
   (b) The Lagrangian L is $\mathcal{C}^2$ and $\det(\partial_v^2 L) \neq 0$.
   (c) The Euler-Lagrange equations associated to L and the Hamiltonian equations $\dot{q}_i = \partial_{p_i} H$, $\dot{p}_i = -\partial_{q_i} H$ are equivalent.

Chapter 2: The N-body problem

1. Prove that the linear momentum is a first integral and that the center of mass moves with constant velocity for the 3 body problem.
2. Prove that if $(a_1, \ldots, a_N)$ is a central configuration with value λ:
   (a) For any $\tau \in \mathbb{R}$ then $(\tau a_1, \ldots, \tau a_N)$ is also a central configuration with value $\frac{\lambda}{\tau^3}$.
   (b) If A is an orthogonal matrix, then $Aa = (Aa_1, \ldots, Aa_N)$ is also a central configuration with the same value λ.
3. (Meyer-Hall-Offin) Draw the complete phase portrait of the collinear Kepler problem. Integrate the collinear Kepler problem.
4. (Meyer-Hall-Offin) Show that $\mu^2(\epsilon^2 - 1) = 2hc^2$ for the Kepler problem.
   (Attention: Meyer-Hall-Offin has a typo)
5. (Meyer-Hall-Offin) The area of an ellipse is $\pi a^2(1 - \epsilon^2)^{1/2}$, where a is the semi-major axis.We have seen in Keplers problem that area is swept out at a constant rate of $c/2$. Prove Keplers third law: The period p of a particle in a circular or elliptic orbit ($\epsilon < 1$) of the Kepler problem is $p = (\frac{2\pi}{\sqrt{\mu}})a^{3/2}$.
6. (Meyer-Hall-Offin) Let
   $$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
   Then:
   $$e^{Kt} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$
   Find a circular solution of the two-dimensional Kepler problem of the form $q = e^{Kt}a$ where a is a constant vector.

7. (Meyer-Hall-Offin) Assume that a particular solution of the N-body problem exists for all $t > 0$ with $h > 0$. Show that $U \to \infty$ as $t \to \infty$. Does this imply that the distance between one pair of particles goes to infinity? (No.)

8. (Meyer-Hall-Offin) Hills lunar problem is defined by the Hamiltonian

$$H = \frac{||y||^2)}{2} - x^T K y - \frac{1}{||x||} - \frac{1}{2}(3x_1^2 - ||x||^2)$$

where $x \in \mathbb{R}^2$, $y \in \mathbb{R}^2$.

(a) Write the equations of motion.

(b) Show that there are two equilibrium points on the x_1-axis.

(c) Sketch the Hills regions for Hills lunar problem.

(d) Why did Hill say that the motion of the moon was bounded? (He had the Earth at the origin, and an infinite sun infinitely far away and x was the position of the moon in this ideal system. What can you say if x and y are small?

Chapter 3: Linear Hamiltonian systems

1. Let $\lambda \neq 0$ be an eigenvalue of a symplectic matrix A. Prove that $\overline{\lambda}$, λ^{-1} and $\overline{\lambda}^{-1}$ are also eigenvalues of A.

2. Prove Lemma 3.3.6 of Meyer-Hall-Offin.

3. Prove Lemma 3.3.7 of Meyer-Hall-Offin.

4. Prove Lemma 3.3.8 of Meyer-Hall-Offin.

5. (Meyer-Hall-Offin) Prove that the two symplectic matrices

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

are not symplectically similar.

6. (Meyer-Hall-Offin) Consider the system

$$M\ddot{q} + Vq = 0, \tag{1}$$

where M and V are $n \times n$ symmetric matrices and M is positive definite. From matrix theory there is a nonsingular matrix P such that $P^T MP = I$ and an orthogonal matrix R such that $R^T(P^T VP)R = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. Show that the above equation can be reduced to $\ddot{p} + \Lambda p = 0$. Discuss the stability and asymptotic behavior of these systems. Write equation 1 as a Hamiltonian system with Hamiltonian matrix $A = J\text{diag}(V, M^1)$. Use the above results to obtain a symplectic matrix T such that

$$T^1 AT = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix}$$

(Hint: Try $T = diag(PR, P^T R)$).

7. (Meyer-Hall-Offin) Let M and V be as in Equation 1.

   (a) Show that if V has one negative eigenvalue, then some solutions of (1) tend to infinity as $t \to \pm\infty$.

   (b) Consider the system

   $$M\ddot{q} + \nabla U(q) = 0, \tag{2}$$

   where M is positive definite and $U : \mathbb{R}^n \to \mathbb{R}$ is smooth. Let q_0 be a critical point of U such that the Hessian of U at q_0 has one negative eigenvalue (so q_0 is not a local minimum of U). Show that q_0 is an unstable critical point for the system (2).

8. (Meyer-Hall-Offin) Hills lunar problem is defined by the Hamiltonian

   $$H = \frac{\|y\|^2}{2} - x^T K y - \frac{1}{\|x\|} - \frac{1}{2}\left(3x_1^1 - \|x\|^2\right)$$

   where $x, y \in \mathbb{R}^2$.

   (a) Write the equations of motion.

   (b) Show that it has two equilibrium points on the x_1–axis.

   (c) Show that the linearized system at these equilibrium points are saddle-centers; i.e., it has one pair of real eigenvalues and one pair of imaginary eigenvalues.

   (d) The linearization matrix of the Restricted 3 Body Problem at the critical point L_2 has two real eigenvalues and two purely imaginary eigenvalues.

   (e) The linearization matrix of the Restricted 3 Body Problem at the critical point L_3 has two real eigenvalues and two purely imaginary eigenvalues.

Chapter 6: Symplectic Transformations

1. (Meyer-Hall-Offin) Show that if you scale time by $t \to \mu t$, then you should scale the Hamiltonian by $H \to \mu^{-1} H$.

2. (Meyer-Hall-Offin) Scale the Hamiltonian of the N-body problem in rotating coordinates so that ω is 1.

3. (Meyer-Hall-Offin) Consider the Restricted 3-body problem. To investigate solutions near ∞, scale by $x \to \varepsilon^{-2} x$, $y \to \varepsilon y$. Show that the Hamiltonian becomes

   $$H(x, y) = -x^T K y + \varepsilon^3 \left(\frac{\|y\|^2}{2} - \frac{1}{\|x\|}\right) + \mathcal{O}(\varepsilon^2).$$

   Justify this result on physical grounds.

4. (Meyer-Hall-Offin) Consider the Restricted 3-body problem. To investigate solutions near one of the primaries first shift the origin to one primary. Then scale by $x \to \varepsilon^2 x$, $y \to \varepsilon^{-1} y$, $t \to \varepsilon^3 t$.

Chapter 8: Geometric Theory

1. Consider the vector fields X and Y and their flows $\phi(t,x)$ and $\psi(t,y)$. Assume there exists an homeomorphism h which gives a topological equivalence between them. Prove that:

   - p is a fixed point of X if and only if $h(p)$ is a fixed point of Y.
   - $\gamma = \{\phi(t,x),\ t \in [0,T]\}$ is a periodic orbit of X if and only if $h(\gamma)$ is a periodic orbit of Y. What can you say about the period of γ and $h(\gamma)$?
   - Prove that if h is a conjugation the periods of γ and $h(\gamma)$ are the same.

2. (Meyer-Hall-Offin) Let $\{\phi_t\}$ be a smooth dynamical system; i.e., $\{\phi_t\}$ satisfies (8.5). Prove that $\phi(t,\xi) = \phi_t(\xi)$ is the general solution of an autonomous differential equation.

3. (Meyer-Hall-Offin) Let ψ be a diffeomorphism of $\mathbb{R}^m$; so, it defines a discrete dynamical system. A non-fixed point is called an ordinary point. So $p \in \mathbb{R}^m$ is an ordinary point if $\psi(p) \neq p$. Prove that there are local coordinates x at an ordinary point p and coordinates y at $q = \psi(p)$ such that in these local coordinates $y_1 = x_1, \ldots y_m = x_m + 1$. (This is the analog of the flow box theorem for discrete systems.)

4. (Meyer-Hall-Offin) Let ψ be as in Problem 2. Let p be a fixed point of ψ. The eigenvalues of $\frac{\partial\psi}{\partial x}(p)$ are called the (characteristic) multipliers of p. If all the multipliers are different from $+1$, then p is called an elementary fixed point of ψ. Prove that elementary fixed points are isolated.

5. (Meyer-Hall-Offin)

   (a) Let $0 < a < b$ and $\xi \in \mathbb{R}^m$ be given. Show that there is a smooth nonnegative function $\gamma : \mathbb{R}^m \to \mathbb{R}$ which is identically $+1$ on the ball $||x - \xi|| < a$ and identically zero for $||x - \xi|| > b$.

   (b) Let O be any closed set in $\mathbb{R}^m$. Show that there exists a smooth, nonnegative function $\delta : \mathbb{R}^m \to \mathbb{R}$ which is zero exactly on O.

6. (Meyer-Hall-Offin) Let $H(q_1, \ldots, q_N, p_1, \ldots, p_N)$, $q_i, p_i \in \mathbb{R}^3$ be invariant under translation; so, $H(q_1 + s, \ldots, q_N + s, p_1, \ldots, p_N) = H(q_1, \ldots, q_N, p_1, \ldots, p_N)$ for all $s \in \mathbb{R}^3$. Show that total linear momentum, $L = \sum p_i$, is an integral. This is another consequence of the Noether theorem.

7. (Meyer-Hall-Offin) An $m \times m$ nonsingular matrix T is such that $T^2 = I$ is a discrete symmetry of (or a reflection for) $\dot{x} = f(x)$ if and only if $f(Tx) = -Tf(x)$ for all $x \in \mathbb{R}^m$. This equation is also called reversible in this case.

   (a) Prove: If T is a discrete symmetry of (1), then $\phi(t, T\xi) = T\phi(-t, \xi)$ where $\phi(t,\xi)$ is the general solution of $\dot{x} = f(x)$.

   (b) Consider the 2×2 case and let $T = \mathrm{diag}(1, -1)$. What does $f(Tx) = -Tf(x)$ mean about the parity of f_1 and f_2? Show that the first item means that a reflection of a solution in the x_1 axis is a solution.

Chapter 9: Continuation of solutions

1. (Meyer-Hall-Offin) Consider a periodic system of equations of the form $\dot{x} = f(t, x, \nu)$ where ν is a parameter and f is T-periodic in t. Let $\phi(t, \xi, \nu)$ be the general solution, $\phi(t, \xi, \nu) = \xi$

   (a) Show that $\phi(t, \xi', \nu')$ is T-periodic if and only if $\phi(T, \xi', \nu') = \xi'$.

   (b) A T-periodic solution $\phi(t, \xi', \nu')$ can be continued if there is a smooth function $\bar{\xi}(\nu)$ such that $\bar{\xi}(\nu') = \xi'$ and $\phi(t, \bar{\xi}(\nu), \nu)$ is T-periodic. The multipliers of the T-periodic solution $\phi(t, \xi', \nu')$ are the eigenvalues of $\partial_\xi \phi(T, \xi', \nu')$. Show that a T-periodic solution can be continued if all its multipliers are different from $+1$.

2. (Meyer-Hall-Offin) Consider the classical Duffing's equation $\ddot{x} + x + \gamma x^3 = A \cos \omega t$, which is Hamiltonian with respect to

$$H(x, y, t) = \frac{1}{2}(y^2 + x^2) + \gamma \frac{x^4}{4} - Ax \cos \omega t$$

   where $y = \dot{x}$. Show that if $\omega^{-1} \neq 0, \pm 1, \pm 2, \pm 3, \ldots$, then for small forcing A and small nonlinearity γ there is a small periodic solution of the forced Duffing equation with the same period as the external forcing, $T = 2\pi/\omega$.

3. (Meyer-Hall-Offin) Hill's lunar problem is defined by the Hamiltonian

$$H = \frac{\|y\|^2}{2} - x^T K y - \frac{1}{\|x\|} - \frac{1}{2}(3x_1^2 - \|x\|^2),$$

   where $x, y \in \mathbb{R}^2$. Show that it has two equilibrium points on the x_1 axis. Linearize the equations of motion about these equilibrium points, and discuss how the Lyapunov's center and the stable manifold theorem apply.