Growth of Sobolev norms for the cubic defocusing nonlinear Schrödinger equation in polynomial time

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The cubic defocusing nonlinear Schrödinger equation

- Consider the equation

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u \\ u(0,x) = u_0(x) \end{cases}$$

  where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.
- Solutions of NLS conserve two quantities:
  - The Hamiltonian

$$E[u](t) = \int_{\mathbb{T}^2} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 \right) dx(t)$$

  - The mass

$$\mathcal{M}[u](t) = \int_{\mathbb{T}^2} |u|^2 dx(t) = \int_{\mathbb{T}^2} |u|^2 dx(0),$$

    the square of the L^2-norm.
- Well posed globally in time (Bourgain 1993).

Transfer of energy

- Fourier series of u,

$$u(x,t) = \sum_{k\in\mathbb{Z}^2} a_n(t)e^{ikx}$$

- Can we have transfer of energy to higher and higher modes as $t \to +\infty$?
- It is possible that a solution u starts oscillating only on scales comparable to the spatial period and eventually oscillates on arbitrarily small scale?

Sobolev norms

- Sobolev norms

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s}|a_n(t)|^2\right)^{1/2},$$

  where $\langle n\rangle = (1+|n|^2)^{1/2}$.
- Thanks to mass and energy conservation,

$$\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C\|u(0)\|_{H^1(\mathbb{T}^2)} \text{ for all } t \geq 0.$$

- The L^2 norm is conserved.
- The energy transfer can be measured with the growth of the Sobolev norms with $s > 1$.
- The only possibility for H^s to grow indefinitely is that the energy of u moves to higher and higher Fourier modes.

Growth of Sobolev norms

- Zakharov-Shabat equation: cubic defocusing NLS for $x \in \mathbb{T}$,

$$-i\partial_t u + \Delta u = |u|^2 u$$

  is integrable.
- There are a priori bounds for all s-Sobolev norms and therefore there cannot be transfer of energy.
- In dimension $d \geq 2$, there are no a priori bounds and growth of s-Sobolev norms may happen.
- It may even be generic.

How fast the energy transfer can be?

Polynomial upper bounds for the growth of Sobolev norms were first obtained by Bourgain (1993),

Theorem

Let us consider a solution u of the cubic defocusing NLS on $\mathbb{T}^2$, then

$$\|u(t)\|_{H^s} \leq t^{2(s-1)+} \|u(0)\|_{H^s} \quad \text{for} \quad t \to +\infty.$$

- $2(s-1)+$ means any number bigger than $2(s-1)$.
- Results improved or applied to other Hamiltonian PDEs by: Bourgain, Catoire, Colliander, Delort, Kenig, Kwon, Oh, Sohinger, Staffilani, Wang, Zong,...

- Question by Bourgain (2000): Are there solutions u such that for $s > 1$,

$$\|u(t)\|_{H^s} \to +\infty \qquad \text{as } t \to +\infty$$

- Moreover, he conjectured that if such solutions exist, the growth should be subpolynomial in time. That is,

$$\|u(t)\|_{H^s} \ll t^{\varepsilon} \|u(0)\|_{H^s} \qquad \text{for } t \to +\infty, \text{ for all } \varepsilon > 0.$$

Previous works

- Kuksin obtained growth of Sobolev norms for NLS with large initial condition.
- For large initial condition, dispersion is much weaker than the nonlinearity.
- Namely, it is equivalent to obtain growth of Sobolev norms for

$$-i\dot{w} = -\delta \Delta w + |w|^2 w, \ \delta \ll 1.$$

Colliander, Keel, Staffilani, Takaoka, Tao proved in [CKSTT10] the following deep result:

Theorem

Fix $s > 1$, $\mathcal{K} \gg 1$ and $\mu \ll 1$. Then there exists a global solution $u(t,x)$ of NLS and T satisfying that

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{K}.$$

- The solutions they obtain have small initial mass and energy.
- They remain small as time evolves whereas the s-Sobolev norm grows considerably.

[CKSTT10] Colliander, Keel, Staffilani, Takaoka, Tao. *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation,* Invent. Math. (2010).

Refining the methods used in that paper, we estimate the speed of the growth of Sobolev norms.

Theorem

Let $s > 1$. Then, there exists $c > 0$ with the following property: for any large $\mathcal{K} \gg 1$ there exists a a global solution $u(t, x)$ of NLS in $\mathbb{T}^2$ and a time T satisfying

$$0 < T \le \mathcal{K}^c,$$

such that for any t with $1 < t < T$ we have

$$\|u(t)\|_{H^s} \ge t^{\frac{1}{c}} \|u(0)\|_{H^s}.$$

In particular,

$$\|u(T)\|_{H^s} \ge \mathcal{K} \|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \le \mathcal{K}^{-\alpha}, \ \alpha > 0.$$

Comparison with Bourgain conjecture

- Bourgain conjecture:

$$\|u(t)\|_{H^s} \ll t^{\varepsilon}\|u(0)\|_{H^s} \qquad \text{for } t \to +\infty, \text{ for all } \varepsilon > 0.$$

- Our result

$$\|u(t)\|_{H^s} \geq t^{\frac{1}{c}}\|u(0)\|_{H^s}. \qquad \text{for } 1 < t < T.$$

- Our result does not contradict Bourgain conjecture about the subpolynomial growth:
  - The theorem deals with arbitrarily large but finite growth in the Sobolev norms.
  - Bourgain conjecture refers to unbounded growth.
- Growth of the s-Sobolev norm may slow down as time grows.

Comments on the theorem

- Our result is valid in any $\mathbb{T}^d$, $d \geq 2$ taking solutions which only depend on two spatial variables.
- We can obtain more detailed information about the distribution of the Sobolev norm of the solution u, among its Fourier modes when $t = T$: we can ensure that there exist $n_1, n_2 \in \mathbb{Z}^2$ such that

$$\|u(T)\|_{H^s}^2 \geq |n_1|^{2s}|a_{n_1}(T)|^2 + |n_2|^{2s}|a_{n_2}(T)|^2 \geq \mathcal{K}^2\|u(0)\|_{H^s}^2.$$

The final Sobolev norm is essentially localized in two modes.

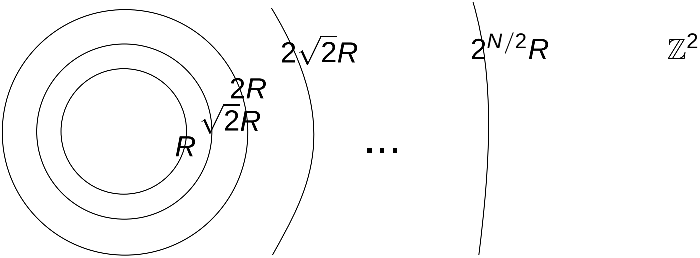
Main ideas of the proof and comparison with the I-team approach

- Choosing carefully a finite set of modes, the *I*-team introduced a finite dimensional (toy) model.
- This toy model approximates well certain solutions of NLS.
- They studied certain solutions of this toy model using Gronwall-like estimates.
- Their methods would lead to bad time estimates

$$T > C^{\mathcal{K}^\alpha}, \;\; C > 0, \alpha \geq 2.$$

- Our main contribution: analysis of the toy model model using
  - Dynamical systems tools (normal forms, Shilnikov boundary problem).
  - A careful choice of the initial conditions
- Last step: prove that solutions of NLS can be approximated by the solutions of the toy model for long time.

Heuristic picture of the transfer of energy



- We start with a set of modes contained in a disk of radius R in Fourier space.
- At each step we *activate* modes in a disk of bigger radius while we put to sleep the previously activated modes.
- We want modes at the *boundary* of the new disk to be activated.
- Since the H^1 norm is almost constant, at each step half of the modes we activate are close to the boundary and the other half closer to zero.

Heuristic picture of the transfer of energy

- We want to make N jumps in the growth of Sobolev norms.
- Here N depends on the growth $\mathcal{K}$: $N \sim \ln \mathcal{K}$.
- At each step only half of the modes cause growth of Sobolev norms.
- This implies that we need $\sim 2^N$ at each disk.
- We need N sets of 2^N modes such that we can construct an energy cascade which at each step has one of these sets activated and the others at rest.

The reduction to the toy model

- We write NLS as an ode (of infinite dimension) for the Fourier coefficients of u:

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- For small a, it can be well approximated by

$$-i\dot{a}_n = |n|^2 a_n - |a_n|^2 a_n + \sum_{n_1,n_2,n_3\in\mathcal{A}(n)} a_{n_1}\overline{a_{n_2}}a_{n_3},$$

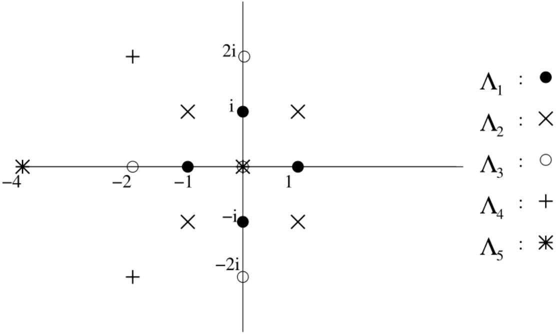
where

$$\mathcal{A}(n) = \Big\{(n_1,n_2,n_3)\in\left(\mathbb{Z}^2\right)^3 : n_1-n_2+n_3=n \\ |n_1|^2-|n_2|^2+|n_3|^2=|n|^2, n_1\neq n, n_3\neq n\Big\}$$

are the resonant terms.

$$\mathcal{A}(n) = \Big\{(n_1, n_2, n_3) \in \left(\mathbb{Z}^2\right)^3 : n_1 - n_2 + n_3 = n$$
$$|n_1|^2 - |n_2|^2 + |n_3|^2 = |n|^2, n_1 \neq n, n_3 \neq n\Big\}.$$

- $(n_1, n_2, n_3) \in \mathcal{A}(n)$ if and only if (n_1, n_2, n_3, n) form a rectangle in $\mathbb{Z}^2$.
- This geometric structure is used in [CKSTT10] to construct a set of modes which gives the energy cascade for the approximated model



- The set they construct is a small modification of the one in the picture.
- The rectangle make transfer of energy from two vertices to the other two.

The finite dimensional set of modes

Summarizing:

- Fix $N \gg 1$
- We construct a finite set $\Lambda \subset \mathbb{Z}^2$ such that
  - It can be split into pairwise disjoint generations $\Lambda = \cup_{j=1}^{N} \Lambda_j$.
  - Each generation has 2^N modes.
  - Only neighboring generations interact.
- Choose the same initial condition for all the modes in each generation. Then, they remain equal under time evolution (for the flow of the approximated model).

The finite dimensional toy model

- Define

$$b_j = a_n \qquad \text{for any} \qquad n \in \Lambda_j$$

  equations of motion become

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right), \; j = 1, \dots N,$$

- It can be seen as a Hamiltonian system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.
- Hamiltonian:

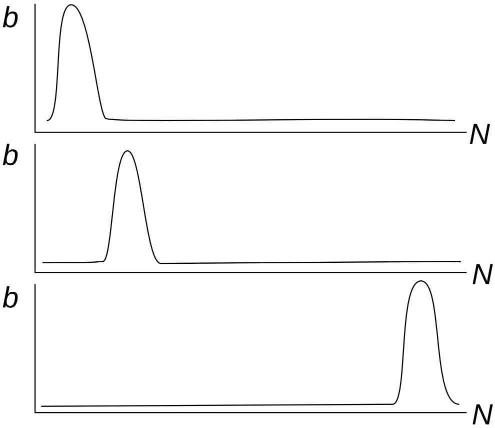
$$h(b) := \frac{1}{4}\sum_{j=1}^{N} |b_j|^4 - \frac{1}{2}\sum_{j=1}^{N}\left(\overline{b}_j^2 b_{j-1}^2 + b_j^2\overline{b}_{j-1}^2\right)$$

The finite dimensional toy model

- The mass conservation now becomes conservation of

$$\mathcal{M}(b) = \sum_{j=1}^{N} |b_j|^2.$$

- We want to look for an orbit $b(t)$ of the toy model such that at $t = 0$ is essentially localized in b_1 and at a certain $t = T \gg 1$ is essentially localized in b_N.



- To estimate the time of instability, we want to obtain this orbit with quantitative estimates with respect to the number of modes N.

The toy model theorem

Theorem

Fix $\gamma \gg 1$. Then, for $N \gg 1$ large enough and $\delta = e^{-\gamma N}$, there exists an orbit of the toy model and $T_0 > 0$ such that

$$\begin{array}{c} |b_1(0)| > 1 - \delta \\ |b_j(0)| < \delta \quad \text{for } j \neq 1 \end{array} \quad \text{and} \quad \begin{array}{c} |b_N(T_0)| > 1 - \delta \\ |b_j(T_0)| < \delta \quad \text{for } j \neq N. \end{array}$$

Moreover, T_0 satisfies

$$T_0 \sim N \ln\left(\frac{1}{\delta}\right) \sim N^2.$$

Dynamics of the toy model

- We analyze the dynamics of the toy model

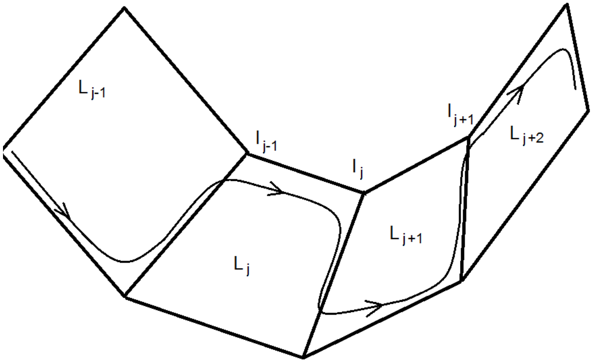
$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0, \dots N,$$

- Each 4-dimensional plane

$$L_j = \{b_1 = \cdots = b_{j-1} = b_{j+2} = \cdots = b_N = 0\}$$

  is invariant.
- The dynamics in L_j is given by a simple Hamiltonian

$$h_j(b_j, b_{j+1}) = \frac{1}{4}\left(|b_j|^4 + |b_{j+1}|^4\right) - \frac{1}{2}\left(b_j^2\overline{b}_{j+1}^2 + \overline{b}_j^2 b_{j+1}^2\right).$$



- We construct solutions that stay close to the planes $\{L_j\}_{j=2}^{N-1}$ and go from one intersection $l_j = L_j \cap L_{j+1}$ to the next one $l_{j+1} = L_{j+1} \cap L_{j+2}$ consequently for $j = 3, \ldots, N-1$.
- In the intersections l_j only b_j is nonzero.
- The planes L_j have normal positive Lyapunov exponents.

Dynamics in L_j

- To construct such orbits, we need to understand the dynamics in each L_j.
- The Hamiltonian h_j and $\mathcal{M}_j(b_j, b_{j+1}) = |b_j|^2 + |b_{j+1}|^2$ are first integrals in involution in L_j.
- This implies that the system in L_j is integrable.
- Away from degeneracies, by Liouville-Arnol'd Theorem, the phase space is foliated by 2-dim. invariant tori.

- It also contains two periodic orbits

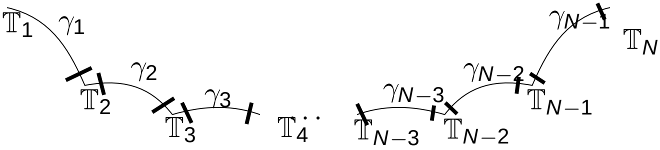
$$\mathbb{T}_j = \{(b_j(t), b_{j+1}(t)) = (e^{-i(t-t_0)}, 0)\}$$

and

$$\mathbb{T}_{j+1} = \{(b_j(t), b_{j+1}(t)) = (0, e^{-i(t-t_0)})\}.$$

- These periodic orbits in L_j are hyperbolic.
- The stable and unstable invariant manifolds of the periodic orbits coincide.
- Call γ_j to the two dimensional manifold asymptotic to $\mathbb{T}_j$ as $t \to -\infty$ and asymptotic to $\mathbb{T}_{j+1}$ as $t \to \infty$.

Key problem: the shadowing



- We put sections transversal to the flow.
- We study:
  - Local maps: study the dynamics close to the periodic orbits $\mathbb{T}_j$.
  - Global maps: study the dynamics close to the heteroclinic connections γ_j.

The local and global maps

- Shadowing for the global map is basically applying (refined) Gronwall estimates.
- The local map is more delicate:
  - The periodic orbits $\mathbb{T}_j$ are of mixed type: hyperbolic and elliptic eigenvalues.
  - Hyperbolic eigenvalues:

$$\lambda, \lambda, -\lambda, -\lambda, \quad \text{for certain } \lambda > 0.$$

  - The resonance complicates the analysis of the local map.

The resonance

- Tipically the orbits deviate from the heteroclinic connections.
- Travel close to $N-1$ periodic orbits: small deviations propagate and become extremely big after $N-1$ periodic orbits.
- This leads to a time for the growth of Sobolev norms

$$T \sim C^{\mathcal{K}^\alpha}, \quad C > 0, \alpha \geq 2.$$

- We restrict the domain of the local map to very particular initial conditions.
- They lead to a cancellation between the resonant terms.
- Orbits stay close to the heteroclinic connections.

Composing the local and the global maps

- We need to compose the local and global maps $\mathcal{B}^j$.
- We define sets $\mathcal{U}_j$ in the transversal sections and we check

$$\mathcal{B}^j(\mathcal{U}_j) \subset \mathcal{U}_{j+1}$$

- To avoid deviations at each local map, we need to impose a restriction at every step.
- To prove that the restrictions are compatible, we consider sets with a product-like structure.

Product-like structure sets

Roughly speaking:

- We start with a product set

$$\mathcal{U}_1 = B(r_1) \times \dots B(r_N)$$

where

$$B(r) = \{|z| < r\}$$

- At each step, we impose a condition on the mode b_{j-1}.
- Inductively, we restrict the domain to

$$\mathcal{U}_j = N(r_1) \times \dots \times N(r_{j-2}) \times N(r_{j-1}) \times B(r_j) \dots B(r_N)$$

with

$$N(r_{j-1}) = B(r_{j-1}) \cap \{g(\mathrm{Re}\, b_{j-1}, \mathrm{Im}\, b_{j-1}) = 0\}$$

- Since the restrictions involve a different mode at each step, the conditions are compatible.

Composing the local and the global maps, we obtain the already stated result.

Theorem

Fix $\gamma \gg 1$. Then, for $N \gg 1$ large enough and $\delta = e^{-\gamma N}$, there exists an orbit of the toy model and $T_0 > 0$ such that

$$\begin{array}{c} |b_1(0)| > 1-\delta \\ |b_j(0)| < \delta \ \ \text{for } j \neq 1 \end{array} \quad \text{and} \quad \begin{array}{c} |b_N(T_0)| > 1-\delta \\ |b_j(T_0)| < \delta \ \ \text{for } j \neq N. \end{array}$$

Moreover, T_0 satisfies

$$T_0 \sim N \ln\left(\frac{1}{\delta}\right) \sim N^2.$$

Approximating solutions of NLS

- Last step: Obtain a solution of NLS close to the solution of the toy model.
- We modify the set Λ from the I-team so that the modes out of Λ only get influenced by few modes in Λ.
- Each b_j is excited in a short period of time.
- A mode out of Λ only receives mass from Λ during a short time.
- Therefore the spreading of mass to modes out of Λ is very slow.
- We obtain an orbit for NLS that undergoes the growth of Sobolev norms in polynomial time.