Nearly integrable systems with orbits accumulating to KAM tori

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Nearly integrable Hamiltonian systems

- Hamiltonian

$$H(\varphi, I) = H_0(I) + \varepsilon H_1(\varphi, I)$$

  where
  - $\varphi \in \mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ are the angles.
  - $I \in V \subset \mathbb{R}^n$ are the actions.
  - $\varepsilon \ll 1$.
- Ergodic hypothesis (Maxwell, Boltzmann): Consider a typical H and a typical energy surface. Almost every point has a dense orbit (in the energy surface).
- (Not only for nearly integrable systems.)
- Disproved by KAM Theory in the 60's

The quasiergodic hypothesis

- Quasiergodic hypothesis (Ehrenfest, Birkhoff): A typical H in a typical energy surface has a dense orbit.
- Disproved by Herman for systems in $\mathbb{T}^{2n} \times [-\delta, \delta]^2$ (existence of codimension 1 tori).
- In systems in $\mathbb{T}^n \times \mathbb{R}^n$: Completely open, believed to be true and out of reach.
- Results for a weak form of such conjecture: orbits accumulating densely in a set of (large) positive measure.
- We deal with the nonautonomous case

$$H(\varphi, I, t) = H_0(I) + \varepsilon H_1(\varphi, I, t), \quad 2\pi - \text{periodic in } t$$

Integrable Hamiltonian systems: $\varepsilon = 0$

- Equations

$$\dot{\varphi} = \partial_I H_0(I)$$
$$\dot{I} = 0$$

- Orbits confined in tori $\mathbb{T}^{n+1} = \{I = I_0\}$.
- Dynamics in these tori is a rigid rotation with frequency $(\omega(I_0), 1) = (\partial_I H_0(I_0), 1)$.
- QEH not possible for these systems.
- What happens for $\varepsilon > 0$ small?
- Stability: KAM

Instability: QEH and Arnold Diffusion

- Is QEH typically true?
- Actions do not move when $\varepsilon = 0$ and we want them to cover the action space densely.
- Easier question (not easy!): Are there orbits whose action component $I(t)$ makes a change independent of ε?

$$\|I(T) - I(0)\| \geq 1$$

- This is called Arnold Diffusion (Arnold 1964).
- Hypotheses:
  - H_0 is strictly convex: $D^{-1}\mathrm{Id} \leq \partial_I^2 H_0(I) \leq D\,\mathrm{Id}$ for some $D > 1$.
  - H_1 is $\mathcal{C}^r$ for some r large enough.
  - H_0 is $\mathcal{C}^{r+3}$.
- Convexity implies that $\omega(I) = \partial_I H_0(I)$ is a global diffeo.

Stability: KAM Theory

- $(\omega, 1) \in U \subset \mathbb{R}^{n+1}$ is (η, τ)-Diophantine if

$$|(\omega, 1) \cdot k| \geq \frac{\eta}{|k|^{n+\tau}}, \quad \text{for all } k \in \mathbb{Z}^{n+1} \setminus \{0\}$$

- Call $\mathcal{D}_{\eta,\tau}$ to the set of such frequencies.
- For any $\tau > 0$, $\mathrm{Meas}(\mathcal{D}_{\eta,\tau}) = 1 - \mathcal{O}(\eta)$.

Theorem (Kolmogorov-Arnold-Moser, Pöschel)

Fix $\eta > 0$, $\tau > 0$. There exists $\varepsilon_0(\mathrm{KAM}) > 0$, such that for all $\varepsilon \in (0, \varepsilon_0(\mathrm{KAM}))$, all tori with frequency in $\mathcal{D}_{\eta,\tau}$ persist for $H_0 + \varepsilon H_1$.

- Call $\mathrm{KAM}_{\eta,\tau}$ to the union of these tori: $\mathrm{Meas}(\mathrm{KAM}_{\eta,\tau}) = 1 - \mathcal{O}(\eta)$.
- One can take $\eta \sim \sqrt{\varepsilon}$.
- Here we stick to fixed $\eta > 0$.

Existence of instabilities

- It depends on the dimension.
- 2 degrees of freedom
  - Energy surface has dimension 3 and KAM tori dimension 2.
  - KAM tori act as a barrier: actions almost constant for all orbits and all time.
  - QEH and Arnold diffusion are not possible
- More than 2 degrees of freedom: QEH and Arnold diffusion expected to be typically true.
- From now on we focus on $2\frac{1}{2}$ degrees of freedom (2 dof plus periodic time dependence).

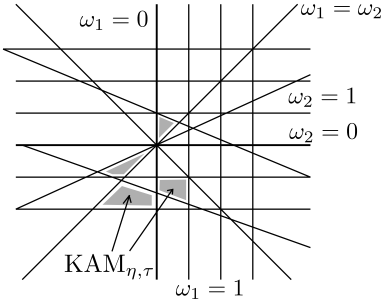
Arnold Diffusion

- Orbits with a change in actions independent of ε in the complement of $\mathrm{KAM}_{\eta,\tau}$.
- Recent results by P. Bernard, V. Kaloshin and K. Zhang.

- Resonance in frequency space: Fix $k \in \mathbb{Z}^3$

$$\Gamma_k = \{(\omega, 1) \cdot k = 0\}.$$

- Main idea: drift along resonances.



Bernard-Kaloshin-Zhang results

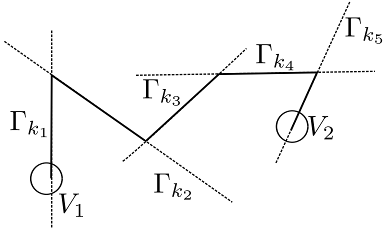
- $2\frac{1}{2}$ dof Hamiltonian

$$H(\varphi, I, t) = H_0(I) + \varepsilon H_1(\varphi, I, t), \quad 2\pi - \text{periodic in } t$$

- Fix two open sets V_1, V_2 in action space and a finite sequence of resonances $\{\Gamma_{k_j}\}_{j=1}^{j=N}$ such that

$$V_1 \cap \Gamma_{k_1} \neq \emptyset, \quad \Gamma_{k_j} \cap \Gamma_{k_{j+1}} \neq \emptyset,\ j = 1, \ldots, N-1, \quad V_2 \cap \Gamma_{k_N} \neq \emptyset$$

- We have a resonant path from V_1 and V_2



Bernard-Kaloshin-Zhang results

Theorem

Consider H_0, two sets V_1, V_2 and a resonant path from V_1 to V_2. Then, for a "$\mathcal{C}^r$-typical" H_1 and $\varepsilon \ll 1$, there exists an orbit $(\varphi(t), I(t), t)$ of $H = H_0 + \varepsilon H_1$ and a time $T > 0$ such that

$$I(0) \in V_1 \qquad \text{and} \qquad I(T) \in V_2.$$

- The orbit drifts along the resonant path: $\mathrm{dist}\left(I(t), \bigcup_{j=1}^{N} \Gamma_{k_j}\right) \lesssim \sqrt{\varepsilon}$ for all $t \in [0, T]$.
- Similar results anounced by J. Mather, J. P. Marco, C. Cheng.

ρ-density and density

- Alternative version of the theorem: Fix $\rho > 0$. Then for $\varepsilon \ll 1$, $H_0 + \varepsilon H_1$ has a ρ-dense orbit (its ρ-neighborhood covers the phase space).
- For QEH we want true density instead of ρ-density.
- Nevertheless if we take $\rho \to 0$, we need $\varepsilon \to 0$.
- Weak form of QEH: obtain an orbit accumulating densely to a set of positive (large) measure.
- Kaloshin-Zhang-Zheng: An example of nearly integrable system with an orbit covering densely a set of positive measure.

Weak form of QEH

- Recall $\mathrm{KAM}_{\eta,\tau}$ is union of tori with Diophantine frequency in $\mathcal{D}_{\eta,\tau}$

Theorem (G.-Kaloshin)

There exists r_0 such that for any $\eta > 0$, $\tau > 0$ small and any $r \geq r_0$ there is a dense set

$$\mathcal{A} \subset \mathcal{S}^r = \{\|H_1\|_{\mathcal{C}^r} = 1\}$$

such that for any $H_1 \in \mathcal{A}$ there exists ε small enough such that there exists an orbit $(\varphi(t), I(t), t)$ of $H_0 + \varepsilon H_1$ satisfying

$$\mathrm{KAM}_{\eta,\tau} \subset \overline{\bigcup_{t\in\mathbb{R}} (\varphi(t), I(t), t)}$$

Therefore $\mathrm{Meas}\left(\overline{\bigcup_{t\in\mathbb{R}} (\varphi(t), I(t), t)}\right) \geq 1 - c\eta$ for some constant $c > 0$ independent of η and ε.

Some remarks

- $\varepsilon < \varepsilon_0(\mathrm{KAM})$.
- Corollary: KAM tori are unstable (also recent results by J. Zhang and Cheng).
- Recall that these tori are Lagrangian (they do not have invariant manifolds).
- With our methods we can probably get $\mathrm{Meas} \geq 1 - c\varepsilon^\alpha$ for some $\alpha < 1/2$.
- Also true for 3 dof in the region $\partial_{I_3} H_0 \geq \nu > 0$: orbits accumulating in a set of positive measure in the energy level.
- We only get a $\mathcal{C}^r$-density result.

Robustness of the result

- Our result is very related to instability of elliptic points.
- R. Douady: instability of elliptic points is a flat phenomenon.
- Take $f_0 \in \mathcal{C}^\infty$ a symplectic mapping (non degenerate) with an elliptic point at the origin.
- Then, $\exists f, g$ such that
  - $f - f_0, g - f_0$ are flat at the origin.
  - The origin is Lyapunov unstable for f and stable for g.
- Lyapunov stability is not an open property.
- Only open if we take perturbations H_1 supported away from $\mathrm{KAM}_{\eta,\tau}$.

Robustness of the result

- Consider the $\mathcal{C}^r$-Whitney topology for H_1's vanishing on $\mathrm{KAM}_{\eta,\tau}$ with some (strong) decay.
- Then if $H_0 + \varepsilon H_1$ satisfies the theorem, so does any Hamiltonian $H_0 + \varepsilon H_1 + \varepsilon \Delta H_1$ with ΔH_1 small with respect to this $\mathcal{C}^r$-Whitney topology.
- Equivalently, take $\Delta H_1 = f \cdot \widetilde{\Delta H_1}$ with $f|_{\mathrm{KAM}_{\eta,\tau}} = 0$ with strong decay.
- Then, the theorem is true for $\widetilde{\Delta H_1}$ in a small ball with respect to the usual $\mathcal{C}^r$ topology.

Some ideas of the proof

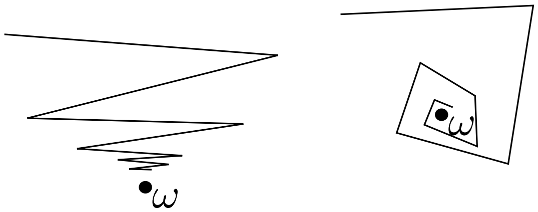
- Construct a set of resonant segments which contain $\mathcal{D}_{\eta,\tau}$ in its closure.
- Adapt the result of Bernard-Kaloshin-Zhang to drift along resonances in a small neighborhood of KAM tori.

Approaching one Diophantine frequency

- Dirichlet theorem: Fix $\omega \in \mathbb{R}^2$, $R \gg 1$. There exists $k \in \mathbb{Z}^3 \setminus \{0\}$, $|k| \leq R$ such that

$$|(\omega, 1) \cdot k| \leq R^{-2}$$

- We need to modify it to avoid angle between segments $\to 0$.

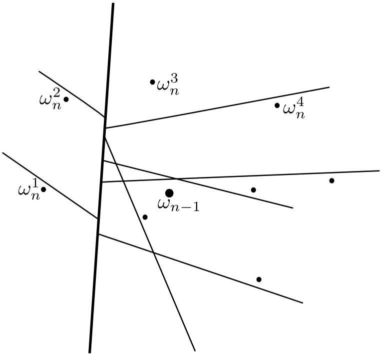


- We cannot apply this idea to all Diophantine frequencies at the same time (we cannot control the intersections between resonant segments).

The tree of resonant segments

- We construct a tree of resonances approaching all frequencies in $\mathcal{D}_{\eta,\tau}$
- Take sequence $\rho_{n+1} = \rho_n^{1+2\tau}$, $\rho_0 \ll 1$.
- By Vitali Covering lemma: take a sequence of grids $\mathcal{D}^n_{\eta,\tau} \subset \mathcal{D}_{\eta,\tau}$, $n \geq 1$ such that
  - $3\rho_n$-balls of $\omega \in \mathcal{D}^n_{\eta,\tau}$ cover $\mathcal{D}_{\eta,\tau}$.
  - ρ_n-balls of $\omega \in \mathcal{D}^n_{\eta,\tau}$ are disjoint.
- We have $\mathcal{D}_{\eta,\tau} \subset \overline{\cup_{n\geq 1}\mathcal{D}^n_{\eta,\tau}}$.
- We make approximation by generations.

The tree of resonant segments

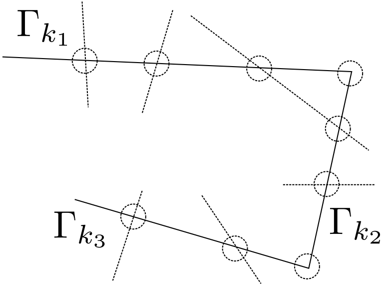


- Generation 0: horizontal and vertical resonant segments which are ρ_0 close.
- Generation $n \geq 1$: we construct resonances
  - ρ_n-close to each $\omega \in \mathcal{D}^n_{\eta,\tau}$.
  - connected to the resonances of the previous generation.
- $\mathcal{D}_{\eta,\tau}$ belongs to the closure of the resonances tree.

Drifting along resonances: Bernard-Kaloshin-Zhang construction

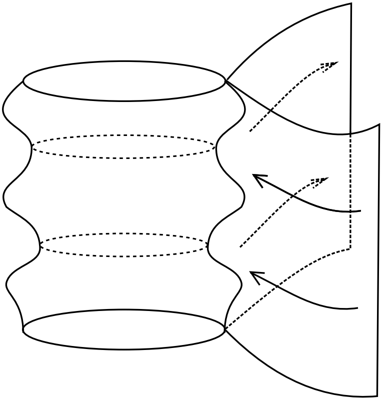
Two regimes

- Single resonance: one resonant relation $\Gamma_k = \{\omega : (\omega, 1) \cdot k = 0\}$.
- Double resonance: ω such that $\exists k_1, k_2$ such that $(\omega, 1) \cdot k_i = 0$.
- Double resonances are dense in the single resonant lines.
- Only matter if k_1 and k_2 are of similar size.



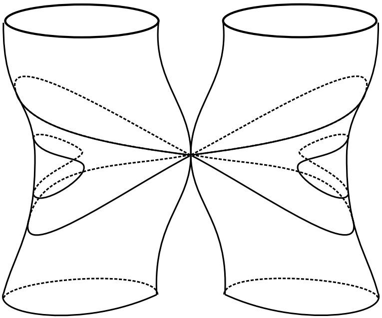
Split the path between single and strong double resonances.

Single resonance regime



- There exists a normally hyperbolic invariant cylinder along the resonance.
- To prove this fact:
  - Normal forms.
  - Normally hyperbolic invariant manifolds theory.
- By adding a small perturbation, the invariant manifolds of the cylinder intersect transversally.

Double resonance regime: the kissing cylinders



- We have a net of cylinders.

- The single resonance cylinders arrive at double resonances.
- Some of them cross the resonance.
- To prove existence of these cylinders
  - Several normal forms as before.
  - Variational methods: we reduce the system to a geodesic flow of a Finsler metric.

Shadowing by variational methods

- We use Mather-Fathi-Bernard variational techniques.
- Single resonance: we drift along the cylinders.
- Double resonance: we use the cylinders that cross it to
  - Cross double resonances: go through one cylinder.
  - Make a turn: jump from one cylinder to another one.
- We analyze the Aubry-Mather sets that belong to the different cylinders.
- There are orbits which shadow these Aubry-Mather sets.