Growth of Sobolev Norms in the Cubic Nonlinear Schrödinger Equation with a Convolution Potential

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Abstract: Fix $s > 1$. Colliander et al. proved in (Invent Math 181:39–113, 2010) the existence of solutions of the cubic defocusing nonlinear Schrödinger equation in the two torus whose s-Sobolev norm undergoes arbitrarily large growth as time evolves. In this paper we generalize their result to the cubic defocusing nonlinear Schrödinger equation with a convolution potential. Moreover, we show that the speed of growth is the same as the one obtained for the cubic defocusing nonlinear Schrödinger equation in Guardia and Kaloshin (Growth of Sobolev norms in the cubic defocusing Nonlinear Schrödinger Equation. To appear in the Journal of the European Mathematical Society, 2012).

1. Introduction

The purpose of this paper is to study the growth of Sobolev norms for the periodic cubic defocusing nonlinear Schrödinger equation with a convolution potential,

$$\begin{cases} -i\partial_t u + \Delta u + V(x) * u = |u|^2 u \\ u(0, x) = u_0(x), \end{cases} \tag{1}$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$, $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$ and $V \in H^{s_0}(\mathbb{T}^2)$, $s_0 > 0$, with real Fourier coefficients. This equation is globally well posed in time in any Sobolev space H^s with $s \geq 1$. Indeed, for the case $V = 0$, the local-in-time well-posedness for any $u_0 \in H^s(\mathbb{T}^2)$, $s > 0$ was proven by Bourgain [Bou93]. His result can be generalized to the equation with a convolution potential using the Trotter product formula (see for instance [EM70]). This along with the two conservation laws, implies the existence of a smooth solution (1) for all time.

If we write the Fourier series

$$u(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t)e^{inx} \qquad \text{and} \qquad V(x) = \sum_{n \in \mathbb{Z}^2} v_n e^{inx},$$

equation (1) becomes an infinite dimensional ordinary differential equation for the Fourier coefficients a_n,

$$-i\dot{a}_n = \left(|n|^2 + v_n\right) a_n + \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a_{n_2}} a_{n_3}. \qquad (2)$$

Note that the assumption that V has real Fourier coefficients implies that for this equation $a = 0$ is an elliptic critical point.

Equation (2) is Hamiltonian since it can be written as

$$\dot{a}_n = 2i \partial_{\overline{a_n}} \mathcal{H}(a, \overline{a}),$$

where

$$\mathcal{H}(a, \overline{a}) = \mathcal{D}(a, \overline{a}) + \mathcal{G}(a, \overline{a}) \qquad (3)$$

with

$$\mathcal{D}(a, \overline{a}) = \frac{1}{2} \sum_{n \in \mathbb{Z}^2} \left(|n|^2 + v_n\right) |a_n|^2$$

$$\mathcal{G}(a, \overline{a}) = \frac{1}{4} \sum_{\substack{n_1, n_2, n_3, n_4 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n_4}} a_{n_1} \overline{a_{n_2}} a_{n_3} \overline{a_{n_4}}.$$

Besides $\mathcal{H}$, Eq. (2) has another conserved quantity,

$$\mathcal{M}(a) = \sum_{n \in \mathbb{Z}^2} |a_n|^2. \qquad (4)$$

It is usually called mass and is just the square of the ℓ^2-norm of the sequence $\{a_n\}_{n \in \mathbb{Z}^2}$, which coincides with the L^2 norm of u.

The problem of growth of s-Sobolev norms in Hamiltonian Partial Differential Equations (PDE) has drawn a wide attention in the past decades and was considered by Bourgain one of the next century's problems in Hamiltonian PDE [Bou00]. The importance of this phenomenon is that solutions undergoing a large growth of s-Sobolev norm with $s > 1$ are solutions which, as time evolves, transfer energy to higher and higher modes. The s-Sobolev norm is defined by

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t, \cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n \in \mathbb{Z}^2} \langle n \rangle^{2s} |a_n|^2\right)^{1/2},$$

where $\langle n \rangle = (1 + |n|^2)^{1/2}$. It follows from conservation of the energy $\mathcal{H}(a, \overline{a})$ that the H^1-norm of any solution of (1) is uniformly bounded. Therefore, if the H^s-norm of a solution grows indefinitely for some given $s > 1$ while the H^1-norm stays bounded, it is clear that the energy of the solution of (1) must be transferred to higher modes.

In [Bou96] (see also [Sta97a]), Bourgain considered the cubic defocusing nonlinear equation

$$\begin{cases} -i\,\partial_t u + \Delta u = |u|^2 u \\ u(0, x) = u_0(x) \end{cases} \tag{5}$$

and obtained upper bounds for the possible growth of Sobolev norms. More concretely, he proved that

$$\|u(t)\|_{H^s} \le t^{C(s-1)}\|u(0)\|_{H^s} \quad \text{for } t \to \infty.$$

However, he did not obtain orbits achieving such growth. In [Bou00], Bourgain posed the following question,

Are there solutions of (5) *with periodic boundary conditions in dimension* 2 *or higher with unbounded growth of H^s-norm for $s > 1$?*

Moreover, he conjectured, that in case this is true, the upper bound that he had obtained in [Bou96] was not optimal and that the growth should be subpolynomial in time, that is,

$$\|u(t)\|_{H^s} \ll t^{\varepsilon}\|u(0)\|_{H^s} \quad \text{for} \quad t \to \infty, \text{ for all } \varepsilon > 0.$$

The question posed by Bourgain is still open. Nevertheless, in the past decades there have been several results in the area, which can be classified in two groups. The first group of works obtains upper bounds for the possible growth of s-Sobolev norms without showing that this growth exists. Such results have been obtained for several Hamiltonian Partial Differential Equations (nonlinear Schrödinger equation with different nonlinearities, nonlinear wave equation, Hartree equation,…) defined in different manifolds (see [Sta97b, CDKS01, Bou04, Zho08, CW10, Soh11a, CKO12, Bou96, Soh11b, Soh12]).

The second group of results is devoted to the obtention of solutions undergoing growth of s-Sobolev norms. Most of the results show arbitrarily large, but finite, growth, as happens in this paper. The first result showing large finite growth is also due to Bourgain. In [Bou96], he constructs orbits with arbitrarily large growth of Sobolev norms for the wave equation with a cubic nonlinearity but with a spectrally defined Laplacian. Similar results are obtained in [GG12, Poc13] for certain nonlinear half-wave equation. The only results dealing with unbounded growth are [GG10, Poc11], which deal with the Szegö equation, which is integrable, and [Han11, Han12], where unbounded growth is shown to exist for a pseudo partial differential equation which is a simplification of (5). Related to this second group of results, there is also [CF12], where a spreading of energy among the modes for certain solutions of (5) is shown. Nevertheless, this spreading does not lead to growth of Sobolev norms.

Concerning growth of Sobolev norms for the nonlinear Schrödinger equation (5), the first result is due to Kuksin. In [Kuk97], he proves the existence of solutions with any prescribed growth of s-Sobolev norm taking initial data large enough (depending on the prescribed growth). To obtain this result he takes advantage of the fact that for large data the nonlinearity in (5) plays a more significant role than the dispersion.

In the present paper we are rather interested in showing growth of Sobolev norms for solutions with small initial data. That is, for solutions close (in some topology) to the solution $u = 0$. From the dynamical systems point of view, $u = 0$ is an elliptic critical point and therefore, showing growth of Sobolev norms for small initial data means that the critical point is unstable in the Sobolev spaces H^s, $s > 1$. The first paper dealing with such setting is the recent paper [CKS+10]. In this paper, the following result is obtained.

Theorem 1.1 ([CKS+10]). *Fix $s > 1$, $\mathcal{C} \gg 1$ and $\mu \ll 1$. Then, there exists a global solution $u(t) = u(t, \cdot)$ of* (5) *and T satisfying that*

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

Note that the initial Sobolev norm gives bounds for the mass and energy of the solution u, which are constant as time evolves, and therefore are small for all time.

The paper [CKS+10] does not give estimates for the time T with respect to the growth of the Sobolev norms, namely estimates of the speed of the growth. These estimates have been obtained recently by the author and Vadim Kaloshin in [GK12]. This paper contains the following two results.

Theorem 1.2 ([GK12]). *Let $s > 1$. Then, there exists $c > 0$ with the following property: for any large $\mathcal{K} \gg 1$ there exists a a global solution $u(t) = u(t, \cdot)$ of* (5) *and a time T satisfying*

$$0 < T \leq \mathcal{K}^c$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{K} \, \|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \leq \mathcal{K}^{-\sigma}.$$

for some $\sigma > 0$ independent of $\mathcal{K}$.

Note that this theorem does not give any information of the initial Sobolev norm but only on its growth. Nevertheless, we want to emphasize that it is dealing with small data since the L_2-norm of the solution is very small. One can impose also that the solution has small initial s-Sobolev norm at the expense of obtaining a slower growth.

Theorem 1.3 ([GK12]). *Let $s > 1$. Then, there exists $c > 0$ with the following property: for any small $\mu \ll 1$ and large $\mathcal{C} \gg 1$ there exists a a global solution $u(t) = u(t, \cdot)$ of* (5) *and a time T satisfying*

$$0 < T \leq \left(\frac{\mathcal{C}}{\mu}\right)^{c \ln(\mathcal{C}/\mu)}$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{C} \quad \text{and} \quad \|u(0)\|_{H^s} \leq \mu.$$

The purpose of the present paper is to show that the stated results from [CKS+10] and [GK12] are also valid if one adds a convolution potential term to (5). Namely, if one considers Eq. (1). The choice of a convolution potential instead of the classical multiplicative potential is, as usual, due to the fact that the term $V(x) * u$ is diagonal in the Fourier basis. This fact simplifies the problem considerably. The existence of solutions of (1) with a multiplicative potential instead of a convolution one undergoing growth of Sobolev norms is still not known.

In this paper, we show that the instability mechanism developed in [CKS+10, GK12] is also valid, with some modifications, for (1) and that therefore for this equation there

also exist solutions with arbitrarily high, but finite, growth of Sobolev norm. Moreover, we obtain the same time estimates as the ones given by Theorems 1.2 and 1.3. Note that the cited results showing growth of Sobolev norms dealt with particular equations. Therefore, it was not clear whether the mechanisms were still valid if one slightly modified the equation. That is, how robust the growth of Sobolev norms was. A positive answer to this question is given in the present paper for the mechanism developed in [CKS+10] and [GK12], since it shows that it is still valid when one adds a convolution potential. We want to emphasize that we obtain results for *any potential*. That is, we do not need any smallness condition. We do not need either any non-resonant condition on its Fourier coefficients as usually happens in stability and KAM results (see for instance [BG03,BG06,Bou98,GL10,EK10]). This implies that the reached growth of Sobolev norms is valid for open sets of potentials.

The main results of this paper are the following.

Theorem 1.4. *Let $s_0 > 0$ and $s > 1$ and take $V \in H^{s_0}(\mathbb{T}^2)$ with real Fourier coefficients. Then, there exists $c > 0$ with the following property: for any large $\mathcal{K} \gg 1$ there exists a global solution $u(t) = u(t, \cdot)$ of (1) and a time T satisfying*

$$0 < T \leq \mathcal{K}^c$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{K} \|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \leq \mathcal{K}^{-\sigma}$$

for some $\sigma > 0$ independent of $\mathcal{K}$.

We can also impose initial small Sobolev norm as in Theorem 1.3. Nevertheless, then we obtain a slower growth, as happened in that theorem, and we need to add the extra hypothesis that the potential satisfies $V \in H^{s_0}$ with $s_0 > 70s/17$ instead of $s_0 > 0$. This hypothesis is certainly not optimal. Nevertheless, it is needed with our methods to impose the smallness of the initial Sobolev norms (see Sect. 6).

Theorem 1.5. *Fix $s > 1$ and Let $s_0 > 70s/17$ and take $V \in H^{s_0}(\mathbb{T}^2)$ with real Fourier coefficients. Then, there exists $c > 0$ with the following property: for any small $\mu \ll 1$ and large $\mathcal{C} \gg 1$ there exists a global solution $u(t) = u(t, \cdot)$ of (1) and a time T satisfying*

$$0 < T \leq \left(\frac{\mathcal{C}}{\mu}\right)^{c \ln(\mathcal{C}/\mu)}$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{C} \quad and \quad \|u(0)\|_{H^s} \leq \mu.$$

Remark 1.6. In Theorems 1.4 and 1.5, the obtained initial conditions u_0, whose trajectories $u(t)$ attain growth of Sobolev norms, depend on the chosen potential V. Nevertheless, from the proofs of these theorems one can easily see that u_0 do not depend fully on the potential V but only on its norm $\|V\|_{H^{s_0}}$. Namely, if we fix $R > 0$, there exists an initial condition u_0 such that if we take any potential $V \in B_{s_0}(R) = \{V \in H^{s_0}(\mathbb{T}^2) : \|V\|_{H^{s_0}} \leq R\}$, then the trajectory of u_0 for Eq. (1) with this potential attains the desired growth of Sobolev norms.

Remark 1.7. As happened with the results in [CKS+10] and [GK12], even if the just stated theorems are for Eq. (5) in the two torus, they can be applied to the d dimensional torus with $d \geq 2$, since the obtained solutions are also solutions of Eq. (5) in the $\mathbb{T}^d$ setting all the other harmonics to zero.

From the dynamical systems point of view, as happened for Eq. (5), $u = 0$ is an elliptic critical point for Eq. (1). For Eq. (5), this critical point is extremely resonant since its eigenvalues are $\lambda_n^\pm = \pm i |n|^2$ [see Eq. (2)]. It is well known that resonances are a source of instabilities and the mechanism developed in [CKS+10] uses them to obtain the growth of Sobolev norms. When one adds the potential and considers Eq. (1), these resonances typically break down. Thus, one might expect that, for well chosen potentials, the growth of Sobolev norms is slower than the one obtained for (5) and, therefore, that the critical point $u = 0$ is stable for longer time. Theorems 1.4 and 1.5 show that this expected longer stability time does not hold and that the instability phenomena of (1) have the same speed as the ones obtained for (5). The reason behind this fact is that for Eq. (1), since $V \in H^{s_0}$, $s_0 > 0$, the eigenvalues of $u = 0$ satisfy

$$\lambda_n^\pm = \pm i \left(|n|^2 + v_n \right) = \pm i |n|^2 + \mathcal{O}\left(\frac{1}{|n|^{s_0}} \right)$$

and therefore, for high enough modes, the potential contribution to the eigenvalues is negligible. Then, for large enough modes, even if the potential might kill the resonances, one has *almost resonant* terms and one can use them to obtain the growth of Sobolev norms (see Sect. 2 for more details).

The instability times obtained in Theorem 1.5 can be compared with the stability results obtained in [BG03] (see also [BG06,GL10]). These papers study the stability of the elliptic critical point $u = 0$ of (1) (for certain potentials) in the H^s topology for $s \geq s_0 > 1$ and some s_0. More concretely, in [BG03] the following result is obtained. They consider Eq. (1) with a potential V, which satisfies certain non-resonance conditions which is fulfilled by a positive measure set of potentials, and an initial condition u_0 satisfying that $\varepsilon = \|u_0\|_{H^s}$ is small enough. Then, they show that the corresponding solution satisfies

$$\|u(t)\|_{H^s} \leq 2\varepsilon \quad \text{for } |t| \leq \frac{C}{\varepsilon^M},$$

for some constants $C, M > 0$ independent of ε. That is, for a positive measure set of potentials, the elliptic critical point $u = 0$ of Eq. (1) is stable for times which are polynomially long with respect to the size of the initial condition in H^s. This implies, that the time estimates that we obtain in Theorem 1.5 are almost optimal, since the instability time cannot be shorter than the time obtained in [BG03].

Equation (1) was also considered in [Bou98,EK10,FG10]. In the first two papers, the authors prove, the existence of small quasi-periodic solutions for the nonlinear Schrödinger equation (1) in the 2-dimensional torus [Bou98] and in a torus of any dimension [EK10]. These results, jointly with Theorems 1.4 and 1.5, show that in any neighborhood of $u = 0$ there is coexistence of stable motion (KAM tori) and unstable motion (orbits undergoing growth of s-Sobolev norms). In [FG10], the authors prove a Nekhoroshev type result for solutions of (1) with small analytic initial condition. In all these papers, the authors need to assume some non-resonance condition on the potential V.

The rest of the paper is devoted to proving Theorems 1.4 and 1.5. These proofs follow the strategy developed to prove Theorem 1.1 in [CKS+10], which was refined in [GK12] to prove Theorems 1.2 and 1.3.

2. Proof of Theorems 1.4 and 1.5

To prove Theorems 1.4 and 1.5 we study Eq. (2) in the Sobolev spaces H^s with $s > 1$ as well as in the ℓ^1-space, which is defined as usual as

$$\ell^1 = \left\{ a : \mathbb{Z}^2 \to \mathbb{C} : \|a\|_{\ell^1} = \sum_{n \in \mathbb{Z}^2} |a_n| < \infty \right\}.$$

The first step is to perform some changes of variables that will simplify certain terms of Eq. (2). These changes will be obtained using the gauge invariance of the equation and performing one step of partial Birkhoff normal form. To take advantage of the gauge invariance we perform the change of coordinates

$$a_n = e^{iGt} r_n, \tag{6}$$

which transforms Eq. (2) into

$$-i\dot{r}_n = \left(|n|^2 + v_n - G \right) r_n + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} r_{n_1} \overline{r_{n_2}} r_{n_3}.$$

We choose G properly to remove some terms of the nonlinear part of the equation. We write the sum as

$$\sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} = \sum_{n_1,n_3 \neq n} + \sum_{n_1 = n} + \sum_{n_3 = n} - \sum_{n_1 = n_3 = n}.$$

The last sum is just one term given by $-|r_n|^2 r_n$. The second and third sums can be written as

$$r_n \sum_{n_2 \in \mathbb{Z}^2} |r_{n_2}|^2.$$

Recall that the ℓ^2 norm of a is a first integral of Eq. (2) and so the same happens with the ℓ^2 norm of r. Therefore, the second and third sums are equal to $2r_n \|r\|_{\ell^2}$ and choosing $G = 2\|r\|_{\ell^2}$ we can remove them. With this choice, we obtain the equation

$$-i\dot{r}_n = \left(|n|^2 + v_n \right) r_n - |r_n|^2 r_n + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n, n_1, n_3 \neq n}} r_{n_1} \overline{r_{n_2}} r_{n_3}. \tag{7}$$

It is a Hamiltonian system with respect to

$$\mathcal{H}'(r, \bar{r}) = \mathcal{D}(r, \bar{r}) + \mathcal{G}'(r, \bar{r}), \tag{8}$$

where $\mathcal{D}$ has been introduced in (3) and

$$\mathcal{G}'(r, \bar{r}) = -\frac{1}{4}|r_n|^4 + \frac{1}{4} \sum_{\substack{n_1,n_2,n_3,n_4 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n_4, n_1, n_3 \neq n_4}} r_{n_1} \overline{r_{n_2}} r_{n_3} \overline{r_{n_4}}.$$

Next step is to perform one step of partial Birkhoff normal form to reduce the size of certain terms in Eq. (7). In the normal form procedure we need to treat in a considerably different way the low and high harmonics. The reason is the following. The eigenvalues of the critical point $r = 0$ of Eq. (7) are given by

$$\lambda_n = \pm i \left(|n|^2 + v_n \right).$$

Recall that we assume that $v_n \in \mathbb{R}$ and therefore the eigenvalues are purely imaginary. Recall also that by hypothesis $V \in H^{s_0}(\mathbb{T}^2)$ and thus its Fourier coefficients decay polynomially in $|n|$. This implies that for low harmonics the term v_n makes a big influence in the eigenvalues whereas for very high harmonics, the eigenvalues λ_n are extremely close to $\pm i |n|^2$. To quantify this fact, we define a constant κ_0 such that

$$|v_n| \leq \frac{1}{100} \quad \text{for } |n| \geq \kappa_0. \tag{9}$$

Throughout the paper we call *low modes* the modes satisfying $|n| < \kappa_0$, and *high modes* the modes satisfying $|n| \geq \kappa_0$. We also use the notation

$$B(\kappa_0) = \left\{ n \in \mathbb{Z}^2 : |n| < \kappa_0 \right\}.$$

We perform a partial normal form procedure treating each monomial differently depending on whether $|n| \leq \kappa_0$ or $|n| \geq \kappa_0$. Before stating the theorem, we introduce the coefficients $\rho_{n_1 n_2 n_3 n_4}$, which are the small divisors that will arise in the normal form,

$$\rho_{n_1 n_2 n_3 n_4} = \sum_{j=1}^{4} (-1)^{j+1} \left(|n_j|^2 + v_{n_j} \right). \tag{10}$$

Theorem 2.1. *Fix $\eta \in (1/4, 1/3)$. There exists a symplectic change of coordinates $r = \Gamma(\alpha)$ in a neighborhood of 0 in ℓ^1 which transforms the Hamiltonian $\mathcal{H}'$ in (8) into the Hamiltonian,*

$$\mathcal{H}' \circ \Gamma = \mathcal{D} + \widetilde{\mathcal{G}}_1 + \widetilde{\mathcal{G}}_2 + \mathcal{R},$$

where $\widetilde{\mathcal{G}}_1$ is defined as

$$\widetilde{\mathcal{G}}_1(\alpha, \overline{\alpha}) = -\frac{1}{4} \sum_{n \in B(\kappa_0)} |\alpha_n|^4 + \frac{1}{4} \sum_{\substack{(n_1, n_2, n_3, n_4) \in \mathcal{I} \\ n_1 - n_2 + n_3 = n_4, \, n_1, n_3 \neq n_4 \\ |\rho_{n_1 n_2 n_3 n_4}| \leq \eta}} \alpha_{n_1} \overline{\alpha_{n_2}} \alpha_{n_3} \overline{\alpha_{n_4}},$$

with

$$\mathcal{I} = \left\{ (n_1, n_2, n_3, n_4) : n_i \in \mathbb{Z}^2 \text{ and at least one of the } n_i\text{'s satisfies } n_i \in B(\kappa_0) \right\},$$

$\widetilde{\mathcal{G}}_2$ is defined as

$$\widetilde{\mathcal{G}}_2(\alpha, \overline{\alpha}) = -\frac{1}{4} \sum_{n \in \mathbb{Z}^2 \setminus B(\kappa_0)} |\alpha_n|^4 + \frac{1}{4} \sum_{\substack{n_i \in \mathbb{Z}^2 \setminus B(\kappa_0), i=1,\ldots,4 \\ n_1 - n_2 + n_3 = n_4, \, n_1, n_3 \neq n_4 \\ \sum_{i=1}^{4} (-1)^i |n_i|^2 = 0}} \alpha_{n_1} \overline{\alpha_{n_2}} \alpha_{n_3} \overline{\alpha_{n_4}}$$

and $X_{\mathcal{R}}$, the vector field associated to the Hamiltonian $\mathcal{R}$, satisfies

$$\|\mathcal{X}_{\mathcal{R}}\|_{\ell^1} \leq \mathcal{O}\left(\|\alpha\|_{\ell^1}^5\right).$$

Moreover, the change Γ satisfies

$$\|\Gamma - \mathrm{Id}\|_{\ell^1} \leq \mathcal{O}\left(\|\alpha\|_{\ell^1}^3\right).$$

The proof of this theorem is postponed to Sect. 3.

Once we have used the gauge invariance and performed one step of partial Birkhoff normal form, we have a new vector field

$$
\begin{aligned}
-i\dot{\alpha}_n = {} & \left(|n|^2 + v_n\right)\alpha_n - |\alpha_n|^2\alpha_n + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_0(n)} \alpha_{n_1}\overline{\alpha_{n_2}}\alpha_{n_3} \\
& + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_1(n)} \alpha_{n_1}\overline{\alpha_{n_2}}\alpha_{n_3} + 2\partial_{\overline{\alpha}_n}\mathcal{R},
\end{aligned}
\tag{11}
$$

where the sets $\mathcal{A}_0(n)$ and $\mathcal{A}_1(n)$ are introduced in the next definition.

Definition 2.2. *We define the sets $\mathcal{A}_0(n)$ and $\mathcal{A}_1(n)$ as follows.*

- *For $n \in B(\kappa_0)$*
  (i) $(n_1, n_2, n_3) \in \mathcal{A}_0(n)$ provided $n_1 - n_2 + n_3 = n$, $n_1 \neq n$, $n_3 \neq n$, $\left|\rho_{n_1 n_2 n_3 n}\right| \leq \eta$
  (ii) $\mathcal{A}_1(n) = \emptyset$.
- *For $n \in \mathbb{Z}^2 \setminus B(\kappa_0)$*
  (i) $(n_1, n_2, n_3) \in \mathcal{A}_0(n)$ provided $n_1 - n_2 + n_3 = n$, $n_1 \neq n$, $n_3 \neq n$, $\left|\rho_{n_1 n_2 n_3 n}\right| \leq \eta$ and at least one of the n_i satisfies $n_i \in B(\kappa_0)$.
  (ii) $(n_1, n_2, n_3) \in \mathcal{A}_1(n)$ provided $n_1 - n_2 + n_3 = n$, $n_1 \neq n$, $n_3 \neq n$, $n_j \in \mathbb{Z}^2 \setminus B(\kappa_0)$, $j = 1, 2, 3$ and $|n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 \neq 0$.

To remove the linear part of this vector field, we perform a change to rotating coordinates

$$\alpha_n = e^{i(|n|^2+v_n)t}\beta_n.\tag{12}$$

Then, we obtain the vector field

$$
\begin{aligned}
-i\dot{\beta}_n = {} & -|\beta_n|^2\beta_n + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_0(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3}e^{i\rho_{n_1 n_2 n_3 n}t} \\
& + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_1(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3}e^{i\rho_{n_1 n_2 n_3 n}t} + \mathcal{R}'_n(\beta),
\end{aligned}
\tag{13}
$$

where $\rho_{n_1 n_2 n_3 n}$ has been defined in (10) and $\mathcal{R}'$ is the vector field associated to the Hamiltonian $\mathcal{R}(\{e^{i(|n|^2+v_n)t}\beta_n\})$.

We want to consider a "good" first order of this vector field. To this end we take into account that for (resonant) nonlinear terms involving only high harmonics, thanks to the decay of the Fourier coefficients of V, one can choose the coefficients $\rho_{n_1 n_2 n_3 n}$ to be very small. Thus, we rewrite this vector field as

$$
\begin{aligned}
-i\dot{\beta}_n = {} & -|\beta_n|^2\beta_n + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_0(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3}e^{i\rho_{n_1 n_2 n_3 n}t} \\
& + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_1(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3} + \widetilde{\mathcal{R}}_n(\beta),
\end{aligned}
\tag{14}
$$

where

$$\widetilde{\mathcal{R}} = \mathcal{R}' + \mathcal{J}, \tag{15}$$

with

$$\mathcal{J}_n(\beta) = \sum_{(n_1,n_2,n_3)\in\mathcal{A}_1(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3}\left(e^{i\rho_{n_1 n_2 n_3 n}t} - 1\right). \tag{16}$$

To obtain orbits undergoing growth of Sobolev norms for this vector field, we first consider the truncated vector field

$$-i\dot{\beta}_n = -|\beta_n|^2\beta_n + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_0(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3}e^{i\rho_{n_1 n_2 n_3 n}t} + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_1(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3}. \tag{17}$$

We will study certain orbits of this vector field with growth of Sobolev norms and later on we will show that close to them are orbits of (14) undergoing the same growth. To obtain such orbits for (17) we will use the instability mechanism developed in [CKS$^+$10]. In that paper, which as we have explained deals with Eq. (5), the authors select a very cleverly chosen finite set of modes. The modes in this set on the one hand do not interact with the modes out of this set and on the other hand interact between themselves in a very particular way that leads to the transfer of energy. To use this mechanism, we need to take into account that in Eq. (17) there are two types of interaction between modes. The first one is through tuples in

$$\widetilde{\mathcal{A}}_0 = \left\{(n_1, n_2, n_3, n_4) \in \left(\mathbb{Z}^2\right)^4 : (n_1, n_2, n_3) \in \mathcal{A}_0(n_4)\right\},$$

which contain at least one low mode (see Definition 2.2). The second one is through tuples in

$$\widetilde{\mathcal{A}}_1 = \left\{(n_1, n_2, n_3, n_4) \in \left(\mathbb{Z}^2\right)^4 : (n_1, n_2, n_3) \in \mathcal{A}_1(n_4)\right\},$$

which only contain high modes. As we have already mentioned, the geometry of the resonances differ considerably depending on whether $(n_1, n_2, n_3, n_4) \in \widetilde{\mathcal{A}}_0$ or $(n_1, n_2, n_3, n_4) \in \widetilde{\mathcal{A}}_1$. In the second case, since we are omitting the influence of the convolution potential, the four points form a rectangle in $\mathbb{Z}^2$. These were the resonances studied in [CKS$^+$10] (and also in [GK12]) since in these papers they consider Eq. (5) and therefore $v_n = 0$. However, now the geometry of tuples involving low harmonics changes considerably. To overcome this problem we take advantage of the fact that the mechanism developed in [CKS$^+$10] deals with orbits supported only in high harmonics and therefore they can be chosen to satisfy $|n| \geq \kappa_0$, where κ_0 is the constant in (9). Nevertheless, we need to slightly modify the support to avoid interactions between high and low harmonics.

We reduce (17) to a finite-dimensional system, which corresponds to an invariant finite-dimensional plane. We explain now the construction of this finite set of modes $\Lambda \subset \mathbb{Z}^2 \backslash B(\kappa_0)$, where κ_0 is the constant defined in (9), and impose additional conditions on Λ from the ones considered in [CKS$^+$10]. Fix $N \gg 1$. Following [CKS$^+$10] we define a set $\Lambda \subset \mathbb{Z}^2 \backslash B(\kappa_0)$ consisting of N pairwise disjoint *generations*:

$$\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_N.$$

Define a *nuclear family* to be a rectangle $(n_1, n_2, n_3, n_4) \in \widetilde{\mathcal{A}}_1$, such that n_1 and n_3 (known as the *parents*) belong to a generation Λ_j and n_2 and n_4 (known as the *children*) live in the next generation Λ_{j+1}. Note that if (n_1, n_2, n_3, n_4) is a nuclear family, then

so are (n_1, n_4, n_3, n_2), (n_3, n_2, n_1, n_4) and (n_3, n_4, n_1, n_2). These families are called trivial permutations of the family (n_1, n_2, n_3, n_4).

The conditions imposed to the set Λ in [CKS$^+$10] are

1_Λ *Closure* If $n_1, n_2, n_3 \in \Lambda$ and $(n_1, n_2, n_3) \in \widetilde{\mathcal{A}}_1(n)$, then $n \in \Lambda$. That is, if three vertices of a rectangle are in Λ so is the last fourth one.

2_Λ *Existence and uniqueness of spouse and children* For any $1 \leq j < N$ and any $n_1 \in \Lambda_j$, there exists a unique nuclear family (n_1, n_2, n_3, n_4) (up to trivial permutations) such that n_1 is a parent of this family. In particular, each $n_1 \in \Lambda_j$ has a unique spouse $n_3 \in \Lambda_j$ and has two unique children $n_2, n_4 \in \Lambda_{j+1}$ (up to permutation).

3_Λ *Existence and uniqueness of sibling and parents* For any $1 \leq j < N$ and any $n_2 \in \Lambda_{j+1}$, there exists a unique nuclear family (n_1, n_2, n_3, n_4) (up to trivial permutations) such that n_2 is a child of this family. In particular each $n_2 \in \Lambda_{j+1}$ has a unique sibling $n_4 \in \Lambda_{j+1}$ and two unique parents $n_1, n_3 \in \Lambda_j$ (up to permutation).

4_Λ *Nondegeneracy* The sibling of a frequency n is never equal to its spouse.

5_Λ *Faithfulness* Apart from the nuclear families, Λ does not contain any other rectangle.

To slow down the spreading of mass to high modes out of Λ. We impose the following condition, which was already considered in [GK12] for the resonant sets of (5).

6_Λ *No spreading to high harmonics* Let us consider $n \in \mathbb{Z}^2 \backslash \Lambda$. Then, there exists at most two tuples $(n_1, n_2, n_3, n_4) \in \widetilde{\mathcal{A}}_1$ (up to permutation) containing n and two modes belonging to Λ.

Now we have to impose the following conditions to avoid interactions between the modes of Λ and the low harmonics in $B(\kappa_0)$, which possess a completely different geometry of resonances.

7_Λ *Absence of low harmonics resonant interactions* Take $n_1, n_2, n_3 \in \Lambda$. Then, for any $n \in B(\kappa_0) \subset \mathbb{Z}^2$, one has that $(n_1, n_2, n_3, n) \notin \widetilde{\mathcal{A}}_0$ (nor any permutation of it).

8_Λ *No spreading to low harmonics* Let us consider $n \in \mathbb{Z}^2 \backslash \Lambda$. Then, there exist no tuples $(n_1, n_2, n_3, n) \in \widetilde{\mathcal{A}}_0$ containing n and two modes belonging to Λ and the same happens for any permutation of the tuple.

Theorem 2.3. *Let $\mathcal{K} \gg 1$. Then, there exists $N \gg 1$ large and a set $\Lambda \subset \mathbb{Z}^2$, with*

$$\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_N,$$

which satisfies conditions $1_\Lambda - 8_\Lambda$ and also

$$\frac{\sum_{n \in \Lambda_{N-1}} |n|^{2s}}{\sum_{n \in \Lambda_3} |n|^{2s}} \geq \frac{1}{2} 2^{(s-1)(N-4)} \geq \mathcal{K}^2. \tag{18}$$

Moreover, we can ensure that each generation Λ_j has 2^{N-1} disjoint frequencies satisfying

$$e^{17N^2} \leq |n| \leq e^{35N^2}, \quad n \in \Lambda, \tag{19}$$

and

$$\frac{\sum_{n \in \Lambda_j} |n|^{2s}}{\sum_{n \in \Lambda_i} |n|^{2s}} \leq C e^{sN} \tag{20}$$

for any $1 \leq i < j \leq N$ and some constant $C > 0$ independent of N.

The construction of such set Λ was first done in [CKS$^+$10] for (5) (see Proposition 2.1 of that paper). In [GK12] the construction was refined adding condition 6_Λ but still for Eq. (5). In Sect. 4 we show how to adapt the construction of Λ to the present setting and we prove Theorem 2.3.

As explained in [CKS$^+$10], thanks to Properties 1_Λ and 7_Λ satisfied by the set Λ, the manifold

$$\mathcal{M} = \left\{ \beta \in \mathbb{C}^{\mathbb{Z}^2} : \beta_n = 0 \text{ for all } n \notin \Lambda \right\}$$

is invariant by the flow associated to (17). This manifold has finite dimension which is equal to $|\Lambda| = N 2^{N-1}$. Equation (17) restricted to $\mathcal{M}$ reads

$$-i\dot{\beta}_n = -\beta_n |\beta_n|^2 + 2\beta_{n_{\text{child}_1}} \beta_{n_{\text{child}_2}} \overline{\beta_{n_{\text{spouse}}}} + 2\beta_{n_{\text{parent}_1}} \beta_{n_{\text{parent}_2}} \overline{\beta_{n_{\text{sibling}}}}. \tag{21}$$

for each $n \in \Lambda$. The presence of parents, children, and the sibling are guaranteed by 2_Λ and 3_Λ. In the first and last generations, the parents and children are set to zero respectively. Note that this system coincides with system (2.14) in [CKS$^+$10] and (16) in [GK12]. To analyze this first order supported on the set Λ, we just recall the results obtained in [CKS$^+$10,GK12]. The first step is to take into account that the manifold $\mathcal{M}$ has a submanifold of considerably lower dimension which is also invariant.

Corollary 2.4 ([CKS$^+$10]). *Consider the subspace*

$$\widetilde{\mathcal{M}} = \left\{ \beta \in \mathcal{M} : \beta_{n_1} = \beta_{n_2} \text{ for all } n_1, n_2 \in \Lambda_j \text{ for some } j \right\},$$

where all the members of a generation take the same value. Then, $\widetilde{\mathcal{M}}$ is invariant under the flow associated to (21).

The dimension of $\widetilde{\mathcal{M}}$ is equal to the number of generations, namely N. To define Eq. (21) restricted to $\widetilde{\mathcal{M}}$, we consider

$$b_j = \beta_n \quad \text{for any } n \in \Lambda_j. \tag{22}$$

Then (21) restricted to $\widetilde{\mathcal{M}}$ becomes

$$\dot{b}_j = -i b_j^2 \overline{b}_j + 2i \overline{b}_j \left(b_{j-1}^2 + b_{j+1}^2 \right), \quad j = 0, \dots N. \tag{23}$$

This model was first derived in [CKS$^+$10] and was studied in great detail in that paper and in [GK12].

Theorem 2.5 ([GK12]). *Fix a large $\gamma \gg 1$. Then for any large enough N and $\delta = e^{-\gamma N}$, there exists an orbit of system* (23), *$\nu > 0$ independent of γ and N and $T_0 > 0$ such that*

$$\begin{array}{ll} |b_3(0)| > 1 - \delta^\nu & \qquad |b_{N-1}(T_0)| > 1 - \delta^\nu \\ |b_j(0)| < \delta^\nu \quad \text{for } j \neq 3 & \text{and} \qquad |b_j(T_0)| < \delta^\nu \quad \text{for } j \neq N - 1. \end{array}$$

Moreover, there exists a constant $\mathbb{K} > 0$ independent of N such that T_0 satisfies

$$0 < T_0 < \mathbb{K} N \ln \left(\frac{1}{\delta} \right) = \mathbb{K} \gamma N^2. \tag{24}$$

This theorem is proved in [GK12] using dynamical systems tools such as normal forms and the Shilnikov boundary problem.

From the orbit obtained in Theorem 2.5 and using the change (22) one can obtain an orbit of Eq. (17). Moreover, both Eqs. (17) and (23) are invariant under rescaling

$$b^\lambda(t) = \lambda^{-1} b\left(\lambda^{-2} t\right).$$ (25)

Then, by Theorem 2.5, the time spent by the solution $b^\lambda(t)$ is

$$T = \lambda^2 T_0 \le \lambda^2 \mathbb{K} \gamma N^2,$$ (26)

where T_0 is the time obtained in Theorem 2.5.

We see now that there are solutions of Eq. (14) which are close to the orbit β^λ of (17) defined as

$$\begin{aligned}
\beta_n^\lambda(t) &= \lambda^{-1} b_j\left(\lambda^{-2} t\right) \quad \text{for each } n \in \Lambda_j \\
\beta_n^\lambda(t) &= 0 \quad \text{for each } n \notin \Lambda,
\end{aligned}$$ (27)

where $b(t)$ is the orbit given by Theorem 2.5. To obtain this approximation result we need to consider a large λ. The reason, as it is explained in [CKS$^+$10,GK12], is that in order for the original system to be well approximated by the truncated one, we need the cubic terms in (14) to dominate over the quintic ones that are in the remainder $\widetilde{\mathcal{R}}$. Nevertheless, due to (26), the bigger λ, the slower the instability time. The choice of λ (with respect to N) that we have to do is different for the proof of Theorems 1.4 and 1.5. Both choices are essentially the same as the ones done in [GK12]. We show here how to prove Theorem 1.4 and at the end of the section we will show how to adapt the proof to deal with Theorem 1.5.

The approximation result needed to prove Theorem 1.4 is the following.

Theorem 2.6. *Let $\beta^\lambda(t) = \{\beta_n^\lambda(t)\}_{n \in \mathbb{Z}^2}$ be the solution of (17) given by (27) and T the time defined in (26). Assume that $V \in H^{s_0}$ with $s_0 > 0$. There exists a constant $\kappa > 0$ independent of N and γ such that, if we take*

$$\lambda = e^{\kappa \gamma N},$$ (28)

then any solution $\widetilde{\beta}(t) = \{\widetilde{\beta}_n(t)\}_{n \in \mathbb{Z}^2}$ of (14) with initial condition $\widetilde{\beta}(0) = \widetilde{\beta}^0 \in \ell^1$ such that $\|\widetilde{\beta}^0 - \beta^\lambda(0)\|_{\ell^1} \le \lambda^{-5/2}$, satisfies

$$\sum_{n \in \mathbb{Z}^2} \left|\widetilde{\beta}_n(t) - \beta_n^\lambda(t)\right| \le \lambda^{-2}$$ (29)

for $0 < t < T$.

The proof of this theorem is deferred to Sect. 5. Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Now we choose the trajectory which will be shown to attain the growth of Sobolev norms. We consider the solution $r(t)$ of system (7) with initial condition $r(0) = \beta^\lambda(0)$ [see (27)]. Note that the change (6), which relates r and the original variable a preserves the Sobolev norms and therefore, to prove Theorem 1.4, one only needs to check the stated properties on $r(t)$ instead of on $a(t)$.

To compute the growth of Sobolev norm of the orbit $r(t)$, we define

$$S_j = \sum_{n \in \Lambda_j} |n|^{2s} \quad \text{for } j = 1, \ldots, N - 1. \tag{30}$$

We want to prove that

$$\frac{\|r(T)\|_{H^s}}{\|r(0)\|_{H^s}} \gtrsim \mathcal{K}$$

and estimate the mass $\|r(0)\|_{L^2}$ of the solution. Since we want to use the estimates obtained in Theorem 2.3, we start by bounding $\|r(T)\|_{H^s}$ in terms of S_{N-1} as

$$\|r(T)\|_{H^s}^2 \geq \sum_{n \in \Lambda_{N-1}} |n|^{2s} |r_n(T)|^2 \geq S_{N-1} \inf_{n \in \Lambda_{N-1}} |r_n(T)|^2,$$

Now it is enough to obtain a lower bound for $|r_n(T)|$ for $n \in \Lambda_{N-1}$. To this end, using the change Γ obtained in Theorem 2.1 and the change of variables (12), we can write $r(t)$ as

$$r(t) = \Gamma\left(\left\{\widetilde{\beta}_n(t) e^{i(|n|^2 + v_n)t}\right\}\right),$$

where $\widetilde{\beta}(t)$ is a solution of system (14). Note that, since $r(0) = \beta^\lambda(0)$, by Theorem 2.1

$$\begin{aligned}
\left\|\widetilde{\beta}(0) - \beta^\lambda(0)\right\|_{\ell^1} &= \left\|\widetilde{\beta}(0) - r(0)\right\|_{\ell^1} \\
&= \left\|\widetilde{\beta}(0) - \Gamma\left(\widetilde{\beta}(0)\right)\right\|_{\ell^1} \\
&\lesssim \left\|\widetilde{\beta}(0)\right\|_{\ell^1}^3.
\end{aligned}$$

To bound this last term, we first compute the ℓ^1 norm of $r(0) = \beta^\lambda(0)$. From the definition of $\beta^\lambda(0)$ in (27) and Theorem 2.5, we know that $\|\beta^\lambda(0)\|_{\ell^\infty} \leq \lambda^{-1}$. Moreover, $\operatorname{supp} \beta^\lambda(0) = \Lambda$. Therefore,

$$\|r(0)\|_{\ell^1} = \|\beta^\lambda(0)\|_{\ell^1} \leq \lambda^{-1}|\Lambda| \leq \lambda^{-1} N 2^{N-1}.$$

From Theorem 2.1, one can easily see that Γ is invertible and that Γ^{-1} satisfies

$$\left\|\Gamma^{-1}(\beta) - \beta\right\|_{\ell^1} \leq \mathcal{O}\left(\|\beta\|_{\ell^1}^3\right).$$

Therefore,

$$\|\widetilde{\beta}(0)\|_{\ell^1} \leq \|\Gamma^{-1}(r(0))\|_{\ell^1} \lesssim \|r(0)\|_{\ell^1} \lesssim \lambda^{-1} N 2^{N-1},$$

which implies, using the definition of λ in Theorem 2.6 and taking N large enough,

$$\left\|\widetilde{\beta}(0) - \beta^\lambda(0)\right\|_{\ell^1} \lesssim \lambda^{-3} N^3 2^{3(N-1)} \leq \lambda^{-5/2}.$$

Therefore, the initial condition $\widetilde{\beta}(0)$ satisfies the hypothesis of Theorem 2.6. We use this fact to estimate the Sobolev norm of $r(t)$ at time $t = T$. Using also the results of Theorems 2.1, we split $|r_n(T)|$ as

$$\begin{aligned}
|r_n(T)| &\geq \left|\widetilde{\beta}_n(T)\right| - \left|\Gamma_n\left(\left\{\widetilde{\beta}_n(T) e^{i(|n|^2 + v_n)T}\right\}\right)(T) - \widetilde{\beta}_n(T) e^{i(|n|^2 + v_n)T}\right| \\
&\geq \left|\beta_n^\lambda(T)\right| - \left|\widetilde{\beta}_n(T) - \beta_n^\lambda(T)\right| \\
&\quad - \left|\Gamma_n\left(\left\{\widetilde{\beta}_n(T) e^{i(|n|^2 + v_n)T}\right\}\right)(T) - \widetilde{\beta}_n(T) e^{i(|n|^2 + v_n)T}\right|. \tag{31}
\end{aligned}$$

We need to obtain a lower bound for the first term of the right hand side and upper bounds for the second and third ones. Using the definition of β^λ in (27), the relation between T and T_0 established in (26) and the results in Theorem 2.5, we have that for $n \in \Lambda_{N-1}$,

$$\left|\beta_n^\lambda(T)\right| = \lambda^{-1}\left|b_{N-1}(T_0)\right| \geq \frac{3}{4}\lambda^{-1}.$$

For the second term in the right hand side of (31), it is enough to use Theorem 2.6 to obtain,

$$\left|\widetilde{\beta}_n(T) - \beta_n^\lambda(T)\right| \leq \left(\sum_{n \in \mathbb{Z}^2}\left|\widetilde{\beta}_n(T) - \beta_n^\lambda(T)\right|\right) \leq \frac{\lambda^{-1}}{8}.$$

For the lower bound of the third term, we use the bound for $\Gamma - \mathrm{Id}$ given in Theorem 2.1. Then,

$$\left|\Gamma_n\left(\left\{\widetilde{\beta}_n(T)e^{i(|n|^2+v_n)T}\right\}\right)(T) - \widetilde{\beta}_n(T)e^{i(|n|^2+v_n)T}\right|$$

$$\leq \left\|\Gamma_n\left(\left\{\widetilde{\beta}_n(T)e^{i(|n|^2+v_n)T}\right\}\right)(T) - \widetilde{\beta}_n(T)e^{i(|n|^2+v_n)T}\right\|_{\ell^1} \leq \frac{\lambda^{-1}}{8}.$$

Thus, we can conclude that

$$\|r(T)\|_{H^s}^2 \geq \frac{\lambda^{-2}}{4}S_{N-1}. \tag{32}$$

Now we prove that

$$\|r(0)\|_{H^s}^2 \lesssim \lambda^{-2}S_3. \tag{33}$$

Let us recall that $r(0) = \beta^\lambda(0)$ and then $\operatorname{supp} r(0) = \Lambda$. Therefore,

$$\|r(0)\|_{H^s}^2 = \sum_{n \in \Lambda}|n|^{2s}|r_n(0)|^2 = \sum_{n \in \Lambda}|n|^{2s}\left|\beta_n^\lambda(0)\right|^2.$$

Then, recalling the definition of β^λ in (27) and the results in Theorem 2.5,

$$\sum_{n \in \Lambda}|n|^{2s}\left|\beta_n^\lambda(0)\right|^2 \leq \lambda^{-2}S_3 + \lambda^{-2}\delta^{2v}\sum_{j \neq 3}S_j$$

$$\leq \lambda^{-2}S_3\left(1 + \delta^{2v}\sum_{j \neq 3}\frac{S_j}{S_3}\right).$$

From Theorem 2.3 we know that for $j \neq 3$,

$$\frac{S_j}{S_3} \lesssim e^{sN}.$$

Therefore, to bound these terms we use the definition of δ from Theorem 2.5 taking $\gamma = \widetilde{\gamma}(s-1)$. Since $s-1 > 0$ is fixed, we can choose such $\widetilde{\gamma} \gg 1$. Then, we have that

$$\|r(0)\|_{H^s} = \sum_{n \in \Lambda}|n|^{2s}\left|\beta_n^\lambda(0)\right|^2 \lesssim \lambda^{-2}S_3.$$

Using inequalities (32) and (33), we have that

$$\frac{\|r(T)\|^2_{H^s}}{\|r(0)\|^2_{H^s}} \gtrsim \frac{S_{N-1}}{S_3},$$

and then, applying Theorem 2.3, we obtain

$$\frac{\|r(T)\|^2_{H^s}}{\|r(0)\|^2_{H^s}} \gtrsim \frac{1}{2} 2^{(s-1)(N-4)} \geq \mathcal{K}^2.$$

To obtain the mass estimate it is enough to use the following bound

$$\|r(0)\|^2_{L^2} \lesssim \lambda^{-2} 2^N.$$

To prove it, one can proceed as for $\|r(0)\|_{H^s}$ taking into account that $|\Lambda_j| = \sum_{n \in \Lambda_j} 1 = 2^{N-1}$.

Finally, it only remains to estimate the diffusion time T. We use Theorem 2.3 to set $\mathcal{K} \simeq 2^{(s-1)N/2}$ and $c = 4\kappa\gamma/(s-1)$, definition (28) to set $\lambda = e^{\kappa\gamma N} \simeq \mathcal{K}^{c/(2\ln 2)}$. Then, we obtain

$$|T| \leq \mathbb{K}\gamma\lambda^2 N^2 \leq \mathbb{K}\gamma\,\mathcal{K}^{c/\ln 2}\frac{4\ln^2\mathcal{K}}{\ln^2 2\,(s-1)^2} \leq \mathcal{K}^c$$

for large $\mathcal{K}$. This completes the proof of Theorem 1.4. $\square$

Finally, we show how to adapt the proof of Theorem 1.4 to prove Theorem 1.5.

Proof of Theorem 1.5. Recall that we have seen in the proof of Theorem 1.4 that the initial Sobolev norm is given by

$$\|r(0)\|^2_{H^s} \lesssim \lambda^{-2} S_3.$$

Using the definition of S_3 in (30) and Theorem 2.3, we have that

$$\|r(0)\|^2_{H^s} \lesssim \lambda^{-2} 2^N e^{70sN^2}.$$

Thus, to obtain that $\|r(0)\|^2_{H^s} \leq \mu$, we have to choose

$$\lambda \sim \mu^{-1} 2^{N/2} e^{35sN^2}. \tag{34}$$

With this choice of λ, one can obtain an Approximation Theorem analogous to Theorem 2.6.

Theorem 2.7. *Let $\beta^\lambda(t) = \{\beta^\lambda_n(t)\}_{n \in \mathbb{Z}^2}$ be the solution of (17) given by (27) and T the time defined in (26). Assume that $V \in H^{s_0}$ with $s_0 > 70s/17$ and take λ as in (34). Then, any solution $\widetilde{\beta}(t) = \{\widetilde{\beta}_n(t)\}_{n \in \mathbb{Z}^2}$ of (14) with initial condition $\widetilde{\beta}(0) = \widetilde{\beta}^0 \in \ell^1$ such that $\|\widetilde{\beta}^0 - \beta^\lambda(0)\|_{\ell^1} \leq \lambda^{-5/2}$, satisfies*

$$\sum_{n \in \mathbb{Z}^2} \left|\widetilde{\beta}_n(t) - \beta^\lambda_n(t)\right| \leq \lambda^{-(1+\sigma)} \tag{35}$$

for $0 < t < T$ and some $\sigma > 0$ independent of N.

In Sect. 6 it is shown how to adapt the proof of Theorem 2.6, given in Sect. 5, to prove this theorem.

With this approximation theorem, the proof of Theorem 1.5 follows the same lines of the proof of Theorem 1.4 recalling that now the growth of Sobolev norms is given by $\mathcal{K} = \mathcal{C}/\mu$. $\square$

3. The Normal Form: Proof of Theorem 2.1

To prove Theorem 2.1, we consider as a change of variables Γ the time one map of a Hamiltonian vector field X_F, where F is the Hamiltonian

$$F = \frac{1}{4} \sum_{n_1, n_2, n_3, n_4 \in \mathbb{Z}^2} F_{n_1 n_2 n_3 n_4} \alpha_{n_1} \overline{\alpha_{n_2}} \alpha_{n_3} \overline{\alpha_{n_4}},$$

with

$$F_{n_1 n_2 n_3 n_4} = \frac{-i}{\sum_{j=1}^{4} (-1)^{j+1} \left(|n_j|^2 + v_{n_j} \right)} \qquad \text{if } (n_1, n_2, n_3, n_4) \in \mathcal{I}'$$

$$F_{n_1 n_2 n_3 n_4} = 0 \qquad\qquad\qquad\qquad \text{otherwise}$$

where $(n_1, n_2, n_3, n_4) \in \mathcal{I}'$ if either one of these two conditions are satisfied:

(i) $(n_1, n_2, n_3, n_4) \in (\mathbb{Z}^2)^4$ satisfies

$$n_1 - n_2 + n_3 - n_4 = 0 \quad \text{and} \quad \left| \sum_{j=1}^{4} (-1)^{j+1} \left(|n_j|^2 + v_j \right) \right| > \eta,$$

where η is the constant given by Theorem 2.1 and at least one of the modes n_j satisfies $n_j \in B(\kappa_0)$, where κ_0 is the constant defined in (9).

(ii) $(n_1, n_2, n_3, n_4) \in (\mathbb{Z}^2)^4$ satisfies

$$n_1 - n_2 + n_3 - n_4 = 0 \quad \text{and} \quad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \neq 0$$

and each mode satisfies $n_j \in \mathbb{Z}^2 \backslash B(\kappa_0)$.

Lemma 3.1. *The Fourier coefficients $F_{n_1 n_2 n_3 n_4}$ are well defined and satisfy*

$$\left| F_{n_1 n_2 n_3 n_4} \right| \leq 4.$$

Thus, the function F is well defined.

Proof. One only needs to prove that the denominators involved in the definition of the coefficients $F_{n_1 n_2 n_3 n_4}$ do not vanish and to compute a lower bound for them. If $(n_1, n_2, n_3, n_4) \in \mathcal{I}'$ and satisfies condition (i), the denominator does not vanish since its absolute value is bigger than η. Using the definition of η in Theorem 2.1, we obtain the wanted bound for $F_{n_1 n_2 n_3 n_4}$. If $(n_1, n_2, n_3, n_4) \in \mathcal{I}'$ and satisfies condition (ii), it is enough to point out that $|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2$ is an integer number at therefore $|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \neq 0$ implies $\left| |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \right| \geq 1$. Then using that $|n_i| \geq \kappa_0$ for $i = 1, \ldots, 4$ and (9), one can easily see that

$$\left| \sum_{j=1}^{4} (-1)^{j+1} \left(|n_j|^2 + v_{n_j} \right) \right| \geq \frac{1}{2}.$$

$\square$

We define Φ_F^t the flow of the vector field associated to the Hamiltonian F. It can be easily seen that for $t \in [0, 1]$ and α with $\|\alpha\|_{\ell^1}$ small enough,

$$\|\Phi_F^t(\alpha)\|_{\ell^1} \leq 2\|\alpha\|_{\ell^1}.$$

We consider as a change of variables the time one map associated to the flow Φ_F^t, that is $\Gamma = \Phi_F^1$. Then, we have that

$$
\begin{aligned}
\mathcal{H}' \circ \Gamma &= \mathcal{H}' \circ \Phi_F^t\big|_{t=1} \\
&= \mathcal{H}' + \{\mathcal{H}', F\} + \int_0^1 (1-t) \{\{\mathcal{H}', F\}, F\} \circ \Phi_F^t dt \\
&= \mathcal{D} + \mathcal{G}' + \{\mathcal{D}, F\} \\
&\quad + \{\mathcal{G}', F\} + \int_0^1 (1-t) \{\{\mathcal{H}', F\}, F\} \circ \Phi_F^t dt,
\end{aligned}
$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket with respect to the symplectic form $\Omega = \frac{i}{2} \sum_{n \in \mathbb{Z}^2} \alpha_n \wedge \overline{\alpha_n}$. We define

$$\mathcal{R} = \{\mathcal{G}', F\} + \int_0^1 (1-t) \{\{\mathcal{H}', F\}, F\} \circ \Phi_F^t dt.$$

Then, it only remains to obtain the desired bounds for $X_{\mathcal{R}}$ and Γ and to see that

$$\mathcal{G}' + \{\mathcal{D}, F\} = \widetilde{\mathcal{G}}_1 + \widetilde{\mathcal{G}}_2.$$

To obtain this last equality it is enough to use the definition for F to see that

$$
\begin{aligned}
\mathcal{G}' + \{\mathcal{D}, F\} &= \frac{1}{4} \sum_{n_1 - n_2 + n_3 = n_4} \left(1 - i \sum_{j=1}^4 (-1)^{j+1} (|n_j|^2 + v_{n_j}) F_{n_1 n_2 n_3 n_4} \right) \alpha_{n_1} \overline{\alpha_{n_2}} \alpha_{n_3} \overline{\alpha_{n_4}} \\
&= \widetilde{\mathcal{G}}_1 + \widetilde{\mathcal{G}}_2.
\end{aligned}
$$

To obtain the bounds for $X_{\mathcal{R}}$ we proceed as in [GK12]. We use the fact that the space ℓ^1 is a Banach algebra with respect to the convolution product. Namely, if $a, b \in \ell^1$ its convolution product $a * b$, which is defined by

$$(a * b)_n = \sum_{n_1 + n_2 = n} a_{n_1} b_{n_2}$$

satisfies

$$\|a * b\|_{\ell^1} \leq \|a\|_{\ell^1} \|b\|_{\ell^1}.$$

We start by bounding $X_{\{\mathcal{G}', F\}}$, the vector field associated to the Hamiltonian $\{\mathcal{G}', F\}$. We have to bound

$$\|X_{\{\mathcal{G}', F\}}\|_{\ell^1} = 2 \sum_{n \in \mathbb{Z}^2} \left| \partial_{\overline{\alpha_n}} \{\mathcal{G}', F\} \right|.$$

Then,

$$
\begin{aligned}
\left\| X_{\{\mathcal{G}',F\}} \right\|_{\ell^1} &\leq 2 \sum_{n,m\in\mathbb{Z}^2} \left| \partial_{\overline{\alpha_n}} \left(\partial_{\overline{\alpha_m}} \mathcal{G}' \partial_{\alpha_m} F \right) \right| + 2 \sum_{n,m\in\mathbb{Z}^2} \left| \partial_{\overline{\alpha_n}} \left(\partial_{\alpha_m} \mathcal{G}' \partial_{\overline{\alpha_m}} F \right) \right| \\
&\leq 2 \sum_{n,m\in\mathbb{Z}^2} \left| \partial_{\overline{\alpha_n \alpha_m}} \mathcal{G}' \right| \left| \partial_{\alpha_m} F \right| + 2 \sum_{n,m\in\mathbb{Z}^2} \left| \partial_{\overline{\alpha_m}} \mathcal{G}' \right| \left| \partial_{\overline{\alpha_n} \alpha_m} F \right| \\
&\quad + 2 \sum_{n,m\in\mathbb{Z}^2} \left| \partial_{\overline{\alpha_n} \alpha_m} \mathcal{G}' \right| \left| \partial_{\overline{\alpha_m}} F \right| + 2 \sum_{n,m\in\mathbb{Z}^2} \left| \partial_{\alpha_m} \mathcal{G}' \right| \left| \partial_{\overline{\alpha_n \alpha_m}} F \right| .
\end{aligned}
$$

All the terms can be bounded analogously. As an example, we bound the first one,

$$
\begin{aligned}
\sum_{n,m\in\mathbb{Z}^2} \left| \partial_{\overline{\alpha_n \alpha_m}} \mathcal{G}' \right| \left| \partial_{\alpha_m} F \right| &\leq 16 \sum_{n,m\in\mathbb{Z}^2} \left| \sum_{n_1+n_2=m+n} \alpha_{n_1} \alpha_{n_2} \right| \left| \sum_{n_1-n_2+n_3=m} \overline{\alpha_{n_1}} \alpha_{n_2} \overline{\alpha_{n_3}} \right| \\
&\leq 16 \sum_{n\in\mathbb{Z}^2} \sum_{n_1+n_2=n} |\alpha_{n_1}||\alpha_{n_2}| \sum_{m\in\mathbb{Z}^2} \sum_{n_1-n_2+n_3=m} |\alpha_{n_1}||\alpha_{n_2}||\alpha_{n_3}| \\
&\leq \mathcal{O}\left(\|\alpha\|_{\ell^1}^5 \right),
\end{aligned}
$$

where, in the first line we have taken into account that $|F_{n_1 n_2 n_3 n_4}| \leq 4$, and to obtain the last line we have used that each sum in the previous line is a convolution product. The other term in the remainder can be bounded analogously taking into account that

$$
\int_0^1 (1-t) \left\{ \{\mathcal{H}', F\}, F \right\} \circ \Phi_F^t \, dt = \int_0^1 (1-t) \left\{ \widetilde{\mathcal{G}}_1 + \widetilde{\mathcal{G}}_2 - \mathcal{G}', F \right\} \circ \Phi_F^t \, dt
$$
$$
+ \int_0^1 (1-t) \left\{ \{\mathcal{G}', F\}, F \right\} \circ \Phi_F^t \, dt.
$$

Analogously, one can obtain bounds for $\Gamma - \mathrm{Id}$ recalling that

$$
\Gamma = \mathrm{Id} + \int_0^1 X_F \circ \Phi_F^t \, dt.
$$

4. Construction of the Set Λ: Proof of Theorem 2.3

We devote this section to the prove the Theorem 2.3. To this end we modify the construction of the set Λ done in [CKS⁺10] and [GK12] for (5). For Eq. (5) the resonant set was defined as

$$
\begin{aligned}
\mathcal{B} = \Big\{ (n_1, n_2, n_3, n_4) \in \left(\mathbb{Z}^2\right)^4 &: n_1 - n_2 + n_3 - n_4 = 0, \\
&|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0, n_1, n_3 \neq n_4 \Big\}.
\end{aligned}
$$

In Sect. 2 we have defined the conditions $1_\Lambda - 6_\Lambda$ which involve tuples in $\widetilde{\mathcal{A}}_1$. To avoid confusions, we denote by $1'_\Lambda - 6'_\Lambda$ the same conditions but referred to the tuples in $\mathcal{B}$ (note that $\widetilde{\mathcal{A}}_1 \subset \mathcal{B}$). In [GK12] the following proposition is proved, which is a quantitative version of Proposition 2.1 of [CKS⁺10]. It constructs a set of modes $\Lambda^{(0)}$ satisfying conditions $1'_\Lambda - 6'_\Lambda$.

Proposition 4.1 ([GK12]). *Let $\mathcal{K} \gg 1$. Then, there exists $N \gg 1$ and a set $\Lambda^{(0)} \subset \mathbb{Z}^2$, with*

$$\Lambda^{(0)} = \Lambda_1^{(0)} \cup \ldots \cup \Lambda_N^{(0)},$$

which satisfies conditions $1'_\Lambda - 6'_\Lambda$ and also

$$\frac{\sum_{n \in \Lambda_{N-1}^{(0)}} |n|^{2s}}{\sum_{n \in \Lambda_3^{(0)}} |n|^{2s}} \geq \frac{1}{2} 2^{(s-1)(N-4)} \geq \mathcal{K}^2. \tag{36}$$

This proposition is proved in Appendix C of [GK12]. As it is stated in Appendix C of that paper, one can choose the set Λ such that its modes satisfy

$$\kappa \leq |n| \leq e^{17N^2}, \quad n \in \Lambda_1, \tag{37}$$

for any constant $\kappa > 0$ independent of N. Moreover, if it is not stated explicitly in [GK12], each generation Λ_j of the obtained set Λ has 2^{N-1} disjoint frequencies satisfying

$$\frac{\sum_{n \in \Lambda_j} |n|^{2s}}{\sum_{n \in \Lambda_i} |n|^{2s}} \leq C e^{sN} \tag{38}$$

for any $1 \leq i < j \leq N$ and some $C > 0$ independent of N.

Furthermore, following the proof, one can easily see that one can impose the following additional conditions.

$9'_\Lambda$ Consider any two modes $n_1, n_2 \in \Lambda^{(0)}$. Then, they satisfy

$$|n_1 + n_2| \neq |n_1 - n_2|. \tag{39}$$

$10'_\Lambda$ Consider any two modes $n_1, n_2 \in \Lambda^{(0)}$. Then, the lines passing through one of these points and perpendicular to the segment between them do not contain the origin. Equivalently,

$$(n_2 - n_1) \cdot n_1 \neq 0. \tag{40}$$

These conditions will be used in the proof of Theorem 2.3.

We use Proposition 4.1 to prove Theorem 2.3, which constructs a set Λ satisfying properties $1_\Lambda - 8_\Lambda$, which refer to the resonant tuples of the Eq. (1), that is to $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_0 \cup \widetilde{\mathcal{A}}_1$. Note that if we take the set $\Lambda^{(0)}$ given by Proposition 4.1 and we define a new set

$$\Lambda' = \left\{ n' \in \mathbb{Z}^2 : n' = Cn, n \in \Lambda^{(0)} \right\},$$

for some $0 < C < e^{\nu N^2}$ with $\nu > 0$, we obtain a new set which also satisfies all the conditions stated in Proposition 4.1 and (38). We will show that performing the appropiate rescaling this new set Λ' satisfies conditions $1_\Lambda - 8_\Lambda$, (20), and also (19).

First, we choose the constant κ in (37) as $\kappa = \kappa_0$, where κ_0 is the constant defined in (9) (recall that κ_0 is independent of N). Then, since the resonant tuples of $\widetilde{\mathcal{A}}_1$ are contained in the resonant tuples of $\mathcal{B}$, one has that $\Lambda^{(0)}$ satisfies conditions $1_\Lambda - 6_\Lambda$.

Now, we show that a suitable blow up of $\Lambda^{(0)}$ satisfies condition 7_Λ for the resonant set $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_0 \cup \widetilde{\mathcal{A}}_1$. Condition 8_Λ will need an additional blow up that will be done later on. To check condition 7_Λ, we consider three modes $n_1, n_2, n_3 \in \Lambda^{(0)}$. By Proposition 4.1,

these three modes cannot make a rectangle with a mode out of $\Lambda^{(0)}$, in particular, they cannot make a rectangle with $n_4 = 0$. This implies

$$|n_1|^2 - |n_2|^2 + |n_3|^2 \neq 0$$

which is equivalent to

$$\left| |n_1|^2 - |n_2|^2 + |n_3|^2 \right| \geq 1$$

since $|n_j|^2$ are natural numbers. We consider the blow up of $\Lambda^{(0)}$,

$$\Lambda^{(1)} = \{n' \in \mathbb{Z}^2 : n' = C_1 n, n \in \Lambda^{(0)}\},$$

with

$$C_1 = \left\lceil \kappa_0^2 + 4\|V\|_{H^{s_0}} + 1 \right\rceil, \tag{41}$$

where $\lceil x \rceil$ denotes the upper integer part of x and κ_0 is the constant introduced in (9). Recall that $V \in H^{s_0}$ by hypothesis. We show that $\Lambda^{(1)}$ satisfies the condition 7_Λ. Note that, by (38) and (38) the modes $n \in \Lambda^{(1)}$ satisfy

$$|n| \leq C_1 e^{17N^2}. \tag{42}$$

We take a mode n_4' satisfying $|n_4'| < \kappa_0$ and we show that they cannot form a resonant tuple in $\widetilde{\mathcal{A}}_0$. Thanks to the blow up, we have that

$$\left| |n_1'|^2 - |n_2'|^2 + |n_3'|^2 \right| \geq \kappa_0^2 + 4\|V\|_{H^{s_0}} + 1.$$

We use this lower bound to prove that the tuple cannot be resonant. We need to check that $|\rho_{n_1' n_2' n_3' n_4'}| \geq \eta$ [see (10) for the definition of $\rho_{n_1' n_2' n_3' n_4'}$ and Theorem 2.1 for the definition of η]. Using $|n_4'| < \kappa_0$ and that the Fourier coefficients v_n satisfy $|v_n| \leq \|V\|_{H^{s_0}}$, we have that

$$\left| \rho_{n_1' n_2' n_3' n_4'} \right| \geq \left| \left| |n_1'|^2 - |n_2'|^2 + |n_3'|^2 \right| - |n_4'|^2 - \sum_{j=1}^{4} |v_{n_j'}| \right|$$

$$\geq \left(\kappa_0^2 + 4\|V\|_{H^{s_0}} + 1 \right) - \kappa_0^2 - 4\|V\|_{H^{s_0}} = 1 \geq \eta,$$

and therefore the tuple cannot be resonant. This completes the proof of condition 7_Λ for $\Lambda^{(1)}$.

Now it only remains to make an additional blow up so that the new set also satisfies 8_Λ. We define

$$\Lambda^{(2)} = \{n' \in \mathbb{Z}^2 : n' = C_2 n, n \in \Lambda^{(1)}\},$$

with

$$C_2 = \left\lceil 16(\kappa_0 + 1)C_1 e^{17N^2} \right\rceil, \tag{43}$$

where C_1 has been defined in (41). One can easily see that after the blow up, the set $\Lambda^{(2)}$ still satisfies conditions $1_\Lambda - 7_\Lambda$ and (38). Moreover, taking into account (42), we have that the points in $\Lambda^{(2)}$ satisfy

$$e^{17N^2} \leq \kappa_0 e^{17N^2} \leq |n| \leq C_2 C_1 e^{17N^2} \leq e^{35N^2}, \tag{44}$$

which gives property (19).

We have to show that there cannot exist tuples in $\widetilde{\mathcal{A}}_0$ involving two modes $n_1', n_2' \in \Lambda$ and $m, \ell \notin \Lambda$.

To belong to $\widetilde{\mathcal{A}}_0$, the tuples must have a mode in $B(\kappa_0)$ and must satisfy either

$$|\rho_{mn_1' \ell n_2'}| \leq \eta \quad \text{or} \quad |\rho_{mn_1' n_2' \ell}| \leq \eta$$

(the other cases are symmetric). We show that this is not possible. We start with the first case. We assume that

$$|\rho_{mn_1' \ell n_2'}| \leq \eta$$

and we show that it implies $|m| > \kappa_0$ and $|\ell| > \kappa_0$.

Using that $m + \ell = n_1' + n_2'$ it is easy to see that the condition $|\rho_{m\ell n_1' n_2'}| \leq \eta$ can be written as

$$\left| 2 \left| m - \frac{n_1' + n_2'}{2} \right|^2 - \frac{|n_1' - n_2'|^2}{2} + v_m - v_\ell + v_{n_1'} - v_{n_2'} \right| \leq \eta$$

which, using that $|v_j| \leq \|V\|_{H^{s_0}}$, implies

$$\left| m - \frac{n_1' + n_2'}{2} \right|^2 \leq \frac{|n_1' - n_2'|^2}{4} + \frac{\eta}{2} + 2\|V\|_{H^{s_0}}.$$

This condition is satisfied provided m is inside the circumference of radius

$$R = \sqrt{\frac{|n_1' - n_2'|^2}{4} + \frac{\eta}{2} + 2\|V\|_{H^{s_0}}}$$

and center $(n_1' + n_2')/2$. We show now that belonging to this circumference implies $|m| > \kappa_0$. The point of the circumference closest to the origin is either one of these two points

$$z_\pm = \frac{n_1' + n_2'}{2} \left(1 \pm \frac{2R}{|n_1' + n_2'|} \right),$$

which, satisfy

$$|z_\pm| \geq \frac{|n_1' + n_2'|}{2} - R$$
$$\geq \frac{\left| |n_1' + n_2'|^2 - |n_1' - n_2'|^2 - 2\eta - 8\|V\|_{H^{s_0}} \right|}{2|n_1' + n_2'| + 4R}.$$

Now recall that $n_1' = C_2 n_1$ and $n_2' = C_2 n_2$ where C_2 has been defined in (43) and $n_1, n_2 \in \Lambda^{(1)}$. Since $\Lambda^{(0)}$, and therefore also $\Lambda^{(1)}$, satisfy condition $9_\Lambda'$, we know that $|n_1 + n_2|^2 - |n_1 - n_2|^2 \neq 0$. Since it is an integer number, we can deduce that

$$\left| |n_1 + n_2|^2 - |n_1 - n_2|^2 \right| \geq 1,$$

which implies

$$\left| |n_1' + n_2'|^2 - |n_1' - n_2'|^2 \right| \geq C_2^2.$$

Then, using the definition of C_2, we have that

$$\left| \left| n_1' + n_2' \right|^2 - \left| n_1' - n_2' \right|^2 - 2\eta - 8\|V\|_{H^{s_0}} \right| \geq \frac{C_2^2}{2}.$$

On the other hand, using (44) and the definition of C_1 in (41), we have that

$$2|n_1' + n_2'| + R \leq 8C_2C_1e^{17N^2}.$$

Therefore, the points $z_{\pm}$ satisfy

$$|z_{\pm}| \geq \frac{C_2}{16C_1e^{17N^2}}.$$

Using the definition of C_2 in (43) we have

$$|z_{\pm}| \geq \kappa_0 + 1.$$

Since one of these points $z_{\pm}$ is the closest point of the circumference to the origin, we can deduce that $|m| > \kappa_0$. Note that the mode ℓ also must be inside the same circumference and therefore it also satisfies $|\ell| > \kappa_0$.

Now we deal with the second case, that is

$$|\rho_{mn_1'n_2'\ell}| \leq \eta.$$

We assume that m satisfies $|m| \leq \kappa_0$ and we reach a contradiction. Using $m + n_2' = n_1 + \ell$, we can write $\rho_{mn_1'n_2'\ell}$ as

$$\rho_{mn_1'n_2'\ell} = -2(n_1' - n_2') \cdot n_1' + 2(n_1' - n_2') \cdot m + v_m - v_{n_1'} + v_{n_2'} - v_\ell.$$

Therefore,

$$\left| \rho_{mn_1'n_2'\ell} \right| \geq 2 \left| (n_1' - n_2') \cdot n_1' \right| - 2 \left| (n_1' - n_2') \cdot m \right| - 4\|V\|_{H^{s_0}}.$$

For the first term, we use the fact that since $n_1', n_2' \in \Lambda^{(2)}$, they are of the form $n_j' = C_2 n_j$ where C_2 is the constant given in (43). Using that $\Lambda^{(0)}$, and also $\Lambda^{(1)}$, satisfy condition 10_Λ, we have that $|(n_1 - n_2) \cdot n_1| \neq 0$. Since it is a natural number, it must satisfy $|(n_1 - n_2) \cdot n_1| \geq 1$, which implies

$$\left| (n_1' - n_2') \cdot n_1' \right| \geq C_2^2.$$

On the other hand, using (44) and that we have assumed that $|m| \leq \kappa_0$, we have that

$$\left| (n_1' - n_2') \cdot m \right| \leq 2\kappa_0 C_2 C_1 e^{17N^2}.$$

Therefore,

$$\left| \rho_{mn_1'n_2'\ell} \right| \geq 2C_2^2 - 4\kappa_0 C_2 C_1 e^{17N^2} - 4\|V\|_{H^{s_0}}.$$

Using the definition of C_2 in (43), one can easily see that

$$\left| \rho_{mn_1'n_2'\ell} \right| \geq 1,$$

which is a contradiction with what we had assumed. This implies that the set $\Lambda^{(2)}$ satisfies condition 8_Λ. Recalled that (19) has already been shown in (44). Finally, recall that the property (20) is invariant by blow ups and therefore, it is a direct consequence of (38). This completes the proof of Theorem 2.3.

5. Proof of Approximation Theorem 2.6

We devote this section to the proof of the Approximation Theorem 2.6, which follows the scheme of the proof of Theorem 4 in [GK12]. Nevertheless, note that now the remainder $\widetilde{\mathcal{R}}$ in (14) has more terms than the corresponding one in [GK12] due to the convolution potential. Throughout this section C denotes any positive constant independent of N and λ.

We write Eq. (14) as

$$-i\dot{\beta}_n = \mathcal{E}_n(\beta) + \widetilde{\mathcal{R}}_n(\beta), \tag{45}$$

where $\mathcal{E}: \ell^1 \to \ell^1$ is the function defined as

$$\mathcal{E}_n(\beta) = -|\beta_n|^2 \beta_n + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_0(n)} .\beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3}e^{i\rho_{n_1n_2n_3n}t} + \sum_{(n_1,n_2,n_3)\in\mathcal{A}_1(n)} \beta_{n_1}\overline{\beta_{n_2}}\beta_{n_3}. \tag{46}$$

Recall that we want to study the closeness of the orbit $\beta^\lambda(t)$ obtained in (27), which is a solution of $-i\dot{\beta}^\lambda = \mathcal{E}(\beta^\lambda)$, with an orbit $\widetilde{\beta}(t)$ of Eq. (45) which satisfies $\|\widetilde{\beta}(0) - \beta^\lambda(0)\|_{\ell^1} \leq \lambda^{-5/2}$. To this end, we define the function ξ as

$$\xi = \widetilde{\beta} - \beta^\lambda, \tag{47}$$

which satisfies $\|\xi(0)\|_{\ell^1} \leq \lambda^{-5/2}$. We apply refined Gronwall-like estimates to bound the ℓ^1 norm of $\xi(t)$.

From Eqs. (45) and (17) and recalling the definition of $\widetilde{\mathcal{R}}$ in (15), one can obtain the equation for ξ, which reads

$$\dot{\xi} = \mathcal{Z}^0(t) + \mathcal{Z}^1(t)\xi + \mathcal{Z}^2(\xi, t) \tag{48}$$

where

$$\mathcal{Z}^0(t) = \mathcal{R}'\left(\beta^\lambda\right) + \mathcal{J}\left(\beta^\lambda\right) \tag{49}$$

$$\mathcal{Z}^1(t) = D\mathcal{E}\left(\beta^\lambda\right) + D\mathcal{J}\left(\beta^\lambda\right) \tag{50}$$

$$\mathcal{Z}^2(\xi, t) = \mathcal{E}\left(\beta^\lambda + \xi\right) - \mathcal{E}\left(\beta^\lambda\right) - D\mathcal{E}\left(\beta^\lambda\right)\xi + \mathcal{R}'\left(\beta^\lambda + \xi\right) - \mathcal{R}'\left(\beta^\lambda\right)$$
$$+ \mathcal{J}\left(\beta^\lambda + \xi\right) - \mathcal{J}\left(\beta^\lambda\right) - D\mathcal{J}\left(\beta^\lambda\right)\xi \tag{51}$$

Applying the ℓ^1 norm to this equation, we obtain

$$\frac{d}{dt}\|\xi\|_{\ell^1} \leq \left\|\mathcal{Z}^0(t)\right\|_{\ell^1} + \left\|\mathcal{Z}^1(t)\xi\right\|_{\ell^1} + \left\|\mathcal{Z}^2(\xi, t)\right\|_{\ell^1}. \tag{52}$$

The next three lemmas give estimates for each term in the right hand side of this equation.

Lemma 5.1. *The function $\mathcal{Z}^0$ defined in* (49) *satisfies* $\left\|\mathcal{Z}^0\right\|_{\ell^1} \leq C\lambda^{-5}2^{5N}$.

Proof. The bound of $\mathcal{R}'(\beta^\lambda)$ can be done as in the proof of Lemma B.1 of [GK12]. We bound now $\mathcal{J}(\beta^\lambda)$, which has been defined in (16). The first step is to obtain upper bound for $\rho_{n_1n_2n_3n}$. Since the terms in $\mathcal{J}_n$ satisfy that $(n_1, n_2, n_3) \in \mathcal{A}_1(n)$ [see (11)], we have that $|n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0$. Then, recalling the definition of $\rho_{n_1n_2n_3n}$ in (10), we have that

$$\rho_{n_1n_2n_3n} = v_{n_1} - v_{n_2} + v_{n_3} - v_n.$$

To bound it, we use the fact that $V \in H^{s_0}(\mathbb{T}^2)$. Then, its Fourier coefficients satisfy $|v_n| \leq C|n|^{-s_0}$. Since by Theorem 2.3 the modes in Λ satisfy $|n| \geq e^{17N^2}$, one has that

$$\left|\rho_{n_1 n_2 n_3 n}\right| \leq Ce^{-17s_0 N^2},$$

and therefore, for $t \in [0, T]$, where T is the time defined in (26),

$$\left|e^{i\rho_{n_1 n_2 n_3 n}t} - 1\right| \leq Ce^{-17s_0 N^2}T \leq Ce^{-17s_0 N^2}\lambda^2 N^2.$$

Using this estimate and that $\mathcal{J}$ is a convolution term, we have that

$$\left\|\mathcal{J}\left(\beta^\lambda\right)\right\|_{\ell^1} \leq Ce^{-17s_0 N^2}\lambda^2 N^2 \left\|\beta^\lambda\right\|_{\ell^1}^3.$$

To bound $\left\|\beta^\lambda\right\|_{\ell^1}$, it is enough to recall that supp$\{\beta^\lambda\} \subset \Lambda$ and the definition of β^λ in (27) and Theorem 2.5 to show that

$$\left\|\beta^\lambda(t)\right\|_{\ell^1} \leq \sum_{n \in \Lambda} \left|\beta_n^\lambda(t)\right| \leq \lambda^{-1}2^N \sum_{j=1}^N \left|b_j\left(\lambda^{-2}t\right)\right| \leq C\lambda^{-1}2^N N.$$

Thus, using the choice of λ done in Theorem 2.6, we can conclude that

$$\|\mathcal{J}\|_{\ell^1} \leq C\lambda^{-1}e^{-17s_0 N^2}2^{3N}N^5 \leq C\lambda^{-5}2^{5N}.$$

$\square$

Lemma 5.2. *The linear operator $\mathcal{Z}^1(t)$ satisfies $\left\|\mathcal{Z}^1(t)\xi\right\|_{\ell^1} \leq \sum_{n \in \mathbb{Z}^2} f_n(t)|\xi_n|$, where $f_n(t)$ are positive functions satisfying*

$$\int_0^T f_n(t)dt \leq C\gamma N, \tag{53}$$

where T is the time given in (26) and γ is the constant given in Theorem 2.5.

The proof of this lemma follows the same lines as the proof of Lemma B.2 in [GK12].

To obtain estimates for $\mathcal{Z}^2(\xi, t)$ defined in (51), we apply bootstrap. Assume that for $0 < t < T^*$ we have

$$\|\xi(t)\|_{\ell^1} \leq C\lambda^{-3/2}2^{-N}. \tag{54}$$

Recall that for $t = 0$ we know that it is already satisfied since $\|\xi(0)\|_{\ell^1} \leq \lambda^{-5/2}$. *A posteriori* we will show that the time T in (26) satisfies $0 < T < T^*$ and therefore the bootstrap assumption holds.

Lemma 5.3. *Assume that condition (54) is satisfied. Then the operator $\mathcal{Z}^2(\xi, t)$ satisfies*

$$\left\|\mathcal{Z}^2(\xi, t)\right\|_{\ell^1} \leq C\lambda^{-5/2}\|\xi\|_{\ell^1}.$$

Proof. The proof of this lemma follows the same lines as the proof of Lemma B.3 in [GK12]. We split $\mathcal{Z}^2$ in (51) as $\mathcal{Z}^2 = \mathcal{Z}^{21} + \mathcal{Z}^{22} + \mathcal{Z}^{23}$ with

$$\mathcal{Z}^{21}(\xi, t) = \mathcal{R}'\left(\beta^\lambda + \xi\right) - \mathcal{R}'\left(\beta^\lambda\right)$$
$$\mathcal{Z}^{22}(\xi, t) = \mathcal{E}\left(\beta^\lambda + \xi\right) - \mathcal{E}\left(\beta^\lambda\right) - D\mathcal{E}\left(\beta^\lambda\right)\xi$$
$$\mathcal{Z}^{23}(\xi, t) = \mathcal{J}\left(\beta^\lambda + \xi\right) - \mathcal{J}\left(\beta^\lambda\right) - D\mathcal{J}\left(\beta^\lambda\right)\xi.$$

The first term can be bounded as done in the proof of Lemma B.3 in [GK12]. The second term too. Nevertheless, we explain here how to bound it, since the bound for the third term is obtained analogously. Using the definition of $\mathcal{E}$ in (46), one can easily see that

$$\left\|\mathcal{Z}^{22}\right\|_{\ell^1} \leq C\left(\|\beta^\lambda\|_{\ell^1}\|\xi\|_{\ell^1}^2 + \|\xi\|_{\ell^1}^3\right).$$

Then, using the bound for $\|\beta^\lambda\|_{\ell^1}$ obtained in the proof of Lemma 5.1 and the bootstrap assumption (54), one obtains

$$\left\|\mathcal{Z}^{22}\right\|_{\ell^1} \leq C\lambda^{-5/2}\|\xi\|_{\ell^1}.$$

For the third term, it is enough to recall the definition of $\mathcal{J}$ in (16) and take into account that for any $t \in \mathbb{R}$,

$$\left|e^{i\rho_{n_1 n_2 n_3 n}t} - 1\right| \leq 2.$$

Therefore, using that $\mathcal{J}$ is a convolution term proceeding as for $\mathcal{E}$, one can see that

$$\left\|\mathcal{Z}^{23}\right\|_{\ell^1} \leq C\left(\|\beta^\lambda\|_{\ell^1}\|\xi\|_{\ell^1}^2 + \|\xi\|_{\ell^1}^3\right),$$

which implies

$$\left\|\mathcal{Z}^{23}\right\|_{\ell^1} \leq C\lambda^{-5/2}\|\xi\|_{\ell^1}.$$

This completes the proof of the lemma. $\square$

The rest of the proof of Theorem 2.6 is analogous to the proof of Theorem 4 in [GK12]. We reproduce it here. Applying Lemmas 5.1, 5.2 and 5.3 to the terms in Eq. (52), one obtains

$$\sum_{n\in\mathbb{Z}^2} \frac{d}{dt}|\xi_n| = \frac{d}{dt}\|\xi\|_{\ell^1} \leq \sum_{n\in\mathbb{Z}^2}\left(f_n(t) + C\lambda^{-5/2}\right)|\xi_n| + C\lambda^{-5}2^{5N}.$$

We apply a Gronwall-like argument to each harmonic of ξ. That is, we consider the following change of coordinates,

$$\xi_n = \zeta_n e^{\int_0^t \left(f_n(s) + C\lambda^{-5/2}\right)ds} \tag{55}$$

to obtain

$$\sum_{n\in\mathbb{Z}^2} e^{\int_0^t \left(f_n(s) + C\lambda^{-5/2}\right)ds}\frac{d}{dt}|\zeta_n| \leq C\lambda^{-5}2^{5N}.$$

Taking into account that $f_n(t) + C\lambda^{-5/2} \geq 0$, we have that

$$\frac{d}{dt}\|\zeta\|_{\ell^1} = \sum_{n \in \mathbb{Z}^2} \frac{d}{dt}|\zeta_n| \leq C\lambda^{-5}2^{5N}. \tag{56}$$

Since $\zeta(0) = \xi(0)$, we know that $\|\zeta(0)\|_{\ell^1} \leq \lambda^{-5/2}$. Then, integrating Eq. (56), and using the bound for T in (26) we obtain that

$$\|\zeta(T)\|_{\ell^1} \leq \|\zeta(0)\|_{\ell^1} + C\lambda^{-3}2^{5N}\gamma N^2 \leq 2\lambda^{-5/2}.$$

To complete the proof we need to deduce from this estimate the bound for $\|\xi\|_{\ell^1}$. It only suffices to use the change (55), the estimate (53) and the definition of T in (26) to obtain

$$\|\xi\|_{\ell^1} \leq 2e^{C\gamma N}\|\zeta\|_{\ell^1} \leq 4e^{C\gamma N}\lambda^{-5/2}.$$

Therefore, using the condition on λ from Theorem 2.6 with any $\kappa > C$ and taking N big enough, we obtain that for $t \in [0, T]$

$$\|\xi\|_{\ell^1} \leq \lambda^{-2}$$

and therefore we can drop the bootstrap assumption (54). This completes the proof of Theorem 2.6

6. Proof of Approximation Theorem 2.7

In this section we show how the proof the Approximation Theorem 2.6 needs to be modified to prove Theorem 2.7. Recall that the only difference is the choice of the constant λ.

We consider again the function ξ defined in (47), which satisfies, by hypothesis, that $\|\xi(0)\|_{\ell^1} \leq \lambda^{-5/2}$ and that is a solution of Eq. (48). Applying the ℓ^1 norm to Eq. (48), we obtain Eq. (52). The next lemma give estimates for the function $\mathcal{Z}_0$ which appears in this equation.

Lemma 6.1. *There exists a constant $\sigma > 0$ independent of N, such that the function $\mathcal{Z}^0$ defined in (49) satisfies*

$$\left\|\mathcal{Z}^0\right\|_{\ell^1} \leq \lambda^{-(3+2\sigma)}.$$

Proof. The only difference is to bound the function $\mathcal{J}(\beta^\lambda)$ defined in (16). Recall that for $t \in [0, T]$, where T is the time defined in (26),

$$\left|e^{i\rho_{n_1 n_2 n_3}nt} - 1\right| \lesssim e^{-17s_0 N^2}T \lesssim e^{-17s_0 N^2}\lambda^2 N^2.$$

Proceeding as in the proof of Lemma 5.1 then, we have that

$$\|\mathcal{J}\|_{\ell^1} \leq C\lambda^{-1}e^{-17s_0 N^2}2^{3N}N^5.$$

Since we are assuming that $s_0 > 70s/17$, there exists $\sigma > 0$ independent of N such that s_0 can be written as

$$s_0 = \frac{70s}{17} + \frac{105s\sigma}{17}.$$

Then, also using the choice of λ in (34) we have that

$$\|\mathcal{J}\|_{\ell^1} \le C\lambda^{-1} \left(e^{35sN^2}\right)^{-(2+3\sigma)} 2^{3N} N^5 \lesssim \lambda^{-(3+3\sigma)} \left(\mu^{-1}2^{N/2}\right)^{2+3\sigma} 2^{3N} N^5.$$

Thus, taking N large enough,

$$\|\mathcal{J}\|_{\ell^1} \le \lambda^{-(3+2\sigma)}.$$

$\square$

The estimate for $\mathcal{Z}_1$ is independent of λ. Therefore, Lemma 5.2 is still valid with the new choice of λ. To obtain estimates for $\mathcal{Z}^2(\xi, t)$ defined in (51), we need to modify the bootstrap assumption. We assume that for $0 < t < T^*$ we have

$$\|\xi(t)\|_{\ell^1} \le C\lambda^{-(1+\sigma/2)}. \tag{57}$$

As in the proof of Theorem 2.6, we show *a posteriori* that the time T in (26) satisfies $0 < T < T^*$ and therefore the bootstrap assumption holds.

Lemma 6.2. *Assume that condition* (57) *is satisfied. Then the operator* $\mathcal{Z}^2(\xi, t)$ *satisfies*

$$\left\|\mathcal{Z}^2(\xi, t)\right\|_{\ell^1} \le C\lambda^{-(2+\sigma/2)}\|\xi\|_{\ell^1}.$$

The proof of this lemma follows the same lines as the proof of Lemma 5.3.
Applying Lemmas 5.2, 6.1 and 6.2 to the terms in Eq. (52), one obtains

$$\sum_{n\in\mathbb{Z}^2} \frac{d}{dt}|\xi_n| = \frac{d}{dt}\|\xi\|_{\ell^1} \le \sum_{n\in\mathbb{Z}^2} \left(f_n(t) + C\lambda^{-(2+\sigma/2)}\right)|\xi_n| + \lambda^{-(3+2\sigma)}.$$

The rest of the proof of Theorem 2.7 is analogous to the proof of Theorem 2.6. Defining ζ as in (55) we obtain Eq. (56). Then, we can deduce that

$$\|\zeta(T)\|_{\ell^1} \le \|\zeta(0)\|_{\ell^1} + C\lambda^{-(1+2\sigma)}\gamma N^2 \le 2C\lambda^{-(1+2\sigma)}\gamma N^2.$$

From this bound and recalling the definition of λ in (34), we can deduce, taking N large enough, that ξ satisfies

$$\|\xi\|_{\ell^1} \le 2e^{C\gamma N}\|\zeta\|_{\ell^1} \le \lambda^{-(1+\sigma)}.$$

Therefore we can drop the bootstrap assumption (57). This completes the proof of Theorem 2.7.

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