Lecture 2: The invariant manifolds of infinity and their intersection

Marcel Guardia

Universitat Politècnica de Catalunya

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Moser approach

- McGehee coordinates send “infinity” to the origin. Then, infinity becomes a parabolic fixed point (linearization equal to the identity).
- This parabolic point has invariant manifolds (Parabolic motions).
- Prove that they intersect transversally.
- Establish symbolic dynamics close to these invariant manifolds.

Outline

- The McGehee transformation and the local invariant manifolds of infinity.
- The transversality between the invariant manifolds: the Poincaré–Melnikov method
- Simó and Llibre result: the transversality of invariant manifolds provided $\mu \ll e^{-\frac{G_0^3}{3}}$ and $G_0 \gg 1$.

The RPC3BP

- The RPC3BP in rotating polar coordinates

$$H(r,\phi,y,G;\mu) = \frac{y^2}{2} + \frac{G^2}{2r^2} - G - U(r,\phi;\mu),$$

  where $U(r,\phi;\mu)$ is the Newtonian potential

$$U(r,\phi) = \frac{(1-\mu)}{\|re^{i\phi}+\mu\|} + \frac{\mu}{\|re^{i\phi}-(1-\mu)\|}.$$

- Two parameters: μ and G_0.
- Now H corresponds to the Jacobi constant and we can identify it with G_0.
- Infinity: $(r,y) = (+\infty, 0)$

The McGehee coordinates

- McGehee change of coordinates $r = \frac{2}{x^2}$ sends infinity to $x = 0$.
- Only the region $x > 0$ is meaningful.
- Integrable case $\mu \to 0$,

$$\dot{x} = -\frac{1}{4}x^3 y \qquad \dot{\phi} = -1 + \frac{1}{4}x^4 G$$

$$\dot{y} = \frac{1}{8}G^2 x^6 - \frac{1}{4}x^4 \qquad \dot{G} = 0$$

- G and

$$K_0(x, y, G) = H_0\left(\frac{2}{x^2}, y, G\right) = \frac{y^2}{2} + \frac{G^2 x^4}{8} - \frac{x^2}{2}.$$

are first integrals.

Infinity in McGehee coordinates

$$\dot{x} = -\frac{1}{4}x^3 y \qquad \dot{\phi} = -1 + \frac{1}{4}x^4 G$$

$$\dot{y} = \frac{1}{8}G^2 x^6 - \frac{1}{4}x^4 \qquad \dot{G} = 0$$

- The cylinder at infinity is $(x, y) = (0, 0)$, which is invariant.
- For any value of G_0:

$$\Lambda_{G_0} = \{(x, y, \phi, G) = (0, 0, \phi, G_0), \phi \in \mathbb{T}\}$$

  is a periodic solution.
- The cylinder of infinity $\Lambda = \cup_{G_0} \Lambda_{G_0}$ is foliated by periodic orbits (one at each level of energy).

Invariant manifolds of infinity when $\mu = 0$

$$K_0(x, y, G) = \frac{y^2}{2} + \frac{G^2 x^4}{8} - \frac{x^2}{2}.$$

- For fixed $G = G_0$, dynamics in the (x, y) plane

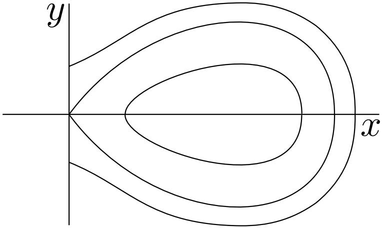
$$\dot{x} = -\frac{1}{4}x^3 y = -\frac{1}{4}x^3 \partial_y K_0$$
$$\dot{y} = \frac{1}{8}G_0^2 x^6 - \frac{1}{4}x^4 = \frac{1}{4}x^3 \partial_x K_0$$

- The vector field has no linear part at $(x, y) = (0, 0)$.
- It is a parabolic point (linear part equal to zero).
- Parabolic points do not always have invariant manifolds.
- In this case it is enough to use the first integral

$$K_0(x, y, G_0) = \frac{y^2}{2} + \frac{G_0^2 x^4}{8} - \frac{x^2}{2}.$$

Invariant manifolds of infinity when $\mu = 0$

- Stable/ unstable manifolds defined by $\dfrac{y^2}{2} + \dfrac{G_0^2 x^4}{8} - \dfrac{x^2}{2} = 0$
- We have a separatrix as for the Duffing equation.
- The vertical axis is infinity.
- Attention: Orbits on the separatrix do not tend to $(x, y) = (0, 0)$ exponentially but polynomially.



Local behavior of the RPC3BP

- Local behavior in McGehee Coordinates

$$
\begin{aligned}
\dot{x} &= -\frac{1}{4}x^3 y \\
\dot{y} &= \frac{1}{8}G^2 x^6 + \partial_r U\left(2x^{-2}, \phi\right) = -\frac{1}{4}x^4 + x^6\mathcal{O}_0 \\
\dot{\phi} &= -1 + \frac{1}{4}Gx^4 \\
\dot{G} &= \partial_\alpha U\left(2x^{-2}, \phi\right) = x^6\mathcal{O}_0
\end{aligned}
$$

  where $\mathcal{O}_k = \mathcal{O}(\|(x,y)\|^k)$.

- Reducing by the energy, we can eliminate G.
- Reparameterizing time we can identify it with ϕ.
- So, we can consider a non-autonomous system in the plane (without modifying the first order).

The local invariant manifolds

- Local behavior

$$x' = -\frac{1}{4}x^3y + x^6 f_1(x, y, \phi)$$
$$y' = -\frac{1}{4}x^4 + x^6 f_2(x, y, \phi)$$

- The change of coordinates

$$q = \frac{1}{2}(x - y)$$
$$p = \frac{1}{2}(x + y)$$

leads to

$$q' = \frac{1}{4}(q + p)^3(q + (q + p)^3\mathcal{O}_0)$$
$$p' = -\frac{1}{4}(q + p)^3(q + (q + p)^3\mathcal{O}_0)$$

Local invariant manifolds

- New system, defined for $q+p>0$,

$$q' = \frac{1}{4}(q+p)^3(q+(q+p)^3\mathcal{O}_0)$$
$$p' = -\frac{1}{4}(q+p)^3(q+(q+p)^3\mathcal{O}_0)$$

- General case of invariant manifolds of parabolic objects is quite open.

- Here just look for invariant manifolds for periodic orbits (fixed points for the Poincaré map).

Local invariant manifolds

- McGehee proved the existence of the local invariant manifolds as graphs for all $\mu \in [0, 1/2]$ and $G_0 > 0$.

Theorem

Fix $G_0 > 0$. The periodic orbit Λ_{G_0} at $(p, q) = (0, 0)$ possesses invariant stable and unstable manifolds, $W^u_{G_0}$ and $W^s_{G_0}$.
More concretely,

$$W^u_{G_0} = \{(q, p, \phi, G) \mid p = \gamma^u(q, \phi_0, G_0),\ q \in [0, q_0)\}$$

where

1. γ^u *is* $\mathcal{C}^\infty$ *with respect to q and analytic with respect to* (ϕ_0, G_0),
2. $\gamma^u(q, \phi_0, G_0) = \mathcal{O}(q^2)$.

- The analogous statement holds for W^s, as a graph over p.

Regularity of the invariant manifolds

- The invariant manifolds are analytic away from $q = 0$.
- We need regularity up to the origin to perform a change of coordinates which straightens the invariant manifolds.
- Hyperbolic case: stable/unstable invariant manifolds of periodic orbits of analytic systems are analytic.
- Parabolic case: they are only $\mathcal{C}^\infty$ at $q = 0$.
- Baldomá-Fontich-Martin: the invariant manifolds are $1/3$-Gevrey at $q = 0$: the Taylor coefficients of the parameterization grow as

$$|P_k| \leq C_1 C_2^k (k!)^{1/3}$$

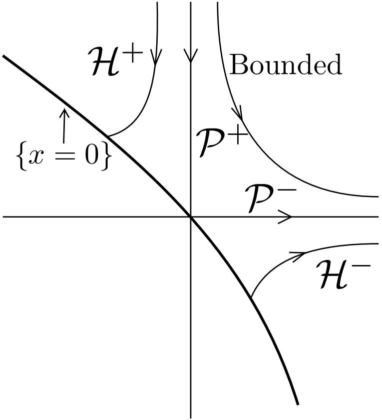
Straightening of the invariant manifolds

- After a change of coordinates

$$\widetilde{q}' = \frac{1}{4}(\widetilde{q}+\widetilde{p})^3\widetilde{q}(1+\mathcal{O}_2)$$
$$\widetilde{p}' = -\frac{1}{4}(\widetilde{q}+\widetilde{p})^3\widetilde{p}(1+\mathcal{O}_2)$$

- The invariant manifolds are the axes.

- We will use this system on Lecture 4 to study the dynamics close to infinity.



The global invariant manifolds

- We want to study the global invariant manifolds and prove that they intersect transversally.
- Poincaré–Melnikov Theory allows to prove the transversality of the invariant manifolds (in some settings).
- We apply Melnikov Theory to the RPC3BP.
- Tomorrow, we will explain how to prove transversality when Melnikov Theory cannot be applied.
- This will be related to exponentially small phenomena.

Melnikov Theory

- Consider a Hamiltonian with $1 + \frac{1}{2}$ degrees of freedom with 2π–periodic time dependence:

$$H(q, p, t; \delta) = H_0(q, p) + \delta H_1(q, p, t; \delta),$$

- Assume that the origin $(q, p) = (0, 0)$ is and equilibrium of saddle type at $H_0 = 0$

- It has associated separatrices included in $H_0^{-1}(0)$.

- Consider one of the separatrices (for instance the one with $p > 0$) and write it as

$$\{(q_0(t), p_0(t)), t \in \mathbb{R}\} =: \Gamma.$$

The easiest possible example: perturbations of the pendulum

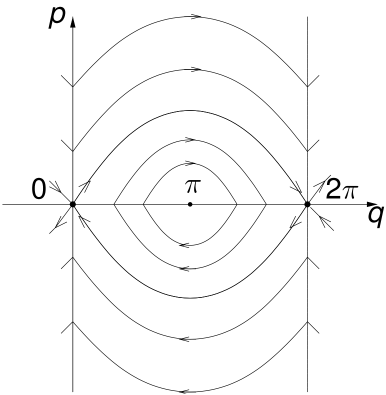
- Example: the pendulum
  $H_0(q,p) = \frac{p^2}{2} + (\cos q - 1)$.

- Parameterization of the upper separatrix

$$q_0(t) = 4\arctan(e^t)$$
$$p_0(t) = \frac{2}{\cosh t}$$

  so that
  $(q_0(t), p_0(t)) \to (0 \bmod 2\pi, 0)$
  for $t \to \pm\infty$.

- So $|q_0(t)| + |p_0(t)| \leqslant Ce^{-|t|}$.



The unperturbed case: the extended phase space

- Extended phase space: add time as variable ($\dot{s} = 1$).
- $\Lambda = \{(q, p) = (0, 0), s \in \mathbb{T}\}$ is a saddle periodic orbit.
- Λ has coincident stable and unstable surfaces.
- Motion on the upper 2-dimensional separatrix

$$W^0(\widetilde{\Lambda}) = \{(q_0(\tau), p_0(\tau), s), \tau \in \mathbb{R}, s \in \mathbb{T}\} = \Gamma \times \mathbb{T}$$

  is

$$\phi(t; q_0(\tau), p_0(\tau), s_0) = (q_0(\tau + t), p_0(\tau + t), s_0 + t)$$

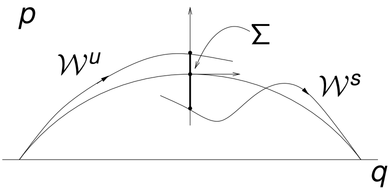
The perturbed case: $\delta \neq 0$ small

$$H(q, p, t; \delta) = H_0(q, p) + \delta H_1(q, p, t; \delta),$$

- For $\delta \ll 1$, there exists a periodic orbit Λ_δ, with stable/unstable manifolds $W^{s,u}_{loc}(\widetilde{\Lambda_\delta})$ and

$$\Lambda_\delta = \Lambda + O(\delta), \quad W^{s,u}_{loc}(\Lambda_\delta) = W^{s,u}_{loc}(\Lambda) + O(\delta)$$

- We want conditions that imply $W^s(\widetilde{\Lambda_\delta}) \neq W^u(\widetilde{\Lambda_\delta})$
- Since $W^s(\Lambda_\delta)$ and $W^u(\Lambda_\delta)$ are close to $W^0(\Lambda)$, they can be also parameterized in terms of τ and s



- We fix Σ, a transversal section to the unperturbed separatrix in order to measure in it the splitting.
- We consider a parameterization γ of the unperturbed separatrix such that $\gamma(0)$ belongs to this section.

- Poincaré–Melnikov Theory: expand the parameterizations of the invariant manifolds in power series in δ and compute the first order of their difference at Σ.
- The distance will be a function periodic in s.

- We define the Melnikov function as:

$$\mathcal{M}(s) = \int_{-\infty}^{+\infty} \{H_0, H_1\}\,(q_0(t), p_0(t), t+s)\,dt$$

- We have that $|q_0(t)| + |p_0(t)| \leqslant Ce^{-\lambda|t|}$ where $\lambda > 0$ is the eigenvalue of the saddle.
- This implies that the integral is convergent.
- M can be computed since H_0, H_1 and γ are known.
- Often is not so easy to compute analytically this integral.
- If the perturbation h is T-periodic, the Melnikov potential and function are also T-periodic.

Poincaré–Melnikov Theory

- The distance between both invariant manifolds for $\delta > 0$ is given by:

$$d(s, \delta) = \delta \frac{M(s)}{\|DH_0(\gamma(0))\|} + \mathcal{O}\left(\delta^2\right)$$

- If there exists s_0 such that

$$(i)\; M(s_0) = 0 \quad (ii)\; \left.\frac{\partial M}{\partial s}\right|_{s=s_0} \neq 0$$

Then the invariant manifolds intersect transversally at Σ for some s' δ-close to s_0.

An example: The perturbed pendulum

$$H\left(p, q, \frac{t}{T}\right) = \frac{p^2}{2} + (\cos q - 1) + \delta(\cos q - 1)\sin\frac{t}{T}$$

- Equations

$$\dot{q} = p$$
$$\dot{p} = \sin q + \delta \sin q \sin\frac{t}{T}$$

- $\Lambda_\delta = \{(0,0)\}$ is a hyperbolic periodic orbit for this system.
- The Melnikov function is

$$\begin{aligned}\mathcal{M}(s) &= \int_{-\infty}^{\infty} \{H_0, H_1\}(p_0(\sigma), q_0(\sigma), s+\sigma)d\sigma \\ &= \int_{-\infty}^{\infty} p_0(\sigma)\sin q_0(\sigma)\sin\frac{s+\sigma}{T}d\sigma \\ &= 4\cos\frac{s}{T}\int_{-\infty}^{\infty}\frac{\sinh\sigma}{\cosh^3\sigma}\sin\frac{\sigma}{T}d\sigma\end{aligned}$$

An example: The perturbed pendulum

- So we only need to compute the integral

$$\int_{-\infty}^{\infty} \frac{\sinh t}{\cosh^3 t} e^{i\sigma/T} d\sigma = \frac{\pi i}{T^2} \frac{1}{2\sinh(\pi/2T)}.$$

- $\cosh t$ has zeros at $t = \pm i\pi/2$.
- Applying Residums Theorem, the Melnikov function is

$$\mathcal{M}(s) = \frac{2\pi}{T^2} \frac{1}{\sinh(\pi/2T)} \cos \frac{s}{T}.$$

- The distance between manifolds is given by

$$d(s,\delta) = \delta \frac{M(s)}{\|DH_0(\gamma(0))\|} + \mathcal{O}\left(\delta^2\right)$$

so

$$d(s,\delta) = \delta \frac{1}{\|DH_0(\gamma(0))\|} \frac{2\pi}{T^2} \frac{1}{\sinh(\pi/2T)} \cos \frac{s}{T} + \mathcal{O}(\delta^2)$$

- Non-degenerate zeros of $\mathcal{M}(s)$ give rise to transversal homoclinic orbits.

What happens if we have a fast forcing?

- Take $T = \varepsilon \ll 1$:

$$H\left(p, q, \frac{t}{\varepsilon}\right) = \frac{p^2}{2} + (\cos q - 1) + \delta(\cos q - 1)\sin\frac{t}{\varepsilon}$$

- Apply Poincaré-Melnikov: fix ε and expand in δ.
- The Melnikov function is:

$$\mathcal{M}(s, \varepsilon) = \frac{2\pi}{\varepsilon^2}\frac{1}{\sinh(\pi/2\varepsilon)}\cos\frac{s}{\varepsilon} \sim \frac{4\pi}{\varepsilon^2}e^{-\frac{\pi}{2\varepsilon}}\cos\frac{s}{\varepsilon}.$$

- The distance between manifolds is given by

$$d(s, \varepsilon, \delta) = \delta\frac{4\pi}{\|DH_0(\gamma(0))\|}e^{-\frac{\pi}{2\varepsilon}}\cos\frac{s}{\varepsilon} + \mathcal{O}(\delta^2)$$

- We need $\delta = \mathcal{O}(e^{-\frac{\pi}{2\varepsilon}})$ to make the error term smaller!
- If we want $\delta \sim \varepsilon$, Poincaré-Melinkov Theory cannot be applied.

What happens if we have a fast forcing?

- Poincaré-Melnikov Theory for periodic fast perturbations only works provided $\delta = \mathcal{O}(e^{-\frac{\pi}{2\varepsilon}})$.
- If we want $\delta \sim \varepsilon$, Poincaré-Melinkov Theory cannot be applied.
- Exponentially small perturbation is extremely restrictive.
- Fast forcing is a very important setting: appears tipically when studying invariant manifolds in the resonances of nearly integrable Hamiltonian systems.
- It appears in many Arnold diffusion problems ("a priori stable" setting).

Melnikov Theory for the PRCRBP

- Recall that we have two parameters μ and G_0.
- Unperturbed separatrix (parabolic motion) was parameterized as

$$
\begin{aligned}
r &= \frac{1}{2} G_0^2 (1 + \tau^2) \\
\alpha &= \alpha_0 + \pi + 2 \arctan \tau \\
y &= \frac{2\tau}{G_0 (1 + \tau^2)} \\
G &= G_0
\end{aligned}
\qquad \text{where } t = \frac{G_0^3}{2} \left(\tau + \frac{\tau^3}{3} \right).
$$

- Recall that $(r, y) = (0, 0)$ is parabolic.
- As long as the local invariant manifolds are defined, we can (try to) apply Melnikov Theory.
- To have an unperturbed problem and a separatrix independent of the parameters, we apply a scaling.

The scaled system

- Fix $G_0 > 0$.
- Rescaling: $r = G_0^2 \widetilde{r}, \quad y = G_0^{-1} \widetilde{y}, \quad \alpha = \widetilde{\alpha}, \quad G = G_0 \widetilde{G}$.
- Time rescaling: $t = G_0^3 s$.
- New Hamiltonian

$$\widetilde{H}(\widetilde{r}, \widetilde{\alpha}, \widetilde{y}, \widetilde{G}, s) = \frac{\widetilde{y}^2}{2} + \frac{\widetilde{G}^2}{2\widetilde{r}^2} - \frac{(1-\mu)}{\|\widetilde{r}e^{i(\widetilde{\alpha} - G_0^3 s} + \mu G_0^{-2}\|} - \frac{\mu}{\|\widetilde{r}e^{i\widetilde{\alpha} - G_0^3 s} - (1-\mu) G_0^{-2}\|}.$$

- Expanding,

$$\widetilde{H}(\widetilde{r}, \widetilde{\alpha}, \widetilde{y}, \widetilde{G}, s) = \frac{\widetilde{y}^2}{2} + \frac{\widetilde{G}^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{\widetilde{r}}} + H_1(\widetilde{r}, \widetilde{\alpha}, \widetilde{y}, \widetilde{G}, G_0^3 s)$$

with $H_1 = \mathcal{O}\left(\mu G_0^{-2}\right)$.

The scaled separatrix

- Scaled separatrix:

$$\begin{aligned} r &= \frac{1}{2}(1+\tau^2) \\ \alpha &= \alpha_0 + \pi + 2\arctan\tau \\ y &= \frac{2\tau}{(1+\tau^2)} \\ G &= 1 \end{aligned} \qquad \text{where } t = \frac{1}{2}\left(\tau + \frac{\tau^3}{3}\right).$$

- In McGehee coordinates: $x = 2\left(1+\tau^2\right)^{-1/2}$.
- We apply Poincaré-Melnikov Theory using μ as a small parameter.
- The perturbation depends on G_0 and so $\mathcal{M}(s, G_0)$ too.
- The computation of this Melnikov function is not so easy.
- We only know how to compute it if we assume $G_0 \gg 1$.

Melnikov Theory applied to the RPC3BP

- Compared with

$$H\left(p, q, \frac{t}{\varepsilon}\right) = \frac{p^2}{2} + (\cos q - 1) + \delta(\cos q - 1)\sin\frac{t}{\varepsilon}$$

we have $\varepsilon = G_0^{-3}$ and $\delta = \mu$.

- Melnikov function

$$\mathcal{M}(s) = CG_0^{3/2} e^{-\frac{G_0^3}{3}} \left(\sin(G_0^3 s) + \mathcal{O}\left(G_0^{-1}\right)\right)$$

for some computable constant $C > 0$.

- Distance between manifolds

$$d(s, \mu, G_0) \sim \mu C G_0^{3/2} e^{-\frac{G_0^3}{3}} \left(\sin(G_0^3 s) + \mathcal{O}\left(G_0^{-1}\right)\right) + \mathcal{O}\left(\mu^2 G_0^{-4}\right).$$

Melnikov Theory applied to the RPC3BP

- So, Melnikov Theory only applied provided $\mu \ll G_0^{-3/2} e^{-\frac{G_0^3}{3}}$
- Simó and Llibre used this condition to prove the transversality of the invariant manifolds.
- Only proved existence of oscillatory motions under this condition.
- Next day,
  - We will explain how to prove the transversality for fast periodic perturbations.
  - I will apply it to the RPC3BP for any $\mu \in (0, 1/2]$ and $G_0 \gg 1$.
  - Construct the symbolic dynamics that leads to oscillatory motions for the RPC3BP.