Chapter 6

Graph Enumeration

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6.1 Introduction

Many references to date on the enumeration of graphs deal with unlabelled graphs, like the monographs of Harary and Palmer [36] and Polya and Read [57]. The main tool there is the cycle index polynomial and the goal is to obtain exact expressions for the number of graphs of a given kind, like trees or connected graphs. Our emphasis in this survey is instead on labelled graphs. One reason is that in many interesting cases the number of unlabelled graphs with \( n \) vertices is asymptotically the number of labelled graphs divided by \( n! \), the number of possible labellings. This is not always so, as in the case of trees, since a random tree has almost surely exponentially many automorphisms.

A class of graphs is a set of labelled graphs closed under isomorphism. Typical examples of interest are bipartite, acyclic, triangle-free or planar graphs. We use \( n \) to denote the number of vertices in a graph. Let \( \mathcal{G} \) be a class of graphs and \( \mathcal{G}_n \) the graphs in \( \mathcal{G} \) with \( n \) vertices. The main problem we consider is to compute or to estimate \( |\mathcal{G}_n| \) when \( n \to \infty \). In very few cases we have exact formulas. Here are some examples:

- The number of labelled graphs is \( 2\binom{n}{2} \). This is because each of the \( \binom{n}{2} \) edges of the complete graph can be chosen independently to be or not in a graph. Likewise, the number of graphs with \( n \) vertices and \( m \) edges is equal to \( \binom{n}{2}^m \).

- The number of labelled even graphs (all vertices have even degree) is \( 2^{\binom{n-1}{2}} \). There is a very simple proof of this fact: given a graph \( G \) on \( n - 1 \) vertices, add a new vertex labelled \( n \) and connect it to all vertices in \( G \) of odd degree. This gives an even graph on \( n \) vertices and it is easy to check that this is in fact a bijection.

- The number of labelled trees is \( n^{n-2} \). This is the well-known Cayley’s formula, that admits many different proofs. There is no simple formula for the number of forests (acyclic graphs). However, the number of labelled forests consisting of \( k \) rooted trees is \( \binom{n}{k}kn^{n-k-1} \).

The main emphasis in this chapter is on asymptotic results. In many cases we use generating functions, which are of the exponential type since graphs are labelled. We start with a basic example. Let \( G_n = 2^{\binom{n}{2}} \) be the number of graphs on \( n \) vertices, with the convention that \( G_0 = 1 \). In order to compute the number \( C_n \) of connected graphs, we introduce the generating functions

\[
G(x) = \sum_{n \geq 0} G_n \frac{x^n}{n!}, \quad C(x) = \sum_{n \geq 0} C_n \frac{x^n}{n!}.
\]

The exponential formula \( G(x) = e^{C(x)} \) implies that

\[
C(x) = \log \left( 1 + \sum_{n \geq 1} 2^{\binom{n}{2}} \frac{x^n}{n!} \right).
\]
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Using the expansion
\[ \log(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots \]
and extracting coefficients, it is easy to see that \( C_n \sim G_n \). In other words, almost all graphs are connected. But this can be proved more easily using the theory of random graphs, as discussed next.

Consider the random graph binomial model \( G(n, p) \) with \( n \) vertices, in which every edge is drawn independently with probability \( p \). The probability of a graph with \( m \) edges is
\[ p^m (1 - p)^{\binom{n}{2} - m}. \]
If \( p = 1/2 \), then every graph has the same probability \( 2^{-\binom{n}{2}} \), and the distribution is uniform among all graphs with \( n \) vertices. We say that almost all graphs in \( G(n, p) \) satisfy property \( \mathcal{A} \) if the probability that a graph with \( n \) vertices satisfies \( \mathcal{A} \) tends to 1, as \( n \) tends to \( \infty \). It is easy to prove (see [10]) that the following properties hold for almost all graphs in \( G(n, 1/2) \), hence they hold for almost all graphs under the uniform distribution.

Let \( R_n \) be a random labelled graph with \( n \) vertices. Then almost surely, as \( n \to \infty \), \( R_n \) satisfies:

- \( R_n \) has diameter two. In particular, it is connected.
- \( R_n \) is \( k \)-connected for every fixed \( k \geq 1 \).
- For every fixed graph \( H \), \( R_n \) contains an induced subgraph isomorphic to \( H \).
- \( R_n \) is Hamiltonian. In particular, it has a perfect matching if the number of vertices is even.
- \( R_n \) has no non-trivial automorphisms.

In the other direction, almost no graph is regular, or planar (since it contains \( K_5 \) as a subgraph), or \( r \)-colourable for fixed \( r \) (since it contains \( K_{r+1} \) as subgraph). One can prove much deeper results using the theory of random graphs. For instance, the chromatic number of a random graph with \( n \) vertices is close to \( n/(2 \log_2 n) \), and the independence number is close to \( 2 \log_2 n \). We will not pursue this topic here, the interested reader can find many more results in the reference textbooks [10, 40] of this fascinating topic.

Throughout the paper we use the following conventions. Given a class of labelled graphs \( \mathcal{G} \), let \( G_n = |\mathcal{G}_n| \) be the number of graphs in \( \mathcal{G} \) with \( n \) vertices. The associated generating function is
\[ G(x) = \sum_n G_n \frac{x^n}{n!}. \]
As a rule we use the same letter for the class, the sequence we want to enumerate and the corresponding generating function. We use the variable \( x \) for marking vertices, and variable \( y \) for edges. If \( G(x, y) \) is a series in two variables, \( G_v(x, y) \) and \( G_e(x, y) \) denote the partial derivatives. We make systematic use of the symbolic method, as in
For instance, if \( \mathcal{G} \) is a class of graphs closed under taking connected components, the exponential formula

\[
G(x) = e^{C(x)}
\]

relates the generating function \( G(x) \) of graphs in \( \mathcal{G} \) with \( C(x) \), the one for connected graphs in \( \mathcal{G} \). As another example, \( xG'(x) = \sum nG_n x^n / n! \) is the generating function of rooted graphs, since a labelled graph with \( n \) vertices can be rooted in \( n \) different ways. Sequences appearing in the Online Encyclopedia of Integer Sequences are referred by their code, for instance A001187.

The notation and terminology for graph theory is standard. Given a graph \( G = (V, E) \), the subgraph induced by \( U \subseteq V \) is the graph \( (U, A) \), where \( A \) is the set of edges joining vertices of \( U \). A set \( U \) of vertices is independent if there are no edges joining vertices of \( G \). A graph is \( k \)-partite if the vertex set can be partitioned as \( V = V_1 \cup \cdots \cup V_k \) such that each \( V_i \) is independent. The chromatic number \( \chi(G) \) is the minimum number \( k \) of colours in a colouring of \( V \) such that adjacent vertices receive different colours. A graph \( G \) is \( k \)-partite if and only if \( \chi(G) \leq k \). A graph is bipartite if and only if has no odd cycles.

The rest of the paper is organized as follows. In Section 6.2 we present basic decompositions of graphs and the corresponding equations for the associated generating functions. This is applied in particular to arbitrary and bipartite graphs. In Section 6.3 we discuss the enumeration of connected graphs with given excess; this is a classical problem with strong connections to the theory of random graphs. Section 6.4 is devoted to regular graphs, another classical problem. In Section 6.5 we cover monotone and hereditary classes, defined in terms of forbidden graphs. This is an active area of research closely related to extremal graph theory. In Sections 6.6 and 6.7 we discuss the enumeration of planar graphs, graphs on surfaces, and classes of graphs defined in terms of excluded minors, which are more recent topics of research. In Section 6.8 we present briefly some results on the enumeration of digraphs. We conclude with several remarks.

### 6.2 Graph decompositions

The results in this section are based on the classical decompositions of graphs into \( k \)-connected components, and on the decomposition of a connected graph into the 2-core and the trees attached to it. As is well-known, a graph decomposes into its connected components, and a connected graph decomposes into its 2-connected components, also called blocks: a block is a maximal 2-connected subgraph or a separating edge. The decomposition into blocks has a tree structure. Furthermore, a 2-connected graph decomposes into its so-called 3-connected components: these are not necessarily subgraphs of the host graph and the decomposition is more difficult to describe (see for instance [16, 31] for a self-contained discussion). Briefly, one decomposes a 2-connected graph \( G \) along its 2-separators in a canonical way into bricks, which are
either cycles, multi-edges or 3-connected graphs. The decomposition is tree-like and can be encoded using generating functions as discussed next.

### 6.2.1 Graphs with given connectivity

Let \( \mathcal{G} \) be a class of graphs with the following properties:

\( C_1 \) A graph \( G \) is in \( \mathcal{G} \) if and only if the connected components of \( G \) are in \( \mathcal{G} \).

\( C_2 \) A connected graph \( G \) is in \( \mathcal{G} \) if and only if the blocks of \( G \) are in \( \mathcal{G} \).

This holds clearly for the class of all graphs, but also for other classes such as planar graphs. The following derivations can be found in [36]. We denote by \( C \) the connected graphs in \( \mathcal{G} \), and by \( B \) the 2-connected graphs in \( \mathcal{G} \). The corresponding generating functions are denoted by

\[
G(x) = \sum G_n \frac{x^n}{n!}, \quad C(x) = \sum C_n \frac{x^n}{n!}, \quad B(x) = \sum B_n \frac{x^n}{n!}.
\]

The decomposition of a graph into connected components, and the decomposition of a connected graph into 2-connected components imply the equations

\[
G(x) = e^{C(x)}, \quad (6.1)
\]

and

\[
C'(x) = e^{B'(xC(x))}. \quad (6.2)
\]

Equation (6.1) is the exponential formula for sets of labelled combinatorial objects, and (6.2) is based on the recursive decomposition of a rooted connected graph into its blocks. The generating function \( \sum nC_n x^n / n! \) enumerates connected graphs rooted at a vertex, since a labelled graph can be rooted at \( n \) different vertices.

It follows that

\[
C(x) = \log G(x),
\]

and \( R(x) = xC'(x) \) satisfies the implicit equation

\[
R(x) = xe^{-B'(R(x))}. \quad (6.2)
\]

We can rephrase the former equation by saying that \( R(x) \) is the functional inverse of \( xe^{-B'(x)} \).

If \( \mathcal{G} \) is the class of all graph then \( G_n = 2^{(n)} \). The former equations determine \( C(x) \) and \( G(x) \) uniquely and we obtain

\[
C(x) = x + \frac{x^2}{2!} + 4 \frac{x^3}{3!} + 38 \frac{x^4}{4!} + 728 \frac{x^5}{5!} + 26704 \frac{x^6}{6!} + \cdots \quad A001187
\]

Using this expansion we also obtain

\[
B(x) = \frac{x^2}{2} + \frac{x^3}{3!} + 10 \frac{x^4}{4!} + 238 \frac{x^5}{5!} + 11368 \frac{x^6}{6!} + \cdots
\]
The previous analysis can be enriched by enumerating graphs also by the number of edges. Let $G_{n,k}$ be the number of graphs with $n$ vertices and $k$ edges and let

$$G(x,y) = \sum G_{n,k} x^n y^k / n!,$$

and define analogously $C(x,y)$. Notice that $G(x,y)$ is an ordinary generating function in $y$, since edges are not labelled.

Equations (6.1) and (6.2) generalize to

$$G(x,y) = e^{C(x,y)}, \quad C_r(x,y) = xe^{B(x,C_r(x,y))},$$

where $R(x,y) = xC_r(x,y)$. These equations hold because connected components and blocks do not share edges.

If again $\mathcal{G}$ is the class of all graphs then

$$G(x,y) = \sum_{n,m} \binom{n}{2} m! \frac{x^n y^m}{n!} = \sum n (1+y) \binom{n}{2} \frac{x^n}{n!}.$$

We obtain

$$C(x,y) = x + y \frac{x^2}{2!} + (3y^2 + y^3) \frac{x^3}{3!} + (16y^3 + 15y^4 + 6y^5 + y^6) \frac{x^4}{4!} + \cdots$$

and

$$B(x,y) = y \frac{x^2}{2!} + y^2 \frac{x^3}{3!} + (3y^4 + 6y^5 + y^6) \frac{x^4}{4!} + \cdots$$

Next we turn to the enumeration of 3-connected graphs using the decomposition of 2-connected graphs into 3-connected components, for which we follow [75]. We add the hypothesis

(C3) A 2-connected graph $G$ is in $\mathcal{G}$ if and only if the 3-connected components of $G$ are in $\mathcal{G}$.

Define a network as a 2-connected graph rooted at a directed edge (which may be deleted or not) and whose endpoints, called poles, are not labelled. Let $D(x,y)$ be the generating functions of networks. Then

$$2(1+y)B(x,y) = x^2 (1 + D(x,y)). \quad (6.3)$$

The left-hand term corresponds to marking and directing an edge, and keeping or not the edge; the right-hand side corresponds to adding the empty network and labelling the poles. We find

$$D(x,y) = y + (y^2 + y^3)x + (2y^3 + 7y^4 + 6y^5 + y^6) \frac{x^2}{2!} + \cdots$$
Let \( T(x, y) \) be the generating function of 3-connected graphs. The decomposition into 3-connected components gives rise to the equation

\[
1 + D(x, y) = (1 + y) \exp \left( \frac{xD(x,y)^2}{1 + xD(x,y)} + \frac{2}{x^2} T_y(x, D(x,y)) \right). 
\]  

(6.4)

Given \( D(x, y) \), this equation determines \( T(x, y) \) and

\[
T(x, y) = y^5 \frac{4}{4!} + (15y^8 + 10y^9 + y^{10}) \frac{y^5}{5!} + \cdots
\]

Setting \( y = 1 \) we obtain the counting series of 3-connected graphs

\[
T(x, 1) = \frac{x^4}{4!} + 26 \frac{x^5}{5!} + 1768 \frac{x^6}{6!} + 225096 \frac{x^7}{7!} + \cdots
\]

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As noticed in [32], it is possible to integrate (6.3) with respect to \( y \) and express \( B(x, y) \) as an explicit function of \( D \) and \( B \):

\[
B(x, y) = T(x, D(x,y)) - \frac{1}{2} x D(x,y) + \frac{1}{2} \log(1 + xD(x,y)) + \frac{x^2}{2} \left( D(x,y) + \frac{1}{2} D(x,y)^2 + (1 + D(x,y)) \frac{1+y}{1+D(x,y)} \right). 
\]

(6.5)

It is worth noticing that all the previous equations can be proved more combinatorially using grammars and the dissymmetry theorem for trees [16]. We summarize the previous results in a single statement.

**Theorem 99** Let \( \mathcal{G} \) be a class of graphs satisfying conditions (C1), (C2) and (C3), and let \( G(x,y), C(x,y), B(x,y), D(x,y) \) and \( T(x, y) \) be the generating functions associated to, respectively, graphs, connected graphs, 2-connected graphs, networks and 3-connected graphs in \( \mathcal{G} \). Then

\[
G(x,y) = e^{C(x,y)}
\]

\[
C(x,y) = e^{B_2(xC(x,y), y)}
\]

\[
B_2(x,y) = \frac{x^2(1 + D(x,y))}{2(1 + y)}
\]

\[
D(x,y) = (1 + y) \exp \left( \frac{xD(x,y)^2}{1 + xD(x,y)} + \frac{2}{x^2} T_y(x, D(x,y)) \right) - 1
\]

\[
B(x,y) = T(x, D(x,y)) - \frac{1}{2} x D(x,y) + \frac{1}{2} \log(1 + xD(x,y)) + \frac{x^2}{2} \left( D(x,y) + \frac{1}{2} D(x,y)^2 + (1 + D(x,y)) \frac{1+y}{1+D(x,y)} \right). 
\]

For the class of all graphs, starting from the easy expression for \( G(x, y) \) we reach an equation defining \( T(x, y) \). In other situations is the opposite way; as we will see in Section 6.6, for planar graphs one starts from the known expression for \( T(x, y) \) and then goes all the way up to \( G(x, y) \).
Differential equations. In addition to the previous system of equations, it is possible to obtain directly $B(x,y)$ and $T(x,y)$ as solutions of second order partial differential equations. This can be done algebraically or more combinatorially as in [79] and [76]. Setting $B = B(x,y)$ and $T = T(x,y)$, one has

\[(2(1+y)B_y - x^2(1 + B_{xx}))(1 - xB_{xx}) - x^3B_{xx}^2 = 0\]

and

\[(1 + y)T_y \frac{x^2}{2} T_{xx} = \frac{x^4F^2}{4Fy} - \frac{x^4y^4}{4(1+xy)^2},\]

where

\[F = \log(1+y) - \frac{xy^2}{1+xy} - \frac{2}{x^2}T_y.\]

6.2.2 Graphs with given minimum degree

In this section we find the generating functions for graphs with minimum degree at least two and at least three. For brevity, we define a $k$-graph as a connected graph with minimum degree at least $k$.

Given a connected graph $G$, its 2-core is the maximum subgraph $C$ with minimum degree at least two. The 2-core $C$ is obtained from $G$ by repeatedly removing vertices of degree one. Conversely, $G$ is obtained by attaching rooted trees at the vertices of $C$. Let $H_n$ be the number of 2-graphs and $H$ the associated generating function. Let $T(x)$ be the generating function of rooted trees, that satisfies $T(x) = xe^{T(x)}$. The generating function of unrooted trees is equal to

\[U(x) = T(x) - \frac{T(x)^2}{2}.\]

The decomposition of a graph into its core and the forest of rooted trees implies the following equation:

\[C(x) = H(T(x)) + U(x).\]

The second summand takes care of trees, which have an empty 2-core. If we change variables $z = T(x)$, the inverse function is $x = ze^{-z}$. This implies

\[H(x) = C(xe^{-x}) - x + \frac{x^2}{2} = \frac{x^3}{3!} + 10\frac{x^4}{4!} + 253\frac{x^5}{5!} + 12058\frac{x^6}{6!} + \cdots\]

If as before $y$ marks edges, then

\[H(x,y) = C(xe^{-xy}) - x + \frac{x^2y}{2} = \frac{y^3x^3}{3!} + (3y^4 + 6y^5 + y^6)\frac{x^4}{4!} + \cdots\]

The kernel of $G$ is obtained by replacing each maximal path of vertices of degree two in the 2-core $C$ by a single edge. The kernel has minimum degree at least three, and $C$ can be recovered from $K$ by replacing edges with paths. The kernel may have
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loops and multiple edges, which must be taken into account since we are dealing with simple graphs. Also, when replacing loops and multiple edge by paths the same graph can be produced several times. This can be dealt with by weighting multigraphs appropriate according to the number of loops and edges of each multiplicity. Given a multigraph \( M \) with \( \alpha_i \) vertices incident to exactly \( i \) loops, and with \( \beta_i \) edges of multiplicity \( i \), the associated weight is

\[
w(G) = \prod_{i \geq 1} \left( \frac{1}{2^i i!} \right)^{\alpha_i} \prod_{i \geq 1} \left( \frac{1}{i!} \right)^{\beta_i}.
\]

The justification is that when replacing an edge of multiplicity \( i \) with \( i \) different paths, the order of the paths is irrelevant, and similarly for the loops. This weight is called the compensation factor in the literature [39].

We have \( 0 < w(G) < 1 \), and moreover \( w(G) = 1 \) if and only if \( G \) is simple. With this definition, the sum of the weights of all 3-multigraphs with \( n \) vertices is finite. Working first with multigraphs and then moving to simple graphs by setting to 0 the variables involving loops and multiedges, one shows the following [54] (this can also be deduced from the results in [38]).

Let \( K(x,y) \) be the generating function of simple 3-graphs. Then

\[
K(x,y) = C(A(x,y),B(x,y)) + E(x,y),
\]

where

\[
A(x,y) = x \exp \left( \frac{x^2 y^3 - 2xy}{2 + 2xy} \right),
\]

\[
B(x,y) = (1 + y) \exp \left( \frac{-xy}{1 + xy} \right) - 1,
\]

\[
E(x,y) = \frac{x^2 y^2}{2 + 2xy} - \frac{x^2 y^2}{4} + \frac{xy}{2} - x - \frac{1}{2} \ln(1 + xy).
\]

In particular

\[
K(x,1) = \frac{x^4}{4!} + 26 \frac{x^5}{5!} + 1858 \frac{x^6}{6!} + 236926 \frac{x^7}{7!} + \cdots
\]

This sequence is not referenced in OEIS.

Finally \( e^H(x) \) and \( e^K(x) \) enumerate graphs (not necessarily connected), with minimum degree at least two and three, respectively.

### 6.2.3 Bipartite graphs

In this section we compute the number of bipartite graphs. The number of bipartite graphs with parts of sizes \( a \) and \( b \) is \( \binom{a+b}{a} \cdot 2^a \), but in this way a graph may be counted more than once. A 2-coloured graph is a bipartite graph equipped with a fixed bipartition. A connected bipartite graph gives rise to two 2-coloured graphs, and a bipartite
graph with $c$ connected components to $2^c$ such 2-coloured graphs. The number $g_n$ of 2-coloured bipartite graphs is

$$g_n = \sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)}.$$ 

Since each connected bipartite graph is counted exactly twice, the generating function $C_b(x)$ for connected bipartite graphs satisfies $\exp(2C_b(x)) = \sum g_n x^n / n!$, and

$$C_b(x) = \frac{1}{2} \log \left( \sum g_n \frac{x^n}{n!} \right) = x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{19x^4}{4!} + \frac{195x^5}{5!} + \cdots$$

Hence the generating function for all bipartite graphs is equal to

$${G_b(x) = e^{C_b(x)} = \left( \sum_{n} \sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)} \frac{x^n}{n!} \right)^{1/2}}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{7x^3}{3!} + \frac{41x^4}{4!} + \frac{376x^5}{5!} + \cdots$$

The dominant term in the sum for $g_n$ is when $k = n/2$, giving $2^{n+1}/4$ as the exponential growth. As for general graphs, almost every bipartite graph is connected.

A graph is bipartite if and only if all its blocks are bipartite. Hence, according to Equation (6.2), the generating function $B_b(x)$ of 2-connected bipartite graphs satisfies

$$C_b'(x) = e^{B_b(x)}.$$ 

This gives

$$B_b(x) = \frac{x^2}{2!} + \frac{3x^4}{4!} + \frac{10x^5}{5!} + \frac{6986x^6}{6!} + \cdots$$

Recurrence relations for the previous numbers are derived in [37]. As far as we know, the problem of counting 3-connected bipartite graphs is open. The difficulty here is to control whether a graph is bipartite in terms of its 3-connected components.

Counting $k$-partite graphs for $k > 2$ is not so simple, since a connected $k$-coloured graph can be coloured in a number of ways that depends on the graph. We discuss this problem in Section 6.5.

### 6.3 Connected graphs with given excess

The excess of a connected graph $G$ is defined as $|E(G)| - |V(G)|$, the number of edges minus the number of vertices. Trees have excess $-1$, and graphs with excess 0 are precisely unicyclic graphs, that is, connected graphs with a unique cycle. For $k \geq -1$, let $C_{n,n+k}$ be the number of connected graphs with $n$ vertices and excess $k$. 


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Computing and estimating $C_{n,n+k}$ is a classical problem that arises in the study of random graphs [23, 10, 39], and is a crucial ingredient in the analysis of the giant component phenomenon.

The number of trees is $C_{n,n} = n^{n-2}$, by the well-known Cayley’s formula. There are several formulas for the number of unicyclic graphs. Wright [83] showed that

$$C(n,n) = \frac{1}{2} \left( \frac{h(n)}{n} - (n-1)n^{n-2} \right),$$

where

$$h(n) = \sum_{s=1}^{n-1} \binom{n}{s} s^2 (n-s)^{n-s}.$$ 


Similar expressions were obtained for larger excess. For instance,

$$C_{n,n+1} = \frac{1}{24} \left( (n-1)(5n^2 + 3n + 2)n^{n-2} - 14h(n) \right).$$

The first values for these sequences are as follows:

<table>
<thead>
<tr>
<th>n</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>OEIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{n,n}$ 1</td>
<td>15</td>
<td>222</td>
<td>3660</td>
<td>68295</td>
<td>1436568</td>
<td>A006351</td>
<td></td>
</tr>
<tr>
<td>$C_{n,n+1}$ 6</td>
<td>205</td>
<td>5700</td>
<td>156555</td>
<td>4483360</td>
<td>A058864</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

These formulas follow from the general recurrence relation found by Wright:

$$2(n+k+1)C_{n,n+k+1} = 2(n(n-1)/2 - n - k)C_{n,n+k}$$

$$+ \sum_{s=1}^{n} \binom{n}{s} s(n-s) \sum_{h=1}^{k+1} C_{s,s+h} C_{n-s,n-s+k-h}.$$ 

The proof is based on analyzing the result of removing one edge from a connected graph, producing either a connected graph or a graph with two connected components. Wright also showed that, for $k \geq 1$,

$$C_{n,n+k} = (-1)^k M_k(n) h(n) + (-1)^{k-1} P_k(n)(n-1)n^{n-2},$$

where and $M_k$ and $P_k$ are polynomials in $n$ that can be computed recursively. This implies that, for $k \geq 1$, the generating function

$$W_k(x) = \sum_{n \geq 1} C_{n,n+k} \frac{x^n}{n!}$$

is a rational function in $T(x) = \sum_{n \geq 1} n^{n-1} \frac{x^n}{n!}$, the generating function of rooted labelled trees, which satisfies the equation $T(x) = xeT(x)$. 


The first values of the $W_k(x)$ are

$$W_{-1}(x) = T(x) - \frac{T(x)^2}{2}$$

$$W_0(z) = \frac{1}{2} \left( \log \frac{1}{1-T(x)} - T(x) - \frac{T(x)^2}{2} \right)$$

$$W_1(z) = \frac{T(x)^4(6 - T(x))}{24(1 - T(x))^3}$$

The expression for $W_{-1}(x)$ is the well-known generating function for unrooted trees. The one for $W_0(x)$ follows from viewing a unicyclic graph as an undirected cycle of Ch. Trees rooted trees of length at least three.

From recurrence (6.7) Wright obtained an estimate for $C_{n,n+k}$ for fixed $k$. In a subsequent paper Wright [84] extended this estimate to values $k = o(n^{1/3})$. The estimate is

$$C_{n,n+k} = f_k n^{(3k-1)/2} n^n \left( 1 + O(n^{-1/2}) \right), \quad (6.8)$$

where

$$f_0 = \frac{\pi}{8}, \quad f_k = \frac{\sqrt{\pi} 3^k (k-1) d_k}{2^{(5k-1)/2} \Gamma(3k/2)}, \quad k \geq 1$$

and the $d_k$ satisfy the recurrence

$$d_1 = d_2 = \frac{5}{36}, \quad d_{k+1} = d_k + \sum_{h=1}^{k-1} \frac{d_h d_{k-h}}{(k+1) \binom{k}{h}}, \quad k \geq 1.$$

Later on, Bender, Canfield and McKay [2] translated recurrence (6.6) into differential equations and extended the estimates to the widest range of parameters. This result is reproved by Pittel and Wormald [56] in a different way, using the 2-core (Section 6.2) of a connected graph. They first estimate the number of connected graphs with minimum degree at least two (called 2-graphs in Section 6.2) with given number of vertices and edges, and then apply it to estimate the number of connected graphs.

A purely analytic approach to the problem was developed in [25]. The authors find an integral representation for the generating function of connected graphs

$$C(x,y) = \log \left( 1 + \sum_{n \geq 1} (1+y)^{(n)} \frac{x^n}{n!} \right),$$

which linearizes the quadratic exponent in $y$. They reprove the fact that

$$W_k(z) = A_k(T(z)) \left( \frac{1 - T(z)}{1 - T(z)} \right)^{3k},$$

where the $A_k$ are polynomials, related to the Airy Ai function. For fixed $k \geq 2$ they obtain complete asymptotic estimates for $C_{n,n+k}$. It is shown that

$$C_{n,n+k} = \frac{A_k(1) \sqrt{\pi}}{2^{(3k-1)/2} n^{(3k-1)/2}} \left( \frac{1}{\Gamma\left(\frac{3k}{2}\right)} + \frac{A_k(1)}{\Gamma\left(\frac{3k-1}{2}\right)} \sqrt{\frac{2}{n}} + O\left(\frac{1}{n}\right) \right),$$

where
Graph Enumeration

and lower order terms depend on the derivatives $A_k^{(j)}(1)$.

The values $A_k(1)$, which give the first order asymptotics, can be computed directly from the following identity

$$
\sum_{k=1}^{\infty} A_k(1)(-x)^k = \log \left( 1 + \sum_{k=1}^{\infty} c_k(-x)^k \right),
$$

where

$$
c_k = \frac{(6k)!}{(3k)!(2k)!^3}.2^k.
$$

The higher derivatives can be computed in terms of the Airy function.

6.4 Regular graphs

A graph is $d$-regular if all the vertices have degree $d$. Let $R_{n,d}$ be the number of (labelled) $d$-regular graphs with $n$ vertices. Read [63] gave an exact formula for $R_{n,d}$ using Pólya’s counting theory, but this formula is not suitable for asymptotic analysis. The key to the enumeration of regular graphs is the pairing model.

6.4.1 The pairing model

In the pairing model for $d$-regular graphs, there are $n$ labelled cells, each with $d$ labelled elements, corresponding to the half-edges at each vertex. A pairing of the $nd$ elements produces a $d$-regular graph by regarding the cells as vertices and the pairs as edges. Notice that loops and multiple edges may be created by choosing a pair from the same cell, or two different pairs joining elements from two distinct cells. The number of such pairings is

$$
\frac{(dn)!}{(dn/2)!2^{dn/2}}.
$$

One can estimate the probability that the resulting graph is simple, by considering the number $X_i$ of cycles of length $i$ created in the pairing. Using the method of moments, it can be shown that the random variables $X_1, \ldots, X_k$ are asymptotically (when $n$ goes to infinity) independent Poisson distributed with means $\lambda_i = (d−1)^i/(2i)$. Hence the probability of being simple, which is the probability of not having cycles of lengths one or two is asymptotically $e^{-\lambda_1}e^{-\lambda_2}$. This gives

$$
P(\text{simple}) \sim e^{(1−d^2)/4}.
$$

Since a simple graph is produced exactly $d^{n}$ times (permuting the $d$ elements at each cell gives the same graph), one obtains for fixed $d$

$$
R_{n,d} \sim \exp \left( \frac{1−d^2}{4} \right) \frac{(dn)!}{(dn/2)!2^{dn/2}d!^n}.
$$
Using Stirling’s estimate this can be rewritten as

$$R_{n,d} \sim \sqrt{2} e^{(1-d^2)/4} \left( \frac{d^d}{e^d (d!)^2} \right)^{n/2} n^{dn/2}. \quad (6.9)$$

The dominating term in the expression above is $n^{dn/2}$, corrected by an exponential term and a constant that depend on $d$. This estimate was obtained independently by several authors (see the historical discussion in [78]).

For $d$ growing with $n$, estimating the probability that the resulting graph is simple is considerably more difficult. McKay [49] introduced a technique called switchings and extended the range of $d$ to $d = o(n^{1/3})$.

The basic idea of a switching can be explained through the following example. Suppose a pairing has a pair $p_1 p_2$ produces a loop in the resulting multigraph. Choose another pair $p_3 p_4$ and replace the two pairs by $p_1 p_3$ and $p_2 p_4$. The pair giving rise to a loop disappears but a new loop can be created. Using double counting one can estimate the probability of these type of events, and then obtain an estimate of the ratio $|S_a|/|S_{a-1}|$, where $S_a$ is the number of pairings having exactly $a$ loops. By telescoping one can estimate $|S_0|$, the number of pairings without loops. More general switchings can involve more than two pairs [78].

The previous analysis usually extends to graphs with a given degree sequence. Let $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ with $\sum_{i=1}^{n} d_i = 2m$. Let $G_{n,d}$ be the number of graphs with degree sequence $d = (d_1, \ldots, d_n)$. One can use again the pairing model with $n$ cells containing $d_1, \ldots, d_n$ elements. The probability that the resulting graph is simple can be estimated (for $d_1$ bounded) as

$$P(\text{simple}) \sim e^{-\lambda/2 - \lambda^2/4},$$

where

$$\lambda = \frac{1}{m} \sum_{i=1}^{m} \binom{d_i}{2}.$$

When all the $d_i$ are equal to $d$, one recovers the previous expression for $d$-regular graphs. This can be extended in some cases when the $d_i$ grow with $n$.

The case of 2-regular graphs is particularly simple, since they consist of a collection of cycles of lengths at least three. If $R(x)$ is the EGF of 2-regular graphs, then

$$R_2(x) = \exp \left[ \frac{1}{2} \left( \log \frac{1}{1-x} - x - \frac{x^2}{2} \right) \right] = \frac{e^{-x/2-x^2/4}}{\sqrt{1-z}} \quad (6.10)$$

$$= \frac{x^3}{3!} + 3\frac{x^4}{4!} + 12\frac{x^5}{5!} + 70\frac{x^6}{6!} + \cdots$$
Singularity analysis leads immediately (see [26, Example IV.1]) to the estimate for the number of 2-regular graphs
\[ e^{-3/4} \frac{n!}{\sqrt{\pi n}} \]
which agrees with (6.9) for \( d = 2 \).

Let us mention that the pairing model is the key for analyzing random regular graphs. Many properties, like the existence of Hamilton cycles or the chromatic number, can be proved using this model [40, 78].

### 6.4.2 Differential equations

Let \( R_d(x) = \sum_n R_{n,d} \frac{x^n}{n!} \) be the generating function of labelled \( d \)-regular graphs. Gessel [29] proved that \( R_d(x) \) is D-finite for each \( d \geq 2 \), that is, satisfies a linear differential equation with polynomial coefficients.

**Theorem 100** For each \( d \geq 2 \), the generating function of \( d \)-regular graphs is D-finite.

For \( d = 2 \), the explicit expression 6.10 immediately gives
\[ 2(1-x)R'_2(x) - x^2 R_2(x) = 0. \]

For \( d = 3 \) it was first proved in [63]. In [77] one finds in addition differential equations for the generating functions of connected, 2-connected and 3-connected cubic graphs, and for \( d = 4 \) it was proved in [65]. In both cases the arguments were combinatorial, based on removing one edge from a \( d \)-regular graph and analyzing the possible resulting configurations.

Let \( R_3(x) = Q(x^2) \). Then
\[ 6x^2(2-2x-x^2) Q''(x) - (x^5 + 6x^4 + 6x^3 - 32x + 8) Q'(x) + \frac{x}{6} (2-2x-x^2)^2 Q(x) = 0. \]

The first terms are
\[ R_3(x) = \frac{x^4}{4!} + 7\frac{x^6}{6!} + 19355\frac{x^8}{8!} + 11180820\frac{x^{10}}{10!} + \cdots \]

A linear second-order differential equation with polynomials coefficients can be obtained also for 4-regular graphs but it is a bit long to write down. The initial terms are
\[ R_4(x) = \frac{x^5}{5!} + 15\frac{x^6}{6!} + 465\frac{x^7}{7!} + 19355\frac{x^8}{8!} + \cdots \]

The problem was revisited by Goulden, Jackson and Reilly in [34], in terms of symmetric functions. Consider labelled graphs on vertices \( \{1,2,\ldots\} \) and let \( t_i \) a
variable marking the degree of vertex $i \geq 1$. Then
\[ T = \prod_{1 \leq i < j} (1 + t_i t_j) \]
is the generating function of labelled graphs marking all possible degrees, because if \{i, j\} is an edge, there is a contribution of one to both the degree of $i$ and $j$. This is a symmetric function in infinitely many variables $t = (t_1, t_2, \ldots)$. The number of $d$-regular graphs on $n$ vertices is the coefficient $[t_1^d \cdots t_n^d] T(i)$. These coefficients (called regular in [34]) can be expressed in terms of the so-called $H$-series of $T$. The authors then showed that the $H$-series satisfies a system of partial differential equations. From here they proved D-finiteness of $R_d(x)$ for $d = 2, 3, 4$, but the general case remained open.

Gessel [29] proved D-finiteness in the general case and at the same time extended it to graphs with loops (a loop contributes 2 to the degree of a vertex) and multiple edges, by considering variations of the $T$ symmetric function. For instance
\[ \prod_{1 \leq i \leq j} (1 + t_i t_j) \]
corresponds to graphs where loops are allowed (by taking $i = j$), and
\[ \prod_{1 \leq i \leq j} \frac{1}{1 - t_i t_j} \]
to graphs where loops and multiple edges are allowed.

### 6.5 Monotone and hereditary classes

In this section we discuss classes of graphs closed under taking subgraphs and under taking induced subgraphs. This is a very active area of research with close connections to extremal graph theory and related areas. Most of the results are asymptotic and only in a few cases one has access to the counting generating functions.

#### 6.5.1 Monotone classes

A class of graphs $\mathcal{G}$ is monotone if it is closed under taking subgraphs. Every monotone class is defined by a collection $\mathcal{F}$, possibly infinite, of minimal forbidden subgraphs. These are the graphs $G$ not in $\mathcal{G}$ but such that every proper subgraph is in $\mathcal{G}$. We denote by $\text{Mon}(\mathcal{F})$ the class of graphs not containing any subgraph isomorphic to a graph in $\mathcal{F}$. Next we briefly describe the relation between monotone classes and extremal graph theory.

Denote by that ex$(n; F)$ the maximum number of edges in a graph with $n$ vertices not containing $F$ as a subgraph. More generally, ex$(n; \mathcal{F})$ is the maximum number
of edges in a graph with $n$ vertices belonging to $\text{Mon}(\mathcal{F})$, that is, containing no subgraph in $\mathcal{F}$. Let $G$ be a graph in $\text{Mon}(\mathcal{F})_n$ with maximal number of edges. Since every subgraph of $G$ is also in $\text{Mon}(\mathcal{F})$, we have

$$|\text{Mon}(\mathcal{F})_n| \geq 2^{\text{ex}(n;\mathcal{F})}.$$ 

It turns out that in many cases this lower bound is not too far from the right answer. This is a consequence of the fundamental Erdős-Stone-Simonovits Theorem in extremal graph theory. Let $\chi(G)$ be the chromatic number of a graph $G$, and let

$$r = \min\{\chi(F) : F \in \mathcal{F}\} - 1.$$ 

Then the extremal function $\text{ex}(n;\text{Mon}(\mathcal{F}))$ satisfies

$$\text{ex}(n;\text{Mon}(\mathcal{F})) = \left(1 - \frac{1}{r}\right) \binom{n}{2} + o(n^2), \quad \text{as } n \to \infty.$$ 

For $r = 1$ this result says that the number of edges is subquadratic, and for $r \geq 2$ it is quadratic and of order $(1 - 1/r)n^2/2$.

A fundamental result of Erdős, Frankl and Rödl [21] gives the rough order of magnitude for the number of graphs in a monotone class, when $r \geq 2$.

**Theorem 101** Let $\text{Mon}(\mathcal{F})$ be a monotone class with $r \geq 2$. Then

$$|\text{Mon}(\mathcal{F})_n| = 2^{\left(1 - \frac{1}{r} + o(1)\right)}.$$ 

Notice that the $o(1)$ term is in the exponent, hence this estimate is very far from being precise. When $r = 1$ (at least one bipartite graph is forbidden) the situation is more complicated. A case that has been much studied is $\text{Mon}(C_4)$. It is proved in [41] that the number $F_n$ of graphs not containing $C_4$ satisfies

$$2^{cn^{3/2}} \leq F_n \leq 2^{c'n^{3/2}}, \quad 0 < c < c',$$

an estimate that is still far from being precise.

In some cases though one has very precise estimates. We start discussing the class of triangle-free graphs. Clearly every bipartite-graph is triangle-free. It was proved by Erdős, Kleitman, and Rothschild [22] that, somehow surprisingly, almost all triangle-free graphs are bipartite. Let $B_n$ be the number of bipartite graphs.

**Theorem 102** Almost all triangle-free graphs are bipartite. More precisely,

$$|\text{Mon}(K_3)| = B_n \left(1 + O\left(\frac{1}{n}\right)\right).$$

Since the number of bipartite graphs is known (see Section 6.2), this gives a precise estimate for the number of triangle-free graphs. This fundamental result was extended by Kolaitis, Prömmel and Rothschild [42].
Theorem 103 Almost all $K_{t+1}$-free graphs are $t$-partite.

The Erdős-Klietman-Rothschild was generalized in a different direction, by showing that almost all graphs in $\text{Mon}(C_{2\ell+1})$ are bipartite [43]. The two previous results were widely generalized by Prömel and Steger [60]. An edge $e$ of a graph $H$ is colour-critical if $\chi(H-e) < \chi(H)$. Notice that both in complete graphs and odd cycles every edge is colour-critical.

Theorem 104 Let $H$ be a graph containing a colour-critical edge with $\chi(H) = r + 1 \geq 3$. Then almost all graphs not containing $H$ as a subgraph are $r$-partite.

In fact the authors show that the previous condition is also necessary: if almost every graph in $\text{Mon}(H)$ is $r$-partite then $H$ contains a colour-critical edge.

We now discuss the problem of counting $k$-partite graphs. A simpler problem is to count $k$-coloured graphs. A $k$-coloured graph is a graph together with a particular $k$-colouring using all $k$ colours. A recursion for the number $c_{n,k}$ of $k$-coloured graphs was obtained in [64]:

$$c_{n,k} = \sum_{j=0}^{n} \binom{n}{k} 2^{j(n-j)} c_{j,k-1},$$

with the initial conditions $c_{0,k} = 1$ and $c_{n,0} = 0$ for $n \geq 1$. Prömel and Steger [58] showed that almost all $k$-partite graphs are uniquely $k$-colourable, hence the number of $k$-partite graphs is asymptotically $k! c_{n,k}$. Estimates for $c_{n,k}$ are given in [80]. The dominant term is

$$2(1-\frac{1}{k})^2 k^n,$$

and the subexponential term depends on a subtle way on the value of $n$ modulo $k$.

6.5.2 Hereditary classes

Recall that a class of graphs is hereditary if it is closed under taking induced subgraphs. Notable examples include perfect graphs, and several subclasses such as chordal graphs, interval graphs or permutation graphs [33]. A graph $G$ is perfect if for every induced subgraph $H$, $\chi(H)$ equals the size of the largest complete subgraph in $H$. A graph is chordal if every cycle of length more than three contains a chord. The theory of perfect graphs is closely related to combinatorial optimization [33].

A hereditary class $\mathcal{G}$ is characterized by the list, not necessarily finite, of forbidden subgraphs, the minimal graphs not in $\mathcal{G}$ all whose proper induced subgraphs are in $\mathcal{G}$. We write $\text{Forb}(H)$ for the class of graphs not containing $H$ as an induced subgraph. The class $\text{Mon}(K_t)$ discussed precisely is hereditary, since a $K_t$ subgraph is always induced. However the class $\text{Mon}(C_4)$ is not hereditary, since a graph can contain copies of $C_4$, none of them induced.

A general asymptotic result is available for hereditary classes, similar to Theorem 101. It is based on a modified version of the extremal function $\text{ex}(n,H)$ that makes sense for induced subgraphs. The following concept was introduced in [62]. Given positive integers $s \leq r$, an $(r,s)$-colouring of a graph $G$ is a colouring $c: V(G) \to [r]$
such that vertices with colour $i$ induce a clique for $1 \leq i \leq s$, and an independent set for $s + 1 \leq i \leq r$. Notice that $(r, 0)$-colourable is the same as $r$-colourable in the classical sense. The colouring number of a hereditary class $\mathcal{G}$ is defined as

$$r(\mathcal{G}) = \max\{r : \text{there exists } s \leq r \text{ such every } (r, s)\text{-colourable graph is in } G\}.$$  

The following was proved in [62] for hereditary classes excluding a single graph, and in the general case in [11].

**Theorem 105** Let $\mathcal{G}$ be a non-trivial hereditary class of graphs with colouring number $r = r(\mathcal{G})$. Then

$$|\mathcal{G}_n| = 2^{(1-\frac{1}{r}+o(1))\binom{n}{2}}.$$  

Observe that again the $o(1)$ term in the exponent makes the estimate very far from being precise.

In a different direction, it is known that the rate of growth of the number of graphs in a hereditary class cannot be arbitrary. This was first established in [68], and later extended by several authors [12]. In the next statement $B(n)$ is the Bell number (the number of partitions of an $n$-set), that grows like $B(n) \sim ((1 + o(1))n/\log n)^n$.

**Theorem 106** Let $\mathcal{G}$ be a non-trivial hereditary class of graphs with colouring number $r = r(\mathcal{G})$. Then one of the following cases holds for sufficiently large $n$.

1. $|\mathcal{G}_n|$ is identically zero, one or two.
2. $|\mathcal{G}_n|$ is a polynomial in $n$.
3. $|\mathcal{G}_n|$ has exponential order of the form $\sum_{i=1}^{k} p_i(n)\alpha^i$ for some $k > 0$ and polynomials $p_i$.
4. $|\mathcal{G}_n| = n^{1-1/r+o(1)}$ for $r > 1$.
5. $B(n) \leq |\mathcal{G}_n| \leq 2^{o(n^2)}$.
6. $|\mathcal{G}_n| = 2^{(1-1/r+o(1))n^2/2}$ for $r > 1$.

These results extend to other combinatorial structures such as oriented graphs, hypergraphs, posets, and others.

Next we turn to particular hereditary classes. A fundamental difference between $\text{Mon}(C_4)$ and $\text{Forb}(C_4)$ can be observed by looking at the extremal functions. It is well-known that $\text{ex}(n; C_4) = \Theta(n^{3/2})$, but a graph in $\text{Forb}(C_4)$ can have $\Theta(n^2)$ edges: consider the disjoint union of $K_n/2$ and the complement of $K_n/2$. The former is an example of a split graph. A graph is split if its vertex set can be partitioned into a clique and an independent set. Clearly a split graph is in $\text{Forb}(C_4)$. The following was proved by Prömel and Steger [59].

**Theorem 107** Almost all graphs in $\text{Forb}(C_4)$ are split.
It follows that the number of graphs in Forb($C_4$) is of the same order as the number of bipartite graphs.

A graph $G$ is generalized split if either $G$ or its complement satisfy the following property: the vertex set can be partitioned into disjoint cliques $H_0, H_1, \ldots, H_k$ such there is no edge between $H_i$ and $H_j$ for $i > j \geq 1$. It is easy to check that generalized split graphs contain no induced $C_5$. The following was proved in [61].

**Theorem 108** Almost all graphs in Forb($C_5$) are generalized split.

The former result has a striking consequence. A graph $G$ is perfect if $\omega(H) = \chi(H)$ for every induced subgraph $H$ of $G$, where $\omega(G)$ is the size of the largest completed subgraph in $G$. A perfect graph does not contain an induced $C_5$ and it can be checked that generalized split graphs are perfect. Hence we have

Generalized split $\subset$ Perfect $\subset$ Forb($C_5$).

It follows that

**Theorem 109** Almost all perfect graphs are generalized split.

This is an unexpected result. Perfect graphs are fundamental in graph theory and some difficult conjectures have been proved only recently. Let us mention the perfect graph conjecture, which says that perfect graphs are precisely those excluding odd cycles of length at least five and their complements as induced subgraphs [17]. However, most perfect graphs have a very simple structure.

**Cographs.** In the last part of this section we analyze hereditary classes that can be enumerated exactly using generating functions. A cograph is a graph not containing the path $P_4$ as an induced subgraph. They can be characterized alternatively as follows:

- The empty graph is a cograph and a single vertex is a cograph.
- If $G$ and $H$, are cographs so are the disjoint union $G \cup H$ and the sum $G + H$ (the sum is the disjoint union plus all edges between $G$ and $H$).

Cographs are relevant in computer science, since several hard computational problems can be solved in polynomial time when the input is a cograph [33].

Let $G(x)$ and $C(x)$ be, respectively, the generating functions of cographs and connected cographs. Since a cographs is connected if and only if its complement is disconnected, we have $G(x) = 1 + x = 2C(x)$. Together with $G(x) = e^{C(x)}$, this gives

$$e^{C(x)} - 2C(x) + x - 1 = 0.$$ 

It follows that the dominant singularity of $C(x)$ is at $\rho = 2\log(2) - 1$. From here it is easy to deduce the estimates

$$C_n \sim cn^{-3/2} \rho^{-n!}, \quad G_n \sim 2C_n.$$
This derivation can be found in several places, for instance [35].

Two subclasses of cographs that have been considered in the literature are the following.

1. A graph is trivially perfect if it does not contain $P_4$ or $C_4$ as induced subgraphs. A similar argument as before gives the equation

$$\left(1 - e^{-x}\right)e^{C(x)} - C(x) = 0.$$  

The dominant singularity is $\rho = 1 - \log(e - 1)$ and

$$C_n \sim cn^{-3/2} \rho^{-n!}, \quad G_n \sim eC_n.$$

2. A graph is a threshold graph if it does not contain $P_4$, $C_4$ or $K_2 \cup K_2$ (two disjoint edges) as induced subgraphs. As for cographs we have $G_n = 2C_n$. In this case we have an explicit expression for $C(x)$:

$$C(x) = \frac{1 - x}{2 - e^x}.$$  

The function $C(x)$ has a simple pole at $\log(2)$ and

$$C_n \sim c \log(2)^{-n!}, \quad G_n \sim 2C_n.$$

The first values for the number of cographs and their subclasses is given in the next table.

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### 6.6 Planar graphs

The enumeration of planar graphs started with Tutte in the 1960s, with his fundamental papers on map enumeration, where a map is a graph with a fixed embedding in the plane. Map enumeration has grown into an important area of research. In this section we are interested in enumerating planar graphs as combinatorial structures, without considering a particular embedding. As we are going to see, the enumeration of planar graphs is based on the the enumeration of planar maps and the graph decompositions described in Section 6.2. Throughout this section, $T(x, y)$, $D(x, y)$, $B(x, y)$, $C(x, y)$ and $G(x, y)$ have the same meaning as in Section 6.2 restricted to planar graphs.
The starting point of the analysis is Whitney’s theorem, claiming that a 3-connected planar graph has a unique embedding in the sphere up to homeomorphism. Let $M_{n,k}$ be the number of rooted 3-connected planar maps with $n$ vertices and $k$ edges, and let $T_{n,k}$ be the number of 3-connected planar graphs with $n$ vertices and $k$ edges. Then we have the relation

$$M_{n,k} n! = 4kT_{n,k}.$$ 

This is because there are $n!$ ways to label a rooted map with $n$ vertices (since vertices in a rooted map are all distinguishable), and there are $4k$ ways of rooting a planar graph with $k$ edges to obtain a rooted map ($2k$ choices for a directed edge and two choices for the root face). Let the generating functions associated to 3-connected maps and graphs, respectively, be

$$M(x,y) = \sum M_{n,k} x^n y^k, \quad T(x,y) = \sum T_{n,k} x^n y^k.$$ 

The former relation implies

$$M(x,y) = 4yT_{y}(x,y).$$

This was first made explicit in [4].

The function $M(x,y)$ is algebraic of degree four and is given by

$$M(x,y) = x^2 y^2 \left( \frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3} \right),$$

(6.11)

where

$$U = xy(1+V)^2, \quad V = y(1+U)^2.$$ (6.12)

From the previous equations one can estimate easily the number $T_n$ of 3-connected planar graphs as

$$T_n \sim c_3 n^{-7/2} \gamma_3 n!,$$

where $\gamma_3 = (272 + 112\sqrt{7})/27 \approx 21.05$.

The series $T(x,y)$ determines the series $B(x,y)$ of 2-connected planar graphs via Theorem 99. From here, Bender, Gao and Wormald [4] obtained the asymptotic number $B_n$ of 2-connected planar graphs as

$$B_n \sim c_2 n^{-7/2} \gamma_2 n!,$$

where $\gamma_2 \approx 26.18$ This was an important step, since it opened the way to the full enumeration of planar graphs.

In order to proceed further one needs an expression for $B(x,y)$ in terms of $D(x,y)$. This was achieved by Giménez and Noy [30].

Lemma 5 Let $D = D(x,y)$ be the generating function of planar networks and let $B(x,y)$ that of 2-connected planar graphs. Let $W = z(1+U)$, where $U$ is defined in (6.12). Then
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\[ B(x,y) = \frac{x^2}{2} B_1(x,y) - \frac{x}{4} B_2(x,y), \] (6.13)

where

\[
\begin{align*}
B_1 &= \frac{D(6x - 2 + xD)}{4x} + (1 + D) \log \left( \frac{1 + y}{1 + D} \right) - \frac{\log(1 + D)}{2} + \frac{\log(1 + xD)}{2x^2}; \\
B_2 &= \frac{2(1 + x)(1 + W)(D + W^2) + 3(W - D)}{2(1 + W)^2} - \frac{1}{2x} \log(1 + xD + xW + xW^2) \\
&\quad + \frac{1 - 4x}{2x} \log(1 + W) + \frac{1 - 4x + 2x^2}{4x} \log \left( \frac{1 - x + xD - xW + xW^2}{(1 - x)(D + W^2 + 1 + W)} \right).
\end{align*}
\]

The former expression is obtained by integrating \( T_y(x,y) \) with respect to \( y \), and using Equations (6.3–6.5). It is remarkable that the same result can be recovered in a purely combinatorial way using grammars [16].

With this explicit expression one can check that the equation

\[ C'(x) = e^{B'_R(xC'(x))} \]

has no branch point. This is equivalent to the fact that \( B''_R(R) < 1/R \). It follows that the singularity \( \rho \) of \( C(x) \) is given by \( \rho = Re^{-B'_R(R)} \), where \( R = 1/\gamma_2 \), and is of the same kind as that of \( B(x) \). Finally, the singularity of \( G(x) = e^{C'(x)} \) is the same as that of \( C(x) \). Applying singularity analysis one obtains estimates for the numbers \( C_n \) and \( G_n \) of the same kind as for \( T_n \) and \( B_n \). The results are summarized as follows.

**Theorem 110** For \( \ell \in \{0, 1, 2, 3\} \), the numbers of \( \ell \)-connected planar graphs are asymptotically of the form

\[ c_\ell n^{-7/2} \gamma_\ell n!, \]

where

\[ \gamma_3 \approx 21.05, \quad \gamma_2 \approx 26.18, \quad \gamma_1 = \gamma_0 \approx 27.23. \]

Moreover, the asymptotic probability that a random planar graph is connected is \( c_1/c_0 \approx 0.95 \).

It is also possible to estimate the number of planar graphs with given edge density.

**Theorem 111** For each \( \alpha \in (1, 3) \), the number of planar graphs with \( n \) vertices and \( \lfloor \alpha n \rfloor \) edges is asymptotically

\[ c(\alpha) n^{-\alpha} \gamma(\alpha)^n n!, \] (6.14)

where \( \gamma(\alpha) \) is an analytic function that achieves its maximum at \( \alpha = \kappa \approx 2.21 \). The expected number of edges in a random planar graph is precisely \( \kappa n \).
The proof is based on a local limit theorem. For each \( \alpha \in (1, 3) \) there is a \( y = y(\alpha) \) such giving a graph with \( k \) edges the weight \( y^k \), the weight is concentrated on graphs with \( \alpha n + O(\sqrt{n}) \) edges. For this value of \( y \), the generating function \( G(x, y) \) has radius of convergence \( \rho(\alpha) \) and we have

\[
[x^n] G(x, y) \sim c(y) n^{-7/2} \gamma(\alpha)^n n!, \quad \gamma(\alpha) = \rho(\alpha)^{-1}.
\]

The extra \( n^{-1/2} \) factor in (6.14) comes from the local limit theorem (for details, see [30]).

The previous enumerative results opened the way to the fine analysis in planar graphs. Basic parameters such as the number of edges and the number of connected components where already analyzed in [30]. Extremal parameters, such as the size of the largest component or the size of the largest block, the maximum degree and the diameter can be analyzed too [53].

Some special classes of planar graphs have been enumerated too, in particular cubic planar graphs [9]. The decomposition into 3-connected components can be adapted for cubic graphs. If \( G \) is a 3-connected planar graph, its dual (which is unique by Whitney’s theorem) is a 3-connected triangulation. Using Tutte’s enumeration of triangulations [72] one can the generating function of 3-connected cubic planar graphs in terms of the (ordinary) generating function \( T(z) \) of triangulations, given by

\[
T(z) = u(z)(1 - 2u(z)), \quad u(z)(1 - u(z))^3 = z.
\]

Theorem 99 can be adapted to this situation and one can show [9] that the number of cubic planar graphs is asymptotically, for even \( n \to \infty \),

\[
cn^{-7/2} \delta^n n!, \quad \delta \approx 3.13, \quad c > 0.
\]

On the other hand, counting 4-regular planar graphs is yet an open problem. No decomposition into 3-connected components is know for 4-regular graphs, and the enumeration of general 4-regular graphs [65] does not seem apply when restricted to planar graphs. Another interesting open problem is to enumerate bipartite planar graphs.

### 6.7 Graphs on surfaces and graph minors

Planar graphs are graphs embeddable on the sphere, hence it is only natural to consider graphs in other surfaces. Graphs on surfaces are a special case of classes of graphs defined in terms of excluded minors.

#### 6.7.1 Graphs on surfaces

The analysis from the previous section can be extended to graphs embeddable in a fixed surface. The starting point is the enumeration of maps on surfaces. After the
seminal work of Tutte, the theory of map enumeration was extended to arbitrary closed surfaces. Using Tutte’s technique of removing the root edge and using induction on the genus, Bender and Canfield [1] proved the following fundamental result.

**Theorem 112** Let $M_{g,n}$ be the number of rooted maps of orientable genus $g \geq 0$ with $n$ edges, and let $N_{h,n}$ be the analogous quantity for non-orientable genus $h \geq 1$. Then

$$M_{g,n} \sim c_g n^{5(g-1)/2} 12^n,$$

and

$$N_{h,n} \sim \tilde{c}_h n^{5(h-2)/4} 12^n,$$

for some positive constants $c_g, \tilde{c}_h$.

As for planar graphs, it took some time to go from enumeration of maps to graphs in a surface. The corresponding result for graphs was obtained independently in [3, 15].

**Theorem 113** Let $G_{g,n}$ be the number of rooted maps of orientable genus $g \geq 0$ with $n$ edges, and let $H_{h,n}$ be the analogous quantity for non-orientable genus $h \geq 1$.

$$G_{g,n} \sim d_g n^{5(g-1)/2-1} \gamma n!$$

and

$$H_{h,n} \sim \tilde{d}_h n^{5(h-2)/4-1} \gamma n!$$

for some positive constants $c_g, c'_h$, and where $\gamma \approx 27.23$ is as in Theorem 110.

The number of graphs that can be embedded in the orientable surface of genus $g$ has the same estimate, since graph of genus less than $g$ are of a smaller order of magnitude than graphs of genus $g$. The same applies to the non-orientable case.

The proof of Theorem 113 is rather technical, here we limit ourselves to give the main ideas. Suppose one wishes, as for planar graphs, to use the enumeration of maps on the surface $S$ for counting graphs (without an embedding) on $S$. There are two main obstacles for this program:

1. No degree of connectivity guarantees a unique embedding.
2. The class of graphs embeddable in $S$ is not closed under taking connected components and blocks (genus is additive on components and blocks).

Hence the basic equations among generating functions from Theorem 99 no longer hold. The road to the solution is to consider a parameter called face-width. The face-width $fw(M)$ of a graph $G$ embedded in $S$ is the minimum number of intersections of $G$ with a simple non-contractible curve $C$ on $S$. It is easy to see that this minimum is achieved when $C$ meets $G$ only at vertices. Face-width is in some sense a measure of local planarity, if the face-width is large then the embedding is planar.
in balls of large radius. The face-width of a graph $G$ is the maximum face-width among all the embeddings of $G$. The key result is that a 3-connected graph with large enough face-width has a unique embedding. It turns out that the generating series of 3-connected graphs of any fixed face-width has a negligible contribution in the asymptotic analysis. Therefore, the enumeration of 3-connected graphs in $S$ can be reduced, up to negligible terms, to the enumeration of 3-connected maps in $S$. It is important to remark that, since maps with small face-width are discarded, one does not work with exact counting series. Instead, if $f(x)$ is the series of interest, one finds computable series $f_1(x)$ and $f_2(x)$ such that $f_1(x) \preceq f(x) \preceq f_2(x)$ (where $\preceq$ means coefficient-wise inequality) and $f_1(x)$ and $f_2(x)$ have the same leading asymptotic estimates.

For the second obstacle one can use a result from [66]: if a connected graph $G$ of genus $g$ has face-width at least two, then $G$ has a unique block of genus $g$ and the remaining blocks are planar. A similar result holds for 2-connected graphs and 3-connected components. Since for planar graphs we have exact expressions for all the generating functions involved, starting from the (asymptotic) enumeration of 3-connected graphs of genus $g$ we can achieve the enumeration of all graphs of genus $g$. To be more precise, let $G^g(x)$ and $C^g(x)$ be the generating functions of graphs and connected graphs of genus $g$, respectively. The usual relation $G^g(x) = \exp C^g(x)$ does no hold, since the union of graphs of genus $g$ can have larger genus. Instead, we have

$$G^g(x) \sim C^g(x)e^{C^g(x)}$$

where the symbol must be understood as that the two functions have the same dominant asymptotic terms. Similarly, the relation between $C^g(x)$ and the generating function $B^g(x)$ of 2-connected graphs of genus $g$ is not an exact equation as in the planar case, since genus is also additive in blocks, but rather an approximate version. The technical details are quite involved but the essence is to discard maps and graphs with small face-width.

### 6.7.2 Graph minors

Graphs on surfaces are one of the main examples of the theory of graph minors. A graph $H$ is a minor of $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. Equivalently, if $H$ can be obtained from $G$ by edge contractions and deletions, and deletion of isolated vertices. A class $\mathcal{G}$ of graphs is minor-closed if whenever $G$ is in $\mathcal{G}$ and $H$ is a minor of $G$, $H$ is also in $\mathcal{G}$. The class is proper if it does not contain all graphs. The class $\mathcal{G}_S$ of graphs that can be embedded in a fixed surface $S$ is minor-closed. Other examples are graphs with bounded tree-width and $\Delta Y$-reducible graphs.

Given a minor-closed class $\mathcal{G}$, an excluded minor for $\mathcal{G}$ is a graph $H$ such that $H$ is not in $\mathcal{G}$ but every proper minor of $H$ is in $\mathcal{G}$. For instance, Kuratowski’s theorem says that $K_5$ and $K_{3,3}$ are the excluded minors for planar graphs. The theory of graph minors is one of main achievements in modern combinatorics, culminating with the great theorem of Robertson and Seymour: for each minor-closed class the number of excluded minors is finite. We write $\mathcal{G} = \text{ex}(H_1, \ldots, H_k)$ if the $H_i$ are the excluded
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minors of \( \mathcal{G} \). A class \( \mathcal{G} \) is said to be decomposable if it satisfies condition (C1) in Section 6.2. It is addable if it is decomposable and in addition satisfies condition (C2). It is easy to see that a class is decomposable if and only if its excluded minors are connected, and is addable if and only if its excluded minors are 2-connected.

Let \( \mathcal{G} \) be a proper minor-closed class of (labelled) graphs. We say that \( \mathcal{G} \) is small if there exists a constant \( c > 0 \) such that

\[ |\mathcal{G}_n| \leq c^n n!. \]

Observe that this is equivalent to the fact that the exponential generating function \( \sum |\mathcal{G}_n| x^n / n! \) has positive radius of convergence. A general enumerative result says that a minor-closed class is small. It was first proved in [52] and later in a more general context in [20].

**Theorem 114** If \( \mathcal{G} \) is a proper minor-closed class of graphs is small.

Hence the possible growth rates are bounded by \( c^n n! \). A classification of the possible growth rates of minor-closed classes of graphs can be found in [6], similar to that for hereditary classes in Section 6.5, although growth rates above \( c^n n! \) are excluded.

To measure the size of a class we consider \( \gamma = \limsup_{n \to \infty} (G_n / n!)^{1/n} \). Because of the previous theorem, if \( \mathcal{G} \) is proper minor-closed, then \( \gamma < +\infty \). We say that \( \mathcal{G} \) has a growth constant if the lim sup is actually a limit, that is,

\[ \gamma = \lim_{n \to \infty} \left( \frac{G_n}{n!} \right)^{1/n}. \]

The class \( \mathcal{G} \) is smooth if the limit \( \lim_{n \to \infty} G_n / (nG_{n-1}) \) exists. This is a stronger condition than having a growth constant and if this is the case then

\[ \lim_{n \to \infty} \frac{G_n}{nG_{n-1}} = \gamma. \]

The following very general theorem was proved by McDiarmid [46].

**Theorem 115** A \( \mathcal{G} \) proper addable minor-closed class of graphs is smooth.

It is an open problem whether every minor-closed class has a growth constant.

Let \( \Gamma \) be the set of real numbers that are growth constants of minor-closed classes of graphs. It is known [7] that 0, 1, 2 belong to \( \Gamma \), and the only other number in \( \Gamma \cap [0, 2] \) is \( \xi \approx 1.76 \), which is the growth constant of forests of caterpillars. Given a minor-closed class \( \mathcal{G} \), one shows that either \( \mathcal{G} \) contains all paths and its growth constant is at least 1, or else it excludes some path and the growth constant is 0. The proof that \( \xi \) is the only growth constant in \( (1, 2) \) uses a similar idea using caterpillars instead of paths. Another result is that if \( \gamma \in \Gamma \), then \( 2\gamma \in \Gamma \). All possible values in \( \Gamma \cap [2, 2.25] \) are also determined [7], where it is shown that there are infinitely many gaps.

\[ \text{Graph Enumeration} \]
6.7.3 Particular classes

There are no general results on the enumeration of minor-closed classes besides those discussed above. But there are precise results for a number of relevant classes. All the following examples satisfy conditions (C1) and (C2) from Section 6.2 (they are addable) and the equations in Theorem 99 apply. The series $G, C, B, D, T$ have the same meaning as in Section 6.2. We use throughout the equations relating these generating functions stated in Theorem 99.

Forests. A forest is an acyclic graph, that is, all its components are trees. Alternatively, the class of forest is $\text{ex}(K_3)$. The generating function of trees is $T(x) = xe^T(x)$ (see Section 6.3). The generating function of forests is

$$G(x) = e^C(x).$$

From here it follows easily that the number of forests is asymptotically (a result first proved by Takács [70])

$$e^{1/2}n^{-2} \sim \left( \frac{e}{2\pi} \right)^{1/2} n^{5/2}e^n n!.$$

Outerplanar graphs. A graph is outerplanar if it can be embedded in the plane so that all the vertices are in the outer face. They are also the same as the class $\text{ex}(K_4, K_{3,2})$. A 2-connected outerplanar graph has a unique Hamilton cycle and it can be drawn in the plane as a polygon plus some non-intersecting diagonals, that is, what in classical enumeration is known as a polygon dissection [24]. From here it follows easily [8] that the generating function $B(x)$ of 2-connected outerplanar graphs satisfies

$$B'(x) = \frac{1 + 5x - \sqrt{1 - 6x + x^2}}{8}.$$

The singularity of $B(x)$ is at $R = 3 - 2\sqrt{2}$. One checks that $B''(R) > 1/R$, which implies that in this case Equation (6.2) has a branch point, as opposed to what happened for planar graphs. In other words, the singularity of $C(X)$ from a branch point and not from the singularity of $C(x)$. From here one gets the estimate for the number of outerplanar graphs with $n$ vertices

$$cn^{-5/2}\gamma^n n!, \quad \gamma \approx 7.32.$$

Here and in the sequel $c$ is a computable positive constant, that depends only on the class under consideration.

If we compare with the numeration of forests, we see the same behaviour. For forests $B(x) = x^2/2$, while for outerplanar graphs it is an algebraic function. However in both cases the singularity comes from the existence of a branch point and the behaviour of $C'(x)$ near it singularity is in both cases of square-root type. This remark applies to the next example too.
Series-parallel graphs. A graph is series-parallel if it does not contain $K_4$ as a minor, that is, series-parallel graphs constitute the class $\text{ex}(K_4)$. Equivalently, series-parallel graphs are those that can be obtained from a forest by repeatedly applying series and parallel operations. Clearly, outerplanar graphs are series-parallel. The starting point for the enumeration is the observation that a series-parallel graph always has a vertex of degree at most two, hence it cannot be 3-connected. Hence the generating function of 3-connected graphs is identically zero. Equation (6.4) reduces to

$$1 + D(x,y) = (1 + y) \exp \left( \frac{xD(x,y)^2}{1 + xD(x,y)} \right).$$

Applying (6.5), this gives a direct expression for $B(x,y)$ in terms of $D(x,y)$. It is not as explicit as for outerplanar graphs, but the situation is very similar. Equation (6.2) has again a branch point and one gets the estimate for the number of series parallel graphs

$$cn^{-5/2} \gamma^n n!, \quad \gamma \approx 9.07.$$

Classes defined by 3-connected components. One can define a class of graphs $\mathcal{G}$ by fixing the class $\mathcal{T}$ of 3-connected members in the class, and letting $\mathcal{G}$ be the graphs whose 3-connected components are in $\mathcal{T}$. For instance, if $\mathcal{T} = \emptyset$, then $\mathcal{G}$ is the class of series-parallel graphs. A detailed analysis of classes defined in terms of 3-connected components is done in [32]. It is shown that the asymptotic enumeration of $\mathcal{G}$ depends essentially on the analytic properties of the generating function $T(x,y)$ of 3-connected graphs at its singularity (for each fixed value of $y$). In a number of cases one shows that the behaviour is similar to the class of series-parallel graphs, with a subexponential term $n^{-5/2}$. In other cases, $\mathcal{G}$ is close to the class of planar graphs and the subexponential term is $n^{-7/2}$. This difference in the asymptotic estimates implies a profound difference in the properties of random graphs from $\mathcal{G}$ with respect to the size of the largest block [32].

We reproduce part of a table from [32] for several minor-closed classes.

<table>
<thead>
<tr>
<th>Class</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Series-parallel</td>
<td>$-5/2$</td>
<td>9.07</td>
</tr>
<tr>
<td>$\text{ex}(W_4)$</td>
<td>$-5/2$</td>
<td>11.54</td>
</tr>
<tr>
<td>$\text{ex}(W_5)$</td>
<td>$-5/2$</td>
<td>14.67</td>
</tr>
<tr>
<td>$\text{ex}(K^-_5)$</td>
<td>$-5/2$</td>
<td>15.65</td>
</tr>
<tr>
<td>$\text{ex}(K_3 \times K_2)$</td>
<td>$-5/2$</td>
<td>14.67</td>
</tr>
<tr>
<td>Planar</td>
<td>$-7/2$</td>
<td>27.2269</td>
</tr>
<tr>
<td>$\text{ex}(K_{3,3})$</td>
<td>$-7/2$</td>
<td>27.2293</td>
</tr>
<tr>
<td>$\text{ex}(K_{3,3}^+)$</td>
<td>$-7/2$</td>
<td>27.2295</td>
</tr>
</tbody>
</table>

The constant $\gamma$ and $\alpha$ are such that the asymptotic estimate is of the form

$$cn^\alpha \gamma^n n!.$$
The classes are defined in terms of excluded minors. $W_n$ is the wheel with $n$ vertices, $K_5^-$ is the graph obtained from $K_5$ by removing one edge, and $K_3^+ 3$ is obtained from $K_3 3$ by adding one edge.

The first five classes in the table are subcritical classes. A class is subcritical if Equation (6.2) relating the generating functions $B(x)$ and $C(x)$ has a branch point. Hence the singularity of $C(x)$ comes from this branch point and not from the singularity of $C(x)$. As shown in [19], for subcritical classes the exponent is always $-5/2$. One may notice that in these cases the excluded minors are planar, as opposed to the cases where the exponent is $-7/2$. It is conjectured in [53] that the class $\text{ex}(H)$ is subcritical precisely when $H$ is planar.

All the previous examples are from addable classes. A first investigation of non-addable classes is carried out in [14]. Here we list some examples.

• The class of graphs that contain at most one cycle. These are trees and unicyclic graphs. The estimate for the number of graphs is

$$c \cdot n^{-3/4} e^n n!, \quad c = \frac{1}{(2e)^{1/4}\Gamma(1/4)}.$$  

The unusual exponent comes from a singularity of type fourth-root.

• The class of forests of caterpillars, that is, all the components are caterpillars. The estimate in this case is

$$cn^{-3/4} e^{2\sqrt{\alpha n}} n!,$$

where $\gamma$ is the inverse of the only positive root $\rho$ of $xe^x = 1$, $\alpha = \frac{1-\rho^2}{(1+\rho)^2}$, and $c$ is expressed in terms of $\rho$ and $\gamma$. The unusual asymptotics are due to the fact that $C(x)$ has a simple pole, so that $G(x)$ has an essential singularity.

• The class of graphs all whose connected components have size at most $k$. The generating function for connected graphs is $C(x) = \sum_{i=1}^{\infty} c_i x^i/i!$, where $c_i$ is the number of connected graphs with $i$ vertices. Hence $G(x) = \exp C(x)$ is the exponential of a polynomial. Applying Haymann admissibility, one obtains

$$\frac{1}{\sqrt{2\pi kn}} \frac{A(\zeta)}{\zeta^a} n!,$$

where $\zeta = \zeta_n$ is the solution to the saddle-point equation $\zeta C'(\zeta) = n$, and satisfies $\zeta = a n^{1/k} + \beta + O(n^{-1/k})$ for suitable constants $a$ and $\beta$.

6.8 Digraphs

A digraph is a labelled directed graph without loops or multiple arcs, but possibly with both arcs $uv$ and $vu$ for a given pair of vertices. The number of digraphs on $n$
vertices is of course \(2^{n(n-1)}\), since there are \(n(n-1)\) possible arcs joining distinct vertices. A source in a digraph is a vertex of indegree zero. A digraph is acyclic if it has no directed cycle, and it is strongly connected if any two vertices are joined by a directed path.

### 6.8.1 Acyclic digraphs

A very classical problem is to enumerate acyclic digraphs; they are clearly loopless and given two distinct vertices \(u\) and \(v\), only one of the two arcs \(uv\) and \(vu\) can be present. Let \(a_n\) be the number of acyclic digraphs on \(n\) labelled vertices, and let \(a_{n,m}\) the number of those having \(m\) arcs. There is a simple recurrence for computing these numbers, found independently in [67, 69] and rediscovered several times. The number of acyclic digraphs such that \(k\) given vertices are sources is equal to \(2^{k(n-k)}a_{n-k}\), since the remaining \(n-k\) vertices induce an acyclic digraph and every arc from the sources to the complement is allowed. Given that an acyclic digraph always has at least one source, inclusion-exclusion gives

\[
a_n = \sum_{t=0}^{n-1} \binom{n}{t} (-1)^{n-t-1} 2^{(n-t)}a_t, \quad a_0 = 1.
\]

This gives the sequence

\[
1, 1, 3, 19, 219, 4231, 130023, \ldots
\]

More generally, if

\[
A_n(y) = \sum_{m=0}^\infty a_{n,m}y^m
\]

is the edge-enumerator of acyclic digraphs with \(n\) vertices, then

\[
A_n(y) = \sum_{t=0}^{n-1} \binom{n}{t} (-1)^{n-t-1}(1+y)^{t(n-t)}A_t(y).
\]

In order to determine the asymptotic behaviour of the \(a_n\), we introduce the generating functions

\[
A(x) = \sum_{n=0}^\infty \frac{a_n x^n}{2^{(n/2)n!}}, \quad A(x,y) = \sum_{n=0}^\infty \frac{A_n(y)x^n}{(1+y)^{(n/2)n!}}.
\]

Define also

\[
\Psi(x) = \sum_{n=0}^\infty \frac{(-1)^{n/2}x^n}{2^{(n/2)n!}}, \quad A(x,y) = \sum_{n=0}^\infty \frac{(-1)^{n/2}x^n}{(1+y)^{(n/2)n!}}.
\]

The previous recurrence relations imply the equations

\[
A(x) = \frac{1}{\Psi(x)}, \quad A(x,y) = \frac{1}{\Psi(x,y)}.
\]
By locating the smallest zero of $\Psi(x)$ one obtains the estimate
\[ a_n \sim \lambda \gamma^n n! 2^{\binom{n}{2}}, \]
where $\gamma \approx 0.67201$ and $\lambda \approx 1.74106$.

The asymptotic enumeration of dense acyclic digraphs (with a quadratic number of arcs) is analyzed in [5].

### 6.8.2 Strongly connected digraphs

It is well-known that almost all digraphs are strongly connected, since with high probability there is a path of length two between any two vertices. Liskovec [44] found a recurrence for the number $s_n$ of strongly connected digraphs on $n$ vertices, later simplified by Wright [81]. Let $b_n$ be defined recursively by
\[
b_n = 2^{n(n-1)} - \sum_{t=1}^{n-1} \binom{n}{t} 2^{(n-1)(n-t)} b_t, \quad b_1 = 1.
\]
Then
\[
s_n = b_n + \sum_{t=1}^{n-1} \binom{n-1}{t-1} s_t b_{n-t} \quad s_1 = 1.
\]
This gives the sequence
\[ 1, 1, 18, 1606, 565080, \ldots \]

Wright [82] also studied the number $s_{n,m}$ of strongly connected digraphs with $n$ vertices and $m$ edges. He showed that
\[ s_{n,n+k} = P_k(n)n!, \]
where $P_k$ is polynomial in $n$ of degree $3k - 1$. Notice the similarity of the previous expression with Equation (6.7) for connected graphs with excess $k$. The first values are
\[
P_1(n) = \frac{1}{4} (n-2)(n+3),
\]
\[
P_2(n) = \frac{1}{2880} (n-2)(51n^4 + 297n^3 - 271n^2 - 1937n - 1020).
\]
This was extended more recently in [55] to cover the range $k = O(n \log n)$. Beyond this range a digraph is almost surely strongly connected.

### 6.9 Unlabelled graphs

An unlabelled graph is an isomorphism class of labelled graphs. The number of ways of labelling an unlabelled graph $G$ is equal to $n!/\text{aut}(G)$, where $\text{aut}(G)$ is the number
of automorphisms of $G$. Clearly counting unlabelled graphs is more demanding than counting their labelled counterparts, since symmetries must be taken into account. In this section we review briefly the theory of counting under symmetries, and also discuss the asymptotic enumeration of unlabelled graphs.

### 6.9.1 Counting graphs under symmetries

The starting point of the theory is the orbit-counting lemma, also known as Burnside-Frobenius lemma. Let $\Gamma$ be a permutation group acting on a finite set $X = \{1, \ldots, n\}$. Then the number of orbits of the action of $\Gamma$ on $X$ is equal to

$$\frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \text{fix}(\sigma),$$

where $\text{fix}(\sigma)$ is the number of points fixed by $\sigma$.

The type of a permutation $\sigma$ is $1^{s_1}2^{s_2}\cdots n^{s_n}$, where $s_i$ is the number of $i$-cycles in the cycle decomposition of $\sigma$. Its cycle index is a monomial in $n$ variables defined as

$$z(\sigma; x_1,\ldots,x_n) = x_1^{s_1} \cdots x_n^{s_n}.$$ 

The cycle index polynomial of $\Gamma$ is

$$Z(\Gamma; x_1,\ldots,x_n) = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} z(\sigma; x_1,\ldots,x_n) = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \prod_{i=1}^n x_i^{s_i(\sigma)}.$$ 

The cycle index polynomial can be used to count inequivalent colourings (in a very general sense) of combinatorial objects under symmetries. Let $C$ be a set of ‘colours’ and $c = |C|$. A colouring is a mapping $f: X \to C$. Two mapping $f, g$ are equivalent if there exists $\sigma \in \Gamma$ such that $g(x) = f(\sigma(x))$. That is, they are the same mapping up to a symmetry. The basic version of Pólya theorem is that the number of inequivalent colourings of $X$ by colours in $C$ under the action of $\Gamma$ is equal to

$$Z(\Gamma; c,\ldots,c).$$

Classical applications of the previous result are, for instance, the enumeration of necklaces (bicoloured cycles) up to cyclic or dihedral symmetry, or counting the number of ways of colourings the faces of a cube with a given number of colours, up to symmetries of the cube.

A useful and important generalization is the following, also known as Redfield-Pólya’s theorem. Let $w: X \to R$ be a weight function into a commutative ring $R$, which in the cases of interest will be either the ring of integers or a polynomial ring. A *colouring* is a mapping $f: X \to R$. The weight of $f$ is defined as $w(f) = \prod_{x \in X} w(f(x))$. Notice that two colourings in the same orbit have the same weight. Then we have the following result.

**Theorem 116 (Redfield-Pólya)** Let $\mathcal{O}$ be the set of orbits of $X$ by $R$ under the action of $\Gamma$. Then

$$\sum_{f \in \mathcal{O}} w(f) = Z(\Gamma; \sum_{c \in \mathcal{C}} w(c), \sum_{c \in \mathcal{C}} w(c)^2, \ldots, \sum_{c \in \mathcal{C}} w(c)^n),$$
When all the weights are equal to 1, one recovers Pólya’s theorem.

We now apply Pólya’s theorem to counting unlabelled graphs. Let \( V = \{1, \ldots, n\} \) and let \( E \) be the set of all 2-subsets of \( V \). A simple graph with vertex set \( V \) can be seen as a mapping \( f : E \to \{0, 1\} \), such that \( f(\{x, y\}) = 1 \) if and only if \( \{x, y\} \) is an edge. Two graphs are isomorphic if there exists a permutation of \( V \) mapping edges of one graph to edges of the other one. Let \( S_n \) be the symmetric group on \( V \) and let \( S'_n = \{ \sigma' : \sigma \in S_n \} \) be the group (isomorphic to \( S_n \)) acting on edges as \( \sigma' : \{x, y\} \to \{\sigma(x), \sigma(y)\} \).

Then \( f, g : E \to \{0, 1\} \) correspond to isomorphic graphs if they are in the same orbit of the action of \( S'_n \) on \( E \). For instance, \( \sigma \in S_5 \) has type \((5)\), then \( \sigma' \) has type \((1^3 1)\). If we perform all the calculations (not a pleasant task if done by hand) then we obtain

\[
Z(S'_5; x_1, x_2, x_3, x_4, x_5, \ldots) = \frac{1}{120} (x_1^{10} + 10x_1^4x_2 + 15x_1^3x_3 + 20x_1^2x_4 + 20x_1x_5 + 30x_2^2 + 24x_3^2).
\]

The number of unlabelled graphs with 5 vertices is then

\[
Z(S'_5; 2, 2, 2, 2) = 34.
\]

The counting sequence of unlabelled graphs, starting with \( n = 1 \), is

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1, 2, 4, 11, 34, 1044, 12346, \ldots

We can also count edges, it suffices to takes as weights \( w(0) = 1, w(1) = x \). Then the enumerator polynomial for edges is

\[
Z(S'_5; 1 + x, 1 + x^2, 1 + x^3, 1 + x^4, 1 + x^5) =
1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 4x^7 + 2x^8 + x^9 + x^{10}.
\]

This is an example that shows the usefulness of the general weighted version. Many other examples can be found in [36]. The case of trees can be handled directly with the symbolic method (for rooted trees) and the dissymmetry theorem (for unrooted trees).

### 6.9.2 Asymptotics

Let \( \mathcal{G} \) be a class of labelled graphs and \( \mathcal{U} \) the corresponding class of unlabelled graphs. Suppose we have a result saying that almost all graphs in \( \mathcal{G} \) have no automorphism as \( n \to \infty \). Then it follows that

\[
|\mathcal{U}_n| \sim \frac{|\mathcal{G}_n|}{n!}.
\]  

(6.15)

Hence an estimate for \( |\mathcal{G}_n| \) provides an estimated for \( |\mathcal{U}_n| \). Of course one can ask for finer asymptotics and investigate the error term.
In order to prove such a result it is enough to show that
\[ E(\text{aut}(G)) \to 1, \quad n \to \infty. \]

This is known in several important cases:

- **General graphs.** As we mentioned in the introduction almost every labelled graph has no automorphism. This statement can be refined by taking into account also the number of edges. Let \( G_{n,m} \) and \( U_{n,m} \) be, respectively, the number of labelled and unlabelled graphs with \( n \) vertices and \( m \) edges. Notice that \( G_{n,m} = \binom{N}{m} \), where \( N = \binom{n}{2} \), and trivially \( U_{n,m} \geq G_{n,m}/n! \). The following holds [10, Chap. 9]:

If \( c > 1 \) is a constant and
\[ cn \log n \leq m \leq N - cn \log n, \]
then
\[ U_{n,m} \sim G_{n,m}/n!. \]

The condition \( m \geq cn \log n \) is necessary as otherwise almost surely there is more than one isolated vertex and there is non-trivial automorphism (the second inequality is also necessary by complementation).

- **Regular graphs.** The fact that \( E(\text{aut}(G)) \to 1 \) for \( d \)-regular graphs was first proved by Bollobás for fixed \( d \), and then extended to \( d = o(\sqrt{n}) \) by McKay and Wormald; see the discussion in [78].

There are situations where the expected number of automorphisms is very large and the relation (6.15) does not hold. The first notable example is the case of trees. It is well-known that a random tree contains almost surely \( \alpha n \) subtrees isomorphic to any fixed subtree \( T_0 \). If we take \( T_0 \) as the tree on three vertices rooted at the vertex of degree two, then \( T_0 \) has a non-trivial automorphism that exchanges the two vertices of degree one. This action can be applied independently to any copy of \( T_0 \), and this implies that almost surely there are at least \( c^n \) automorphisms, where \( c = 2^\alpha \).

This phenomenon is true more generally for addable minor-closed classes (Section 6.7), since again almost surely there are linearly many pendant copies of each connected graph in the class. For subcritical families there is a general result [19] saying the the number of unlabelled graphs in the class is asymptotically
\[ c_u n^{-5/2} \gamma_u^{5/2}, \]
where \( \gamma_u \) is the unlabelled growth constant. Because almost surely the number of automorphisms is exponential, necessarily \( \gamma_u > \gamma \), the labelled growth constant. In some cases (for example for outerplanar graphs), \( \gamma_u \) has been determined. But for planar graphs this is a remarkable open problem. As we have seen in this case \( \gamma \approx 27.23 \), which is a lower bound for \( \gamma_u \). The best upper bound for \( \gamma_u \) is
\[ \gamma_u < 30.06, \]
which is proved by encoding unlabelled planar graphs with \( \alpha n \) bits, and \( \alpha \approx 4.91 \).

The main obstacle for extending the techniques of [19] to planar graphs is the enumeration of unlabelled planar graphs.
References


References


