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Irreducibility of the Tutte Polynomial of a Connected Matroid

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We solve in the affirmative a conjecture of Brylawski, namely that the Tutte polynomial of a connected matroid is irreducible over the integers. © 2001 Elsevier Science

If M is a matroid over a set E, then its Tutte polynomial is defined as

$$T(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)},$$

where $r(A)$ is the rank of A in M. This polynomial is an important invariant as it contains much information on the matroid; see [2, 3] for useful surveys.

One of the basic properties of $T(M; x, y)$ is that, if M is the direct sum of two matroids M_1 and M_2, then

$$T(M; x, y) = T(M_1; x, y)\, T(M_2; x, y).$$

In particular, this implies that $T(M; x, y)$ has a non-trivial factor in $\mathbb{Z}[x, y]$ if M is disconnected. Brylawski [1] conjectured that the converse also holds; this paper is devoted to a proof of this conjecture.

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THEOREM 1. *If M is a connected matroid, then $T(M; x, y)$ is irreducible in $\mathbb{Z}[x, y]$.*

Actually, an analysis of our proof shows that $T(M; x, y)$ is irreducible even in $\mathbb{C}[x, y]$.

COROLLARY 1. *If a matroid M has c connected components $M_1, ..., M_c$, then the factorization of $T(M; x, y)$ in $\mathbb{Z}[x, y]$ is exactly*

$$T(M; x, y) = T(M_1; x, y) \cdots T(M_c; x, y).$$

The main tool in proving our result is the following set of linear equations (B_k) that are satisfied by the coefficients of the Tutte polynomial of any matroid, and that were proved in [1].

LEMMA 1. *Let $T(M; x, y) = \sum b_{ij} x^i y^j$ be the Tutte polynomial of a matroid M and let m be the number of elements in M. Then*

$$\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^t \binom{k-s}{t} b_{st} = 0, \qquad (B_k)$$

for $k = 0, 1, ..., m-1$.

We also need the following basic properties [2]:

(1) $b_{00} = 0$ if $|E| \geqslant 1$;
(2) $b_{10} \neq 0$ if and only if M is connected;
(3) $x^k \mid T(M; x, y)$ if and only if M has at least k coloops;
(4) $y^k \mid T(M; x, y)$ if and only if M has at least k loops;
(5) if $i \geqslant r(M)$ or $j \geqslant n(M)$, then $b_{ij} = 0$, except if $i = r(M)$ and $j = 0$, or if $i = 0$ and $j = n(M)$, where $r(M)$ is the rank of M and $n(M) = m - r(M)$ is the nullity of M.

Suppose now M is a connected matroid on a set of m elements, and that there is a non-trivial factorization

$$T(M; x, y) = \sum b_{ij} x^i y^j = A(x, y)\, C(x, y), \qquad (1)$$

where $A(x, y) = \sum a_{ij} x^i y^j$ and $C(x, y) = \sum c_{ij} x^i y^j$.

Since $b_{00} = 0$, either a_{00} or c_{00} is zero; we may assume $a_{00} = 0$. Since

$$0 \neq b_{10} = a_{00} c_{10} + a_{10} c_{00},$$

the assumption implies that $c_{00} \neq 0$. We will prove that $c_{00} = 0$, thus obtaining a contradiction.

Since M is connected, by properties (3) and (4) above, neither x nor y are factors of $A(x, y)$ or $C(x, y)$. Define for a polynomial $P(x, y) = \sum p_{ij} x^i y^j$,

$$r_P(x) = \max\{i : p_{i0} \neq 0\}, \qquad r_P(y) = \max\{j : p_{0j} \neq 0\}, \tag{2}$$

and let

$$m(P) = r_P(x) + r_P(y).$$

Clearly, from (1), $m = m(T) = m(A) + m(C)$. As we are supposing a non-trivial factorization, it follows that $r_A(x), r_A(y) \leqslant m(A) < m(T)$. Also useful is the following property.

LEMMA 2. *Let M be a connected matroid and $T(M; x, y)$ be its Tutte polynomial with a factorization as in* (1). *Then the polynomial $A(x, y)$ also satisfies property* 5, *that is, if $r_A(x) \leqslant i$ or $r_A(y) \leqslant j$, then $a_{ij} = 0$, except if $i = r_A(x)$ and $j = 0$, or if $i = 0$ and $j = r_A(y)$.*

Proof. Let $\alpha = \max\{i : a_{ij} \neq 0 \text{ for some } j\}$ and $\beta = \max\{j : a_{\alpha j} \neq 0\}$; define analogously α' and β' for the polynomial $C(x, y)$. The monomial $a_{\alpha\beta} c_{\alpha'\beta'} x^{\alpha+\alpha'} y^{\beta+\beta'}$ appears in $T(M; x, y)$, as it cannot be cancelled, and it is the term with maximum degree of x in $T(M; x, y)$. Using property (5) we see that $\alpha + \alpha' = r(M)$ and $\beta + \beta' = 0$, so $\beta = 0$ and $\alpha = r_A(x)$. Thus, the maximum degree of x in $A(x, y)$ has coefficient $a_{r_A(x), 0}$. A similar argument shows that the maximum degree of y in $A(x, y)$ has coefficient $a_{0, r_A(y)}$. ∎

We next prove two lemmas that together imply Theorem 1.

Let (A_k), $k = 0, 1, ..., m(A)$ be the same set of equations as the (B_k), but with the a_{st} replacing the b_{st}, that is,

$$\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^t \binom{k-s}{t} a_{st} = 0. \tag{A_k}$$

Note that we do *not* assume $A(x, y)$ to be the Tutte polynomial of a matroid, hence we do not know whether equations (A_k) hold or not. In fact, we have the following result.

LEMMA 3. *With hypothesis as in Lemma* 2, *there is at least one equation (A_l) with $r_A(x) \leqslant l \leqslant m(A)$ that does not hold.*

Proof. First, for $r_A(x) \leqslant k \leqslant m(A)$ and $i \geqslant 0$ we define the equation (A_{ki}) as

$$\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{t+i} \binom{k-s}{t} a_{s,\,t+i} = 0. \tag{A_{ki}}$$

Note that (A_{k0}) is the same equation as (A_k). Now we prove a recurrence relation involving these equations.

Observe that for $i > 0$ and $k > r_A(x)$ the left-hand side of equation $(A_{k,\,i-1})$ is

$$\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{t+i-1} \binom{k-s}{t} a_{s,\,t+i-1}. \tag{3}$$

Using the fact that $\binom{k-s}{t} = \binom{k-s-1}{t} + \binom{k-s-1}{t-1}$, and assuming $\binom{a}{-b} = 0$ for $a \geqslant 0$, $b > 0$, and also $\binom{a}{b} = 0$ if $a < b$, we can rewrite (3) in the following way:

$$\begin{aligned}
&\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{t+i-1} \left[\binom{k-1-s}{t} + \binom{k-1-s}{t-1} \right] a_{s,\,t+i-1} + (-1)^{i-1} a_{k,\,i-1} \\
&\quad = \sum_{s=0}^{k-1} \sum_{t=0}^{k-1-s} (-1)^{t+i-1} \binom{k-1-s}{t} a_{s,\,t+i-1} \\
&\quad\quad + \sum_{s=0}^{k-1} \sum_{t=1}^{k-s} (-1)^{t+i-1} \binom{k-1-s}{t-1} a_{s,\,t+i-1} + (-1)^{i-1} a_{k,\,i-1}.
\end{aligned}$$

The last term appears because $\binom{0}{0}$ cannot be decomposed into two binomial coefficients. But by Lemma 2 for this last term we have $a_{k,\,i-1} = 0$, as $r_A(x) < k$. Also, the first and second terms in the second row of the last expression (after a change of variables) are, respectively, the left-hand side of equations $(A_{k-1,\,i-1})$ and $(A_{k-1,\,i})$. So we can write symbolically

$$(A_{k,\,i-1}) = (A_{k-1,\,i}) + (A_{k-1,\,i-1})$$

or

$$(A_{k-1,\,i}) = (A_{k,\,i-1}) - (A_{k-1,\,i-1}) \tag{4}$$

for $r_A(x) < k \leqslant m(A)$ and $i > 0$.

Let us suppose now that all equations (A_k) hold for $r_A(x) \leqslant k \leqslant m(A)$ and we will find a contradiction. Consider equation $(A_{r_A(x),\,r_A(y)})$. By Lemma 2, the only term a_{ij} involved in this equation that is not zero is

$a_{0, r_A(y)}$. Then the left-hand side of $(A_{r_A(x), r_A(y)})$ reduces to $\binom{r_A(x)}{0} a_{0, r_A(y)} = a_{0, r_A(y)}$, which is different from zero. On the other hand, using Eq. (4) repeatedly $r_A(y)$ times, we can express this nonzero term as a sum of the left-hand sides of equations (A_{k0}) for $r_A(x) \leqslant k \leqslant m(A) = r_A(x) + r_A(y)$, that we are assuming to be all equal to zero. Therefore we obtain a contradiction and we conclude that not all of the $(A_{k0}) = (A_k)$ hold for $r_A(x) \leqslant k \leqslant m(A)$. ∎

LEMMA 4. *If the coefficients a_{ij} do not satisfy equation (A_k) for some $k \leqslant m(A)$, then $c_{00} = 0$.*

Proof. Let (A_k) be the first equation that does not hold. Equation (B_k) holds because $k \leqslant m(A) < m$. First, we rewrite this equation taking into account that

$$b_{st} = \sum_{h \leqslant s} \sum_{l \leqslant t} c_{hl} a_{s-h,\, t-l}.$$

Then we have the following equalities for the left-hand side of (B_k).

$$\begin{aligned}
\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^t \binom{k-s}{t} b_{st} &= \sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^t \binom{k-s}{t} \sum_{h \leqslant s} \sum_{l \leqslant t} c_{hl} a_{s-h,\, t-l} \\
&= c_{00} \sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^t \binom{k-s}{t} a_{st} \\
&\quad + \sum_{0 < h+l \leqslant k} c_{hl} \left[\sum_{s=h}^{k} \sum_{t=l}^{k-s} (-1)^t \binom{k-s}{t} a_{s-h,\, t-l} \right]. \qquad (5)
\end{aligned}$$

Note that each c_{hl} has as coefficient an expression similar to the left hand side of (A_k); in particular, for c_{00} this coefficient is exactly the left-hand side of equation (A_k). More precisely, we introduce the equation (A'_{ni}).

$$\sum_{s=0}^{n-i} \sum_{t=0}^{n-i-s} (-1)^{t+i} \binom{n-s}{t+i} a_{st} = 0. \qquad (A'_{ni})$$

Observe that the left-hand side of equation $(A'_{k-h,\, l})$ is the coefficient of c_{hl} in (5) above: change indices $s \leftarrow s+h$, $t \leftarrow t+l$, and note that for $s > k-l$ and $t \geqslant l$ the binomial $\binom{k-s}{t}$ vanishes. Also note that (A'_{n0}) is precisely equation (A_n), which we are assuming holds for $0 \leqslant n < k$. Now, we prove that (A'_{ni}) holds for $1 \leqslant n \leqslant k$ and $1 \leqslant i \leqslant n$ using induction on n.

If $n = 1$, the only possible value for i is 1 and (A'_{11}) reduces to $a_{00} = 0$, which was supposed from the beginning. Assuming the result for all values less than n, we use again a formula for the binomial coefficients to decompose the left-hand side of equation (A'_{ni}) into a sum of previous equations:

$$\sum_{s=0}^{n-i} \sum_{t=0}^{n-i-s} (-1)^{t+i} \binom{n-s}{t+i} a_{st}$$
$$= \sum_{s=0}^{n-i} \sum_{t=0}^{n-i-s} (-1)^{t+i} \left[\binom{n-s-1}{t+i-1} + \binom{n-s-2}{t+i-1} + \cdots + \binom{t+i-1}{t+i-1} \right] a_{st}.$$

Each binomial coefficient $\binom{n-s}{t+i}$ is partitioned into exactly $n-s-t-i+1$ terms, so that the last expression equals

$$\sum_{s=0}^{n-1-(i-1)} \sum_{t=0}^{n-1-(i-1)-s} (-1)^{t+i} \binom{n-1-s}{t+i-1} a_{st}$$
$$+ \sum_{s=0}^{n-2-(i-1)} \sum_{t=0}^{n-2-(i-1)-s} (-1)^{t+i} \binom{n-2-s}{t+i-1} a_{st}$$
$$+ \cdots + \sum_{s=0}^{0} \sum_{t=0}^{0} (-1)^{t+i} \binom{t+i-1}{t+i-1} a_{st}.$$

Now it is easy to check that the pth term in the last sum is equal (up to the sign) to the left hand side of equation $(A'_{n-p,\,i-1})$, for $1 \leqslant p \leqslant n-i+1$. Thus we obtain the following relation:

$$(A'_{ni}) = -((A'_{n-1,\,i-1}) + (A'_{n-2,\,i-1}) + \cdots + (A'_{i-1,\,i-1})).$$

If $i=1$, the equations on the right are (A_{n-1}), ..., (A_0), all of which hold because $n-1<k$. If $i>1$, equations $(A'_{n-1,\,i-1})$, ..., $(A'_{i-1,\,i-1})$ hold by inductive hypothesis. In both cases (A'_{ni}) holds, and this concludes the induction.

Using this result we see from (5) that equation (B_k) reduces to $c_{00}(A_k) = 0$. As (A_k) does not hold, c_{00} must be zero and the lemma is proved. ▮

The above two lemmas show that $c_{00}=0$ and this establishes the theorem.

Remark. The assumption of characteristic zero is necessary, since otherwise property 2 after Lemma 1 does not hold, that is, b_{10} can be zero because of the characteristic. For example,

$$T(M(K_4); x, y) = 2x + 2y + 3x^2 + 4xy + 3y^2 + x^3 + y^3$$
$$= (x+y)(x+y+x^2+xy+y^2) \qquad \bmod 2,$$

whereas $M(K_4)$ is a connected matroid.

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