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The maximum degree of random planar graphs

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Abstract

McDiarmid and Reed ['On the maximum degree of a random planar graph', *Combin. Probab. Comput.* 17 (2008) 591–601] showed that the maximum degree Δ_n of a random labeled planar graph with n vertices satisfies with high probability (w.h.p.)

$$c_1 \log n < \Delta_n < c_2 \log n,$$

for suitable constants $0 < c_1 < c_2$. In this paper, we determine the precise asymptotics, showing in particular that w.h.p.

$$|\Delta_n - c \log n| = O(\log \log n),$$

for a constant $c \approx 2.52946$ that we determine explicitly. The proof combines tools from analytic combinatorics and Boltzmann sampling techniques.

1. *Introduction*

Let $\mathcal{G}_n$ be the family of labeled planar graphs with n vertices and let $g_n = |\mathcal{G}_n|$. By a random planar graph G_n, we mean a graph drawn from $\mathcal{G}_n$ with uniform probability $1/g_n$. Let Δ_n be the random variable equal to the maximum vertex degree in G_n. McDiarmid and Reed [21] showed that there exist constants $0 < c_1 < c_2$ such that with high probability†

$$c_1 \log n < \Delta_n < c_2 \log n.$$

These bounds were obtained by combinatorial arguments, using double counting and basic properties of random planar graphs from [22].

The main goal of this paper is to determine exactly the asymptotics of the maximum degree in random planar graphs. We show that Δ_n is concentrated around $c \log n$ for a well-determined constant c and the second-order term is $O(\log \log n)$.

Theorem 1.1. *There exists a constant $c > 0$ such that w.h.p.*

$$|\Delta_n - c \log n| = O(\log \log n). \tag{1.1}$$

Moreover, as $n \to \infty$

$$\mathbb{E}\Delta_n = (1 + o(1))c \log n. \tag{1.2}$$

We have $c = 1/\log w_0$, where w_0 is defined in Lemma 3.1, and is approximately $c \approx 2.52946$. The same result holds with the same constant for connected and for 2-connected planar graphs.

This model differs radically from the classical Erdős–Rényi model, where edges are drawn independently. For analyzing constrained classes of graphs, such as triangle-free graphs,

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†We say that a graph property $\mathcal{P}$ holds with high probability (w.h.p.) if the probability that $G_n \in \mathcal{P}$ tends to 1 as $n \to \infty$.

or, more generally, graphs without forbidden subgraphs [**27**] or without induced forbidden subgraphs [**6**], one has to resort to counting arguments. In particular, for planar and other related classes of graphs precise asymptotic estimates on the number of graphs are necessary.

The random planar graph model was introduced by Denise, Vasconcellos and Welsh [**8**] in 1996, and since then it has been studied intensively. One of the first results in this area is that a random planar graph of size n has at least $3n/2$ edges w.h.p. This was improved over the years showing that w.h.p. the number of edges is in the interval $(1.85n, 2.44n)$. It required the use of advanced tools from analytic combinatorics [**19**] to show finally that the number of edges is asymptotically normally distributed and strongly concentrated around $c'n$, where $c' \approx 2.21$.

A graph of size n with m edges has average degree $2m/n$. Thus, the average degree in a random planar graph is very close to 4.42. What can be said about the distribution of the vertex degrees? A basic result from [**22**] is that w.h.p. for every integer $k > 0$ there are linearly many vertices of degree k. This indicates the possibility of a discrete limit law for vertex degrees: if we show that the expected number of vertices of degree k is asymptotically $p_k n$ for some quantity p_k, then the probability that a random vertex has degree k is, up to lower order terms in n, equal to p_k. The existence of such a limit distribution has been established independently in [**11**, **25**]. The exact solution is quite involved: several pages are needed to write down the explicit expression for the probability generating function $\sum_{k\geqslant 1} p_k w^k$, but the values p_k are computable.

One of the results from [**11**] is an asymptotic estimate on the tail of the distribution. In particular, it was shown that, as $k \to \infty$

$$p_k = (1 + o(1))\, c \cdot k^{-1/2} w_0^{-k}, \tag{1.3}$$

where $c > 0$ and $w_0 \approx 1.48$ are constants that were determined. Note that this quantity becomes $\Theta(1/n)$ when k is close to $\log_{w_0} n$. Hence, this suggests that the expected number of vertices of that degree is $O(1)$, and that this is the right order of magnitude for the maximum degree Δ_n. More precisely, let $X_{n,k}$ denote the number of vertices of degree k in a random planar graph of size n and let

$$Y_{n,k} = \sum_{\ell > k} X_{n,\ell}$$

denote the number of vertices of degree larger than k. Clearly, we have

$$\Delta_n > k \iff Y_{n,k} > 0,$$

and consequently

$$\Pr[\Delta_n > k] = \Pr[Y_{n,k} > 0] \leqslant \mathbb{E}Y_{n,k}.$$

Suppose now that the estimate (1.3) is valid for any $0 < k < n$. This would then imply that $\mathbb{E}\, Y_{n,k}$ is $o(1)$ when $k = (1 + o(1)) \log n / \log w_0$. Thus, we would expect that, w.h.p.,

$$\Delta_n \sim \frac{\log n}{\log w_0}.$$

The aim of this work is to confirm this intuitive argument. Such a result has been proved recently using complex analytic methods by Drmota, Giménez and Noy [**12**] (with a different value of w_0) for series-parallel graphs, an important subclass of planar graphs. The result for series-parallel graphs was previously conjectured by Bernasconi, Panagiotou and Steger [**2**], where the authors prove strong concentration results for the number of vertices of degree up to $(c - \epsilon) \log n$, where $c = 1/\log w_0$. The results in [**2**] are obtained using so-called Boltzmann samplers, a framework that reduces the study of vertex degrees to properties of sequences of independent and identically distributed (i.i.d.) random variables.

In this paper, we combine both methods, complex analytic and probabilistic, to prove Theorem 1.1. As we will see, a combination of the two methods is in fact necessary to achieve the desired goal. The upper bound is proved using the tail estimate (1.3) and the first moment method. In fact, we perform a careful analysis of the singular structure of the multivariate generating function $G(x, y, w)$ of planar graphs rooted at a vertex, enumerated according to the number vertices, the number of edges and the degree of the root. It turns out that there is a computable critical value $w_0 > 1$ such that bounding $\mathbb{E}\, Y_{n,k}$ amounts to estimating the coefficients of $G(x, 1, w_0)$. This is achieved by first analyzing the corresponding generating function $B(x, y, w)$ for 2-connected planar graphs, that has the same critical value w_0 (this is why we get the same results for arbitrary and 2-connected planar graphs). The equations defining $B(x, y, w)$ and $G(x, y, w)$ are very involved (see Subsection 2.6), and one needs several technical results on the representation of the generating functions around their singularities.

In principle, the lower bound could be proved in a similar way using the second moment method, by rooting at a secondary vertex in addition to the root vertex and working with $G(x, y, w, t)$, where t marks the degree of the secondary vertex. This is done in [**12**] for series-parallel graphs, which is already very demanding. However, the technical difficulties with this approach for planar graphs appear insurmountable, since the equations defining $G(x, y, w, t)$ are just too complicated.

In order to obtain the lower bound, we use a different approach: Boltzmann samplers. They were introduced by Duchon, Flajolet, Louchard and Schaeffer [**13**] for the random generation of combinatorial objects. The basic principle is to sample as follows according to a control parameter. Let $\mathcal{A}$ be a class of combinatorial objects, let $\mathcal{A}_n$ be the set of objects of size n and let $a_n = |\mathcal{A}_n|$. Let also $A(x) = \sum_{n\geqslant 0} a_n x^n/n!$ be the exponential generating function (e.g.f.) of the class, and let x_0 be a real number for which $A(x_0)$ is convergent. Then any object $\alpha \in \mathcal{A}_n$ is assigned the probability $x_0^n/n!A(x_0)$. Note that the objects generated fluctuate in size, but all the objects of size n have the same probability. This framework has been applied successfully since then, in particular in the efficient generation of random planar graphs [**17**], and also to objects under the action of symmetries [**4**].

Boltzmann samplers have proved useful not only for random generation, but also for the *analysis* of random combinatorial objects. This approach was started in [**26**] and later pursued in [**2**, **3**] and in [**16**, **24**, **25**]. In the present paper, Boltzmann samplers also play a key role. A crucial fact, proved independently using probabilistic [**24**] and analytic methods [**20**], is that w.h.p. a connected random planar graph has a unique block (2-connected component) of linear size, and the remaining blocks are of order at most $n^{2/3}$. Thus, a typical random planar graph G can be thought of as a large block B together with small planar graphs attached to its vertices. If we condition on the total size of G being n, then the graphs attached to B are drawn *independently* from the set of all connected planar graphs. Thus, we recover the power of independent samples and are able to use techniques closer to the classical theory of random graphs. As we show later, the maximum degree comes from within the largest block.

We find then ourselves in an unexpected (and satisfactory) situation: analytic methods yield only the upper bound and probabilistic methods yield only the lower bound, with the same multiplicative constant. This can be considered as the culmination of two parallel and independent approaches for analyzing random planar graphs, one based on generating functions and analytic methods [**10**–**12**, **19**, **20**], the other one based on Boltzmann samplers and concentration inequalities [**2**, **3**, **16**, **24**–**26**].

Outline. The rest of the paper is structured as follows. In Subsection 2.1, we introduce the basic methods, combinatorial constructions, generating functions, Boltzmann samplers and analytic tools, that are required for our further analysis. This section also serves as a gentle and concise introduction to the techniques mentioned above. Section 3 contains the proof of the upper bound on the maximum degree given in Theorem 1.1, equation (1.1), and Section 4 provides the matching lower bound. Finally, in Section 5, we show the proof of (1.2). Some further research directions and discussion are provided in Section 6.

2. *Tools and techniques*

2.1. *Basic notation*

Let $\mathcal{G}$ be a class of graphs, let $\mathcal{G}_{n,m}$ be the graphs in $\mathcal{G}$ with n (labeled) vertices and m edges, and write $g_{n,m} = |\mathcal{G}_{n,m}|$. Let also $\mathcal{G}_n = \bigcup_{m\geqslant 0} \mathcal{G}_{n,m}$ and set $g_n = |\mathcal{G}_n|$. In particular, in the remainder of the paper we write $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{T}$, respectively, for the classes of connected, 2-connected and 3-connected planar graphs. Given a class of graphs $\mathcal{G}$, define $\mathcal{G}^\bullet = \bigcup_{n\geqslant 1}\{1,\ldots,n\} \times \mathcal{G}_n$ as the class of *vertex-rooted* graphs, so that every graph $G \in \mathcal{G}_n$ is contained n times in $\mathcal{G}_n$, and each copy contains a different distinguished vertex. Similarly, the *vertex-derived* class $\mathcal{G}'_{n-1,m}$ is obtained by removing the label n from each graph in $\mathcal{G}_{n,m}$, so that the resulting graphs have $n-1$ labeled vertices and one distinguished vertex bearing no label. Consequently, there is a bijection between the classes $\mathcal{G}'_{n-1}$ and $\mathcal{G}_n$. We set $\mathcal{G}' := \bigcup_{n\geqslant 0} \mathcal{G}'_n$. It will also be necessary to distinguish edges. To this end, define $\mathcal{G}_e = \bigcup_{n,m\geqslant 1}\{1,\ldots,m\} \times \mathcal{G}_{n,m}$ as the class of *edge-rooted* graphs, which contains each graph in $\mathcal{G}$ a number of times equal to the number of edges. As for vertex-rooted graphs, every graph in $\mathcal{G}_e$ has a specific distinguished edge. For technical reasons, we assume that the marked edge does not contribute to the total number of edges in each graph in $\mathcal{G}_e$. In other words, we may think that this edge is removed, but its former endpoints are distinguished, so that the graph can be fully recovered.

The main parameter of study in this paper is the maximum degree of random planar graphs. Let $\mathcal{C}^\bullet_{n,m,k}$ be the class of vertex-rooted planar graphs with n vertices and m edges, such that the degree of the root vertex is k. Define $\mathcal{B}^\bullet_{n,m,k}$ and $\mathcal{T}^\bullet_{n,m,k}$ similarly. Moreover, for $\mathcal{G} \in \{\mathcal{C},\mathcal{B},\mathcal{T}\}$ let

$$G^\bullet(x,y,w) = \sum_{n,m,k\geqslant 0} \frac{|\mathcal{G}^\bullet_{n,m,k}|}{n!} x^n y^m w^k$$

denote the *e.g.f.* enumerating the sequence $(|\mathcal{G}^\bullet_{n,m,k}|)_{n,m,k\geqslant 0}$. We shall omit any of the parameters x, y, w if the corresponding value is equal to 1; for example, we write $G^\bullet(x) = G^\bullet(x,1,1)$. Similar, we write $G'(x,y,w)$ for the e.g.f. enumerating $(|\mathcal{G}'_{n,m,k}|)_{n,m,k\geqslant 0}$, where $\mathcal{G}'_{n,m,k}$ are the graphs in $\mathcal{G}'_{n,m}$, whose unlabeled vertex has degree k. Observe that

$$G^\bullet(x,y,1) = x\frac{\partial}{\partial x}G(x,y), \quad G_e(x,y,1) = \frac{\partial}{\partial y}G(x,y).$$

We mention already at this point that all generating functions considered in this work have (at least) one finite dominant non-zero singularity on the real axis. For a generating function G enumerating a graph class $\mathcal{G}$, we write ρ_G for this singularity.

2.2. *Combinatorial constructions, generating functions and Boltzmann samplers*

In this section, we describe a collection of five universal constructions (disjoint union, product, set, vertex- and edge substitution), together with the associated relations for the generating functions and the resulting Boltzmann sampling algorithms, that we use to formulate a decomposition of the class of all connected planar graphs. We first define *Boltzmann samplers*, introduced in [**13**]. Let $\mathcal{G}$ be a class of labeled combinatorial objects (in our case graphs, where possibly vertices or/and edges might be distinguished), enumerated by the function $G(x,y)$. A Boltzmann sampler is a randomized algorithm that draws graphs from $\mathcal{G}$ under a certain probability distribution that is spread over the whole class. More precisely, suppose that $x, y \in \mathbb{R}_+$ are such that $G(x,y)$ exists. Then, the Boltzmann distribution with parameters

x, y assigns to each $\gamma \in \mathcal{G}$ the weight

$$\Pr[\gamma] = \frac{x^{v(\gamma)} y^{e(\gamma)}}{v(\gamma)! G(x, y)}, \tag{2.1}$$

where $v(\gamma)$ denotes the number of labeled vertices in γ, and $e(\gamma)$ denotes the number of edges of γ. A Boltzmann sampler $\Gamma G(x, y)$ is an algorithm that generates graphs according to the distribution in (2.1).

Note that Boltzmann samplers are not *a priori* suited for studying the distribution of graphs that are drawn uniformly at random from $\mathcal{G}_n$, as (2.1) defines a distribution over the whole of $\mathcal{G}$. However, observe that if we set $y = 1$ in (2.1), then the Boltzmann distribution is actually the uniform distribution on graphs of the same size. More precisely, if we denote by G_n a graph drawn uniformly at random from $\mathcal{G}_n$ and abbreviate $\gamma = \Gamma G(x, 1)$, then for any $\mathcal{P} \subseteq \mathcal{G}$ we have

$$\Pr[G_n \in \mathcal{P}] = \Pr[\gamma \in \mathcal{P} \mid \gamma \in \mathcal{G}_n] = \Pr[\gamma \in \mathcal{P} \text{ and } \gamma \in \mathcal{G}_n] \cdot \Pr[\gamma \in \mathcal{G}_n]^{-1}. \tag{2.2}$$

Boltzmann samplers can be constructed explicitly, and provide essentially 'recipes', which translate sequences of i.i.d. random variables into random graphs. So, if the Boltzmann probability of getting a desired graph of size n is not too small, then the study of random graphs boils down with (2.2) to studying properties of sequences of i.i.d. random variables. This approach is essential in Section 4.

We now define the combinatorial constructions and the associated generating functions and Boltzmann samplers. For a detailed exposition of the combinatorial constructions and the symbolic method, we refer the reader to the book of Flajolet and Sedgewick [**15**]. The proofs for the given relations of the generating functions and the validity of the Boltzmann samplers can all be found in [**13**, **17**].

Singleton class. We denote by $\mathcal{X}$ the class containing a single object (in our case a graph) of size 1, that is, a graph with one vertex. Using the notation from Subsection 2.1, the e.g.f. enumerating $\mathcal{X}$ is given by x. A Boltzmann sampler ΓX for $\mathcal{X}$ simply returns with probability 1 the graph in $\mathcal{X}$.

Disjoint union. The disjoint union of two classes $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{G} = \mathcal{A} + \mathcal{B}$, and its e.g.f. is $G(x, y) = A(x, y) + B(x, y)$. A Boltzmann sampler ΓG for $\mathcal{G}$ can be described in terms of Boltzmann samplers $\Gamma A(x, y)$ for $\mathcal{A}$ and $\Gamma B(x, y)$ for $\mathcal{B}$, where we denote by $\mathsf{Be}(p)$ a Bernoulli random variable with success probability p.

$\Gamma G(x, y)$:
  $b \leftarrow \mathsf{Be}(\frac{A(x,y)}{G(x,y)})$
  if $b = 1$ **return** $\Gamma A(x, y)$
  else return $\Gamma B(x, y)$

In other words, the Boltzmann sampler for $\mathcal{G}$ first makes a Bernulli choice between $\mathcal{A}$ and $\mathcal{B}$, and then resorts to the Boltzmann sampler for the chosen class.

Product. The labeled product $\mathcal{G} = \mathcal{A} \times \mathcal{B}$ of two classes $\mathcal{A}$ and $\mathcal{B}$ is obtained by taking all ordered pairs (a, b) with $a \in \mathcal{A}_n$ and $b \in \mathcal{B}_{n'}$, and relabeling them in all possible order-preserving ways such that all labels in $\{1, \ldots, n + n'\}$ are used. This means that when vertices are relabeled it is assumed that the order of the labels in a and b is preserved. We will denote any such relabeling as *canonical* in the sequel. The e.g.f. enumerating $\mathcal{G}$ is given by $G(x, y) = A(x, y)B(x, y)$ (this is why our generating functions are of the exponential type, see [**15**, Definition II.3]). A Boltzmann sampler ΓG for $\mathcal{G}$ can be described in terms of Boltzmann samplers for $\mathcal{A}$ and $\mathcal{B}$ as follows. The algorithm *RandomLabels*(G) assigns random labels to the vertices of G from the set $\{1, \ldots, v(G)\}$.

$\Gamma G(x, y)$:
  $\gamma_A \leftarrow \Gamma A(x, y)$
  $\gamma_B \leftarrow \Gamma B(x, y)$
  return RandomLabels$((\gamma_A, \gamma_B))$

Note that the Boltzmann sampler performs independent calls to the samplers associated to $\mathcal{A}$ and $\mathcal{B}$, and composes a graph from $\mathcal{G}$ by assembling them and distributing randomly the labels.
Set. Let $\mathcal{A}$ be a class of graphs that contain at least one labeled vertex.† The class $\mathcal{G} = \mathrm{Set}(\mathcal{A})$ of sets of $\mathcal{A}$ is defined as follows. A graph in $\mathcal{G}$ consists of a finite set (unordered collection) of graphs from $\mathcal{A}$, whose vertices, as before, are relabeled canonically. In addition, for each non-negative integer k we write $\mathcal{G} = \mathrm{Set}_{\geqslant k}(\mathcal{A})$ for the class of sets of size at least k. The associated e.g.f. is given by

$$G_k(x,y) = e^{A(x,y)} - \sum_{i=0}^{k-1} \frac{A(x,y)^i}{i!}.$$

Let $\mathsf{Po}_{\geqslant k}(\lambda)$ be a Poisson distributed random variable with expectation λ, conditioned on being at least k. That is, for $j \geqslant k$,

$$\Pr[\mathsf{Po}_{\geqslant k}(\lambda) = j] = e^{-\lambda}\frac{\lambda^j}{j!} \cdot \left(1 - \sum_{i=0}^{k-1} e^{-\lambda}\frac{\lambda^i}{i!}\right)^{-1}.$$

A Boltzmann sampler for the ($\geqslant k$)-set construction is given by the following algorithm.

$\Gamma G(x,y)$:
 $j \leftarrow \mathsf{Po}_{\geqslant k}(A(x,y))$
 for $\ell = 1 \ldots j$ **do** $\gamma_i \leftarrow \Gamma A(x,y)$
 return RandomLabels$((\gamma_1, \ldots, \gamma_j))$

Note that the number of 'components' in a (Boltzmann) random graph from $\mathrm{Set}(\mathcal{A})$ is Poisson distributed with parameter $A(x,y)$; this can be easily verified by observing that the subclass of $\mathcal{G}$ containing all graphs with exactly j components from $\mathcal{A}$ is enumerated by $A(x,y)^j/j!$.
Vertex substitution. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes such that the graphs in $\mathcal{B}$ are vertex rooted. Then the class $\mathcal{G} = \mathcal{A} \circ \mathcal{B}$ obtained by vertex substitution from the core class $\mathcal{A}$ and the replacement class $\mathcal{B}$ is defined as follows. Given any $a \in \mathcal{A}$ and any family $(b_v)_{v \in V(a)}$ of graphs in $\mathcal{B}$, where $V(a)$ denotes the set of labeled vertices of a, identify each $v \in V(a)$ with the root of b_v and relabel the vertices in $(b_v)_{v \in V(a)}$ canonically. This creates a graph in $\mathcal{G} = \mathcal{A} \circ \mathcal{B}$. The e.g.f. enumerating $\mathcal{G}$ is given by $G(x,y) = A(B(x,y),y)$, so that vertex substitution corresponds formally to the substitution of the variable marking vertices.

The Boltzmann sampler for $\mathcal{G}$ first samples a core object from the Boltzmann distribution for $\mathcal{A}$, and then replaces independently each vertex with a random graph from $\mathcal{B}$, as follows:

$\Gamma G(x,y)$:
 $\gamma \leftarrow \Gamma A(B(x,y),y)$
 for each vertex $v \in V(\gamma)$ **do**
  $\gamma_v \leftarrow \Gamma B(x,y)$
 identify all $v \in V(\gamma)$ with the root of γ_v
 return RandomLabels(γ)

Edge substitution. The setting is as before, but now graphs in $\mathcal{B}$ have two distinct ordered vertex roots. The class $\mathcal{G} = \mathcal{A} \,\tilde{\circ}\, \mathcal{B}$ is obtained by edge substitution from the core class $\mathcal{A}$ and the replacement class $\mathcal{B}$. Given any $a \in \mathcal{A}$ and any family $(b_e)_{e \in E(a)}$ of graphs in $\mathcal{B}$, where $E(a)$ denotes the set of edges in a, identify each $e \in E(a)$ with the roots of b_e according to a specific rule, for example, defined in terms of the labels in e, and relabel the vertices in $(b_e)_{e \in E(a)}$ canonically. This creates a graph in $\mathcal{G} = \mathcal{A} \tilde{\circ} \mathcal{B}$. The e.g.f. enumerating $\mathcal{G}$ is given by

†Note that the derivative operator from Subsection 2.1 allows us to construct classes in which all graphs bear no labels, for example, $\mathcal{X}'$. Such classes are not allowed to be used within the Set construction.

$G(x, y) = A(x, B(x, y))$, so that edge substitution corresponds formally to the substitution of the variable marking edges. The Boltzmann sampler for $\mathcal{G}$ proceeds analogously:

$\Gamma G(x, y)$:
$\gamma \leftarrow \Gamma A(x, B(x, y))$
for each edge $e \in E(\gamma)$ **do**
$\gamma_e \leftarrow \Gamma B(x, y)$
identify for all $e \in E(\gamma)$ the endpoints of e with the roots of γ_e
return RandomLabels(γ)

2.3. *Grammars and generating functions for planar graphs*

A connected graph is uniquely specified in terms of its 2-connected components, each of which is further decomposed into 3-connected components. We describe this decomposition here, tailored to the specific setting of planar graphs and using the notation from the previous section. See the classical reference [**29**], or [**7**] for a modern exposition.

We start with the well-known decomposition of a graph into 2-connected components. A *block* of a vertex-derived connected graph $C' \in \mathcal{C}'$ is either a maximal 2-connected subgraph of C' or an isthmus (an edge whose removal disconnects C'). Note that C' can be obtained recursively as follows. Start with a set $\{B'_1, \ldots, B'_\ell\}$ of vertex-derived 2-connected graphs or vertex-derived edges whose distinguished vertices are all identified in a single vertex (the root of C'), and substitute every other vertex in $B'_1, \ldots, B'_\ell$ with a vertex-rooted connected graph. Note that the B'_i correspond to the blocks incident to the distinguished vertex of C'. This description gives us the combinatorial relation

$$\mathcal{C}' = \mathrm{Set}(\mathcal{B}' \circ \mathcal{C}^\bullet), \tag{2.3}$$

where an equality between classes must be understood (here and in the sequel) as a bijection preserving the size of the objects. This combinatorial equality translates into the following equation for the associated generating functions:

$$\frac{\partial C(x, y)}{\partial x} = \exp\left(\frac{\partial B}{\partial x}\left(x\frac{\partial C(x, y)}{\partial x}, y\right)\right). \tag{2.4}$$

The decomposition of 2-connected planar graphs into 3-connected components is more involved. We describe it in sufficient detail, as it is crucial for our further analysis. Following Trakhtenbrot [**28**] and Tutte [**29**], we define a (*planar*) *network* as a connected graph with two 'special' vertices, called the left pole and the right pole, such that after adding the edge between the poles and ignoring possible multiple edges, what results is a 2-connected planar graph. The poles do not bear labels, and thus in the e.g.f. enumerating networks the variable x marks the number of non-pole vertices. The above description provides us an explicit relation between the class $\mathcal{B}$ and the class of networks $\mathcal{D}$. Note that every edge-rooted 2-connected planar graph $B_e \in (\mathcal{B}_e)_n$, where $n \geqslant 3$, gives rise to two networks with $n - 2 \geqslant 1$ non-pole vertices: one is obtained by removing the labels from the endpoints of the root-edge (and relabeling the remaining vertices with $\{1, \ldots, n - 2\}$ canonically), and the other one is obtained by adding the root-edge to B_e. Moreover, if $n = 2$ and $(B_e)_n$ consists of a single edge, then we obtain the network that consists of a single edge or an auxiliary structure that consists of two poles and no edge. If e is the network consisting of a single edge and $\mathcal{X}_2$ the class of graphs that contains a single graph with two vertices and no edge, then we infer that $\mathcal{D}$ and $\mathcal{B}$ are related through

$$\mathcal{X}_2 + (\mathcal{D} \times \mathcal{X}_2) = (1 + e) \times \mathcal{B}_e.$$

Since the e.g.f. of $\mathcal{X}_2$ is equal to $x^2/2$, this translates into the following equation among the generating functions:

$$(1 + D(x, y))\frac{x^2}{2} = (1 + y)\frac{\partial B(x, y)}{\partial y}. \tag{2.5}$$

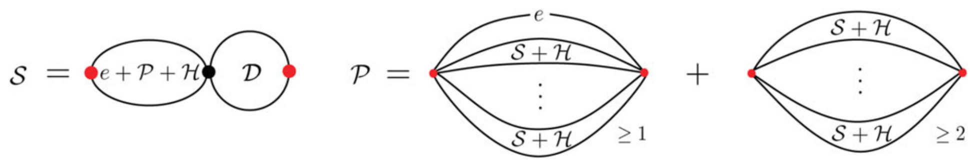

FIGURE 2.1. *The decomposition of series and parallel networks.*

We next describe the decomposition of networks in terms of 3-connected planar graphs. Following Tutte [**29**], a network is either an edge, whose end-vertices are the poles, or is in the class $\mathcal{S}$ (*series network*), or in the class $\mathcal{P}$ (*parallel network*), or in the class $\mathcal{H}$ (*core network*). This classes are disjoint and we obtain the combinatorial composition

$$\mathcal{D} = e + \mathcal{S} + \mathcal{P} + \mathcal{H}. \tag{2.6}$$

Writing $S(x, y)$, $P(x, y)$ and $H(x, y)$ for the e.g.f. enumerating $\mathcal{S}$, $\mathcal{P}$ and $\mathcal{H}$, respectively, we have

$$D(x, y) = y + S(x, y) + P(x, y) + H(x, y). \tag{2.7}$$

The decomposition of series networks is as follows (see Figure 2.1, taken from [**16**]). A network in $\mathcal{S}$ consists of two networks D_1 and D_2, such that the right pole of D_1 is identified with the left pole of D_2. Here, D_1 is restricted to be either an edge, or in $\mathcal{P}$ or in $\mathcal{H}$, and $D_2 \in \mathcal{D}$. Hence,

$$\mathcal{S} = (e + \mathcal{P} + \mathcal{H}) \times \mathcal{X} \times \mathcal{D} \quad \text{and} \quad S(x, y) = x(y + P(x, y) + H(x, y))D(x, y). \tag{2.8}$$

A parallel network (see Figure 2.1) consists either of an edge and a non-empty set of networks, either in $\mathcal{S}$ or in $\mathcal{H}$, where their right poles (left poles) are identified into a single right pole (left pole), or of a set of at least two networks, either in $\mathcal{S}$ or in $\mathcal{H}$ where the identification of the poles is as before. Thus,

$$\mathcal{P} = e \times \mathrm{Set}_{\geqslant 1}(\mathcal{S} + \mathcal{H}) + \mathrm{Set}_{\geqslant 2}(\mathcal{S} + \mathcal{H}), \tag{2.9}$$

and consequently

$$P(x, y) = y(e^{S(x,y)+H(x,y)} - 1) + (e^{S(x,y)+H(x,y)} - S(x, y) - H(x, y) - 1). \tag{2.10}$$

Finally, we define the class of core networks. Recall that $\mathcal{T}$ denotes the class of 3-connected planar graphs. Let $\bar{\mathcal{T}}$ be the class of networks obtained by taking a graph in $\mathcal{T}$, deleting an edge, and turning its former end-vertices into poles. A network in $\mathcal{H}$ (see Figure 2.2) consists of a network from $\bar{\mathcal{T}}$, where each edge is replaced by a network whose poles are identified in a unique way with the end-vertices of the edges. We thus obtain the relations

$$\mathcal{H} = \bar{\mathcal{T}} \tilde{\circ} \mathcal{D} \quad \text{and} \quad H(x, y) = \bar{T}(x, D(x, y)). \tag{2.11}$$

This concludes the definition of the networks and the setup for the associated generating functions. By a simple elimination procedure [**28**], equations (2.7)–(2.11) can be reduced to a single equation for $D(x, y)$:

$$D(x, y) = (1 + y) \exp\left(\frac{xD(x, y)^2}{1 + xD(x, y)} + \bar{T}(x, D(x, y))\right) - 1. \tag{2.12}$$

It is also known [**19**] that $B(x, y)$ can be computed explicitly in terms of $D(x, y)$, that is, the integration in (2.5) can be made explicit. In particular, setting $D = D(x, y)$ we obtain

$$B(x, y) = T(x, D) - \frac{xD}{2} + \frac{1}{2}\log(1 + xD) + \frac{x^2}{2}\left(D + \frac{D^2}{2} + (1 + D)\log\left(\frac{1 + y}{1 + D}\right)\right). \tag{2.13}$$

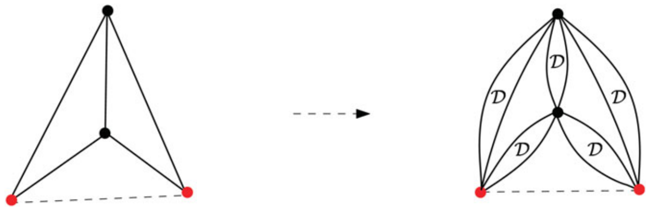

FIGURE 2.2. *The decomposition of core networks.*

The last step needed to complete the decomposition of the class of all connected planar graphs is to specify the class $\mathcal{T}$. We will not describe the decomposition here, as it is not needed for our further analysis, and refer to [**5**, **23**]. However, we need the associated generating functions, which satisfy the following equations:

$$\bar{T}(x,y) = \frac{y}{2}\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U(x,y))^2(1+V(x,y))^2}{(1+U(x,y)+V(x,y))^3}\right), \tag{2.14}$$

where $U(x,y)$ and $V(x,y)$ are given by

$$U(x,y) = xy(1+V(x,y))^2 \quad \text{and} \quad V(x,y) = y(1+U(x,y))^2. \tag{2.15}$$

Similarly, there is an explicit expression for $T(x,y)$ in terms of $U(x,y)$ of $V(x,y)$; see [**19**].

2.4. *Singular expansions and asymptotics*

A main feature of our approach is the fact that analytic properties, in particular the local behavior around singularities of a function $f(x) = \sum_n f_n x^n$, translate into asymptotic expansions for the coefficients $f_n = [x^n]f(x)$. We use in particular the so-called Transfer Lemma by Flajolet and Odlyzko [**14**]. Let x_0, ϵ and δ be positive real numbers. The region

$$\Delta = \Delta(x_0,\epsilon,\delta) = \{x \in \mathbb{C} : |x| < x_0 + \epsilon,\ |\arg(x/x_0 - 1)| > \delta\}$$

is called a Δ-region. Suppose that a function $f(x)$ is analytic in $\Delta(x_0,\epsilon,\delta)$ and satisfies

$$f(x) = C(1-x/x_0)^\alpha + O((1-x/x_0)^{\alpha+1}), \quad x \in \Delta(x_0,\epsilon,\delta),$$

where C is a constant. Then we have

$$[x^n]f(x) = C\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}x_0^{-n} + O(x_0^{-n}n^{-\alpha-2}). \tag{2.16}$$

It important to observe that the implicit constants are effective, in the sense that the O-constant in the expansion of $f(x)$ provides an explicit O-constant for the expansion of $[x^n]f(x)$; see [**14**]. In particular, singular expansions that are uniform in some parameter also translate into asymptotic expansions of the form (2.16) with a uniform error term.

A typical situation where the Transfer Lemma applies, is a generating function with a so-called *square-root singularity*, that is, with a local representation of the form

$$f(x) = g(x) - h(x)\sqrt{1-x/x_0} \tag{2.17}$$

that holds in a complex neighborhood U of x_0 with $x_0 \neq 0$, cutting the half line $\{x \in \mathbb{C} : \arg(x/x_0 - 1) = 0\}$ in order to have an unambiguous value of the square root. The functions $g(x)$ and $h(x)$ are assumed to be analytic in U. We also assume that $f(x)$ has an analytic continuation to the region $\{x \in \mathbb{C} : |x| < x_0 + \varepsilon\} \setminus U$ for some $\varepsilon > 0$. These assumptions imply that

$$f(x) = g(x_0) - h(x_0)\sqrt{1-x/x_0} - x_0 g'(x_0)(1-x/x_0) + O((1-x/x_0)^{3/2}),$$

uniformly in a Δ-region. It follows that

$$f_n = [x^n]\, f(x) = \frac{h(x_0)}{2\sqrt{\pi}}\, n^{-3/2} x_0^{-n} + O(n^{-5/2} x_0^{-n}). \tag{2.18}$$

Note that a function $f(x)$ of the form (2.17) can also represented as

$$f(x) = \sum_{\ell \geqslant 0} a_\ell \left(1 - \frac{x}{x_0}\right)^{\ell/2} = \sum_{\ell \geqslant 0} a_\ell X^\ell, \tag{2.19}$$

where $X = \sqrt{1 - x/x_0}$ and the coefficients are complex numbers. Moreover, the power series $\sum_{\ell \geqslant 0} a_\ell X^\ell$ converges for $|X| < r$ (for a suitable $r > 0$), so that it represents an analytic function of X. It is also clear that a representation of the form (2.19) can be rewritten into (2.17). We refer to both representations as *singular expansions* of $f(x)$. If we are only interested in the first few terms, then we write

$$f(x) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + O(X^4).$$

We also encounter several situations where $a_1 = 0$. Then $f(x)$ can be represented as

$$f(x) = \bar{g}(x) + \bar{h}(x) X^3 = \bar{g}(x) + \bar{h}(x)(1 - x/x_0)^{3/2},$$

where $\bar{g}, \bar{h}$ are analytic in a complex neighborhood of x_0. In this case, the corresponding asymptotic expansion for the coefficients is of the form

$$f_n = \frac{3h(x_0)}{4\sqrt{\pi}}\, n^{-5/2} x_0^{-n} + O(n^{-7/2} x_0^{-n}).$$

Functions $f(x)$ with a square-root singularity appear naturally as solutions of functional equations $\Phi(x, f(x)) = 0$, where $\Phi(x, y)$ is an analytic function (see [**9**]). More precisely, if we know that there is x_0 and $y_0 = y(x_0)$ such that (x_0, y_0) is a regular point of $\Phi(x, y)$ with

$$\Phi(x_0, y_0) = 0 \quad \text{and} \quad \Phi_y(x_0, y_0) = 0, \tag{2.20}$$

and the conditions

$$\Phi_x(x_0, y_0) \neq 0 \quad \text{and} \quad \Phi_{yy}(x_0, y_0) \neq 0, \tag{2.21}$$

then x_0 is a singularity of $f(x)$ and there is a local representation of the form (2.17) with $g(x_0) = y_0$ and $h(x_0) = \sqrt{2x_0 \Phi_x(x_0, y_0)/\Phi_{yy}(x_0, y_0)}$.

Usually, it is easy to verify that $f(x)$ has an analytic continuation to a Δ-region. A basic example is the following. Suppose that $\Phi(x, y)$ is of the form $\Phi(x, y) = y - F(x, y)$, where $F(0, y) = 0$ and $F(x, y) = \sum_{i,j} F_{ij} x^i y^j$ has non-negative coefficients F_{ij}, and the system (2.20) has a positive solution (x_0, y_0) inside the region of convergence of $F(x, y)$. Then there is a unique power series solution $y = f(x) = \sum_n f_n x^n$ of $y = F(x, y)$ with $f(0) = 0$, and $f_n \geqslant 0$ for all n. If in addition there exist two non-zero coefficients f_{n_1}, f_{n_2} with $\gcd(n_1, n_2) = 1$, then $|F_y(x, f(x))| < F_y(|x|, f(|x|))$ if x is not real and positive. Consequently, it is impossible that $F_y(x, f(x)) = 1 = F(x_0, y_0)$ for $|x| \leqslant x_0$ and $x \neq x_0$. By the implicit function theorem, there are no singularities in this range, and thus there is an analytic continuation to a Δ-region. Similar properties hold for solutions $\mathbf{f}(x) = (f_1(x), \ldots, f_d(x))$ of a system of equations $\mathbf{f}(x) = \mathbf{F}(x, \mathbf{f}(x))$, where $\mathbf{F}$ is *positive* and *strongly connected*. For details, see [**9**].

If the functional equation has an additional analytic *parameter* u, that is, $y = f(x, u)$ satisfies $\Phi(x, u, y) = 0$, then we are in a situation that is relevant in this paper (the additional parameter will typically mark the number of edges and/or the degree of the root vertex). Then we (usually) have a local representation of the form

$$f(x, u) = g(x, u) - h(x, u)\sqrt{1 - x/\rho(u)} \tag{2.22}$$

that holds in a (complex) neighborhood U of (x_0, u_0) with $x_0 > 0$, $u_0 > 0$ and with $\rho(u_0) = x_0$ (as before we slit the line $\{x \in \mathbb{C} : \arg(x - \rho(u)) = 0\}$). The functions $g(x, u)$ and $h(x, u)$

are analytic in U and $\rho(u)$ is analytic in a neighborhood of u_0. As above, it is usually easy to establish that $f(x,u)$ has an analytic continuation to the region $\{(x,u) \in \mathbb{C}^2 : |x| < x_0 + \varepsilon,\ |u| < u_0 + \varepsilon\} \setminus U$ for some $\varepsilon > 0$. Moreover, in complete analogy to the case without the additional parameter, a function $f(x,u)$ of the form (2.22) can be represented as

$$f(x,u) = \sum_{\ell \geqslant 0} a_\ell(u) \left(1 - \frac{x}{\rho(u)}\right)^{\ell/2} = \sum_{\ell \geqslant 0} a_\ell(u) X^\ell, \tag{2.23}$$

where $X = \sqrt{1 - x/\rho(u)}$ and the coefficients $a_\ell(u)$ are analytic in u (for u close to u_0).

We recall that square-root singularities appear if we consider solutions $f(x)$ with $f(x_0) = y_0$ of a functional equation $\Phi(x,y) = 0$, where (x_0, y_0) is a regular point of $\Phi(x,y)$. Of course, this will not remain true if (x_0, y_0) is a singularity of $\Phi(x,y)$. Nevertheless, we will encounter several situations, where a special singular structure appears. The following lemma is [**9**, Theorem 2.31].

Theorem 2.1. *Suppose that $F(x,y,u)$ has a local representation of the form*

$$F(x,y,u) = G(x,y,u) + H(x,y,u)\left(1 - \frac{y}{r(x,u)}\right)^{3/2}, \tag{2.24}$$

with functions $G(x,y,u)$, $H(x,y,u)$, $r(x,u)$ that are analytic in a neighborhood of (x_0,y_0,u_0) and satisfy $G_y(x_0,y_0,u_0) \neq 1$, $H(x_0,y_0,u_0) \neq 0$, $r(x_0,u_0) \neq 0$ and $r_x(x_0,u_0) \neq G_x(x_0,y_0,u_0)$. Furthermore, suppose that $y = f(x,u)$ is a solution of the functional equation

$$y = F(x,y,u),$$

with $f(x_0,u_0) = y_0$. Then $f(x,u)$ has a local representation of the form

$$f(x,u) = g(x,u) + h(x,u)\left(1 - \frac{x}{\rho(u)}\right)^{3/2}, \tag{2.25}$$

where $g(x,u)$, $h(x,u)$ and $\rho(u)$ are analytic at (x_0,u_0) and satisfy $h(x_0,u_0) \neq 0$ and $\rho(u_0) = x_0$.

2.5. *Asymptotics for the number of planar graphs*

The system of equations for the generating functions $B(x,y)$ and $C(x,y)$, as described in Subsection 2.3, can be used to obtain asymptotic formulas for the numbers b_n and c_n of 2-connected and connected planar graphs [**19**]. Since we use some of the proof methods in the analysis of the root degree we include a sketch of the proof.

Lemma 2.2. *The generating functions $B(x)$ and $C(x)$ for planar graphs have finite radii of convergence ρ_B and ρ_C, respectively, and have local representations of the forms*

$$B(x) = g_2(x) + h_2(x)(1 - x/\rho_B)^{5/2}, \quad C(x) = g_4(x) + h_4(x)(1 - x/\rho_C)^{5/2},$$

with functions $g_2(x)$, $h_2(x)$ and $g_4(x)$, $h_4(x)$ that are non-zero and analytic at ρ_B and ρ_C, respectively, and $B(x)$ and $C(x)$ have analytic continuations to proper Δ-regions. In particular, if $t(y)$ is given by the equation

$$y = \frac{1+2t}{(1+3t)(1-t)} \exp\left(-\frac{t^2(1-t)(18+36t+5t^2)}{2(3+t)(1+2t)(1+3t)^2}\right) - 1, \tag{2.26}$$

then $\rho_B = (1+3t(1))(1-t(1))^3/(16t(1)^3)$ and $\rho_C = \rho_B e^{-B'(\rho_B,1)}$.

Consequently, there are constants $b, c > 0$ such that

$$b_n = b \cdot n^{-7/2} \rho_B^{-n} n!(1 + O(n^{-1})) \quad \text{and} \quad c_n = c \cdot n^{-7/2} \rho_C^{-n} n!(1 + O(n^{-1})).$$

Proof. The main part of the proof is to characterize the kind of singularities of the generating functions. The analytic continuation to proper Δ-regions is always straightforward to establish (see also [**1**]). First, since U and V satisfy the positive systems of equations (2.15) it follows (see [**9**, Theorem 2.33]) that U and V have a singular expansion of the form

$$U(x,z) = U_0(x) + U_1(x)Z + U_2(x)Z^2 + U_3(x)Z^3 + O(Z^4),$$
$$V(x,z) = V_0(x) + V_1(x)Z + V_2(x)Z^2 + V_3(x)Z^3 + O(Z^4),$$

where $Z = \sqrt{1 - z/\tau(x)}$ and $\tau(x)$ is given by

$$\tau(x) = \frac{1}{(4x^2(1+U_0(x)))^{2/3}}.$$

Moreover, $U_0(x)$ is the solution of the equation

$$x = \frac{(1+u)(3u-1)^3}{16u}.$$

The functions $U_j(x)$ and $V_j(x)$ are also analytic and can be explicitly given in terms of $U_0(x)$. With the help of these expansions, it follows that there is a cancelation of the coefficient of Z in the expansion of

$$\frac{(1+U)^2(1+V)^2}{(1+U+V)^3} = E_0 + E_2Z^2 + E_3Z^3 + O(Z^4).$$

Thus, using (2.14) we infer that $\bar{T}(x,z)$ can be represented as

$$\bar{T}(x,z) = T_0(x) + T_2(x)Z^2 + T_3(x)Z^3 + O(Z^4).$$

Next we use (2.12) to obtain

$$D = (1+y)\exp\left(\frac{xD^2}{1+xD} + \bar{T}(x,D)\right) - 1 =: F(x,y,D),$$

and assume that $y = 1$ or that y is very close to 1. Due to the singular structure of the right-hand side we can apply Theorem 2.1 and obtain a local expansion for $D = D(x,y)$ of the form

$$D(x,y) = D_0(y) + D_2(y)X^2 + D_3(y)X^3 + O(X^4), \tag{2.27}$$

where $X = \sqrt{1 - x/\rho_D(y)}$ for some function $\rho_D(y)$. In fact, we can be much more precise. Let $t = t(y)$ be defined by (2.26), that exists in a suitable neighborhood of $y = 1$. Then $\rho_D(y)$ is given by

$$\rho_D(y) = \frac{(1+3t(y))(1-t(y))^3}{16t(y)^3},$$

in particular $\rho_D = \rho_D(1) = 0.038191\ldots$. There are several ways to show that $D(x,y)$ extends analytically to a Δ-region. One way is to rewrite the system of equations (2.8), (2.10), (2.11) explicitly into one equation of the form $f(x,y) = F(x,y,f(x,y))$ for the function $f(x,y) = S(x,y) + H(x,y)$, where F has non-negative coefficients. It is easy to check that $F_f(x_0,1,f(x_0,1)) < 1$, which implies that $|F_f(x,y,f(x,y))| < 1$ for $|x| \leqslant x_0$ and $|y| \leqslant 1$. By the implicit function theorem, there is an analytic continuation to a proper Δ-region for $f(x,y) = S(x,y) + H(x,y)$, and consequently also for $D(x,y) = y + f(x,y) + y(e^{f(x,y)} - 1) + e^{f(x,y)} - 1 - f(x,y)$.

The representation (2.27) provides a local expansion for $\partial B(x,y)/\partial y$ of the form

$$\frac{\partial B(x,y)}{\partial y} = \bar{B}_0(y) + \bar{B}_2(y)X^2 + \bar{B}_3(y)X^3 + O(X^4) = g_1(x,y) + h_1(x,y)X^3,$$

with certain analytic functions $g_1(x,y)$ and $h_1(x,y)$. Hence, by integration (see [**9**]) or by using the representation (2.13), where one has to check that the coefficients of X and X^3 disappear,

$B(x,y)$ and consequently $\partial B(x,y)/\partial x$ have an expansions of the form

$$B(x,y) = g_2(x,y) + h_2(x,y)X^5 \quad \text{and} \quad \frac{\partial B(x,y)}{\partial x} = g_3(x,y) + h_3(x,y)X^3,$$

with certain analytic functions $g_2(x,y)$, $g_3(x,y)$ and $h_2(x,y)$, $h_3(x,y)$. Note that $\rho_B = \rho_D$ and the analytic continuation property of $D(x,y)$ implies a corresponding property for $B(x,y)$.

Finally, we have to solve (2.4). For simplicity, set $y = 1$. Since $\rho_D\, B''(\rho_D) \approx 0.0402624 < 1$, the singularity of the right-hand side induces the singular behavior of the solution $xC'(x)$. Actually, we just have to apply Theorem 2.1 and obtain a local expansion for $C'(x)$ of the form

$$C'(x) = g_3(x) + h_3(x)(1 - x/\rho_C)^{3/2}, \tag{2.28}$$

where $\rho_C = \rho_B e^{-B'(\rho_B)} = 0.0367284\ldots$, and consequently we obtain corresponding representations for

$$C(x) = g_4(x) + h_4(x)(1 - x/\rho_C)^{5/2}.$$

Note that the condition $\rho_D\, B''(\rho_D) \approx 0.0402624 < 1$ also ensures that $C'(x)$ (and also $C(x)$) has no other singularity for $|x| \leqslant \rho_C$ which implies that $C'(x)$ (and $C(x)$) has an analytic continuation to a Δ-region. Using these representations, the asymptotic expansion for b_n and c_n follow immediately by the Transfer Lemma of Flajolet and Odlyzko [**14**]. □

2.6. *Generating functions for the root degree*

In this section, we extend the results from Subsection 2.3 to incorporate the root degree into the generating functions. We start with connected planar graphs. Recall (2.3), which says that a vertex-derived connected planar graph C' can be decomposed as a set $\{B'_1, \ldots, B'_\ell\}$ of vertex-derived 2-connected graphs or vertex-derived edges, whose roots are identified into a single vertex, and where each other vertex is substituted by a vertex-rooted connected graph. Since the root degree of C' equals the sum of the root degrees of the $(B'_i)_{1\leqslant i\leqslant \ell}$, we obtain

$$C'(x,y,w) = \exp(B'(C^\bullet(x,y),y,w)). \tag{2.29}$$

It was shown in [**11**] that the remaining steps of the decomposition can be translated into corresponding relations for the generating functions that also take into account the root degree. We omit the lengthy details here, and just state the results. The generating functions for $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ satisfy the relations

$$\frac{\partial B'(x,y,w)}{\partial w} = xy\frac{1 + D(x,y,w)}{1 + yw}, \tag{2.30}$$

$$D(x,y,w) = (1 + yw)\exp\left(\frac{xD(x,y,w)D(x,y,1)}{1 + xD(x,y,1)} + \bar{T}\left(x, D(x,y,1), \frac{D(x,y,w)}{D(x,y,1)}\right)\right) - 1, \tag{2.31}$$

$$\bar{T}(x,y,w) = \frac{yw}{2}\left(\frac{1}{1 + wy} + \frac{1}{1 + xy} - 1 - \frac{(U+1)^2(-w_1(U,V,w) + (U - w + 1)\sqrt{w_2(U,V,w)})}{2w(Vw + U^2 + 2U + 1)(1 + U + V)^3}\right), \tag{2.32}$$

with polynomials $w_1 = w_1(U,V,w)$ and $w_2 = w_2(U,V,w)$ given by

$$\begin{aligned} w_1 &= -UVw^2 + w(1 + 4V + 3UV^2 + 5V^2 + U^2 + 2U + 2V^3 + 3U^2V + 7UV) \\ &\quad + (U+1)^2(U + 2V + 1 + V^2), \\ w_2 &= U^2V^2w^2 - 2wUV(2U^2V + 6UV + 2V^3 + 3UV^2 + 5V^2 + U^2 + 2U \\ &\quad + 4V + 1) + (U+1)^2(U + 2V + 1 + V^2)^2. \end{aligned}$$

Again it is possible to integrate $\partial B'(x,y,w)/\partial w$ and one obtains the following expression (see [11]):

$$B'(x,y,w) = x\left(D - \frac{xED}{1+xE}\left(1+\frac{D}{2}\right)\right) - x(1+D)\bar{T}(x,E,D/E) + x\int_0^D \bar{T}(x,E,t/E)\,dt, \tag{2.33}$$

where for simplicity we let $D = D(x,y,w)$ and $E = E(x,y) = D(x,y,1)$. The remaining integral is equal to the following lengthy expression (note that our $T(x,E,t/E)$ equals $T^\bullet(x,E,t/E)/tx^2$ in [11]):

$$\begin{aligned}
&\int_0^D \bar{T}(x,E,t/E)\,dt \\
&\quad = -\frac{xED^2 - 2D - 2xED + (2+2xE)\log(1+D)}{4(1+xE)} - \frac{uv}{2x(1+u+v)^3} \\
&\quad\times\Bigg(\frac{D(2u^3+(6v+6)u^2+(6v^2-vD/E+14v+6)u+4v^3+10v^2+8v+2)}{4v(v+1)^2E} \\
&\quad+\frac{(1+u)(1+u+2v+v^2)(2u^3+(4v+5)u^2+(3v^2+8v+4)u+2v^3+5v^2+4v+1)}{4uv^2(v+1)^2} \\
&\quad-\frac{\sqrt{Q}(2u^3+(4v+5)u^2+(3v^2-vD/E+8v+4)u+5v^2+2v^3+4v+1)}{4uv^2(v+1)^2} \\
&\quad+\frac{(1+u)^2(1+u+v)^3\log(Q_1)}{2v^2(1+v)^2}+\frac{(u^3+2u^2+u-2v^3-4v^2-2v)(1+u+v)^3\log(Q_2)}{2v^2(1+v)^2u}\Bigg),
\end{aligned}$$

where the expressions Q, Q_1 and Q_2 are given by

$$\begin{aligned}
Q &= u^2v^2D^2/E^2 - 2uvD/E(u^2(2v+1)+u(3v^2+6v+2)+2v^3+5v^2+4v+1) \\
&\quad + (1+u)^2(u+(v+1)^2)^2, \\
Q_1 &= \frac{1}{2(Dv/E+(u+1)^2)^2(v+1)(u^2+u(v+2)+(v+1)^2)} \\
&\quad\times(-uvD/E(u^2+u(v+2)+2v^2+3v+1)+(u+1)(u+v+1)\sqrt{Q} \\
&\qquad+(u+1)^2(2u^2(v+1)+u(v^2+3v+2)+v^3+3v^2+3v+1)) \\
Q_2 &= \frac{-Duv/E+u^2(2v-1)+u(3v^2+6v+2)+2v^3+5v^2+4v+1-\sqrt{Q}}{2v(u^2+u(v+2)+(v+1)^2)},
\end{aligned}$$

and u and v abbreviate $u = U(x,E)$ and $v = V(x,E)$.

2.7. *A Boltzmann sampler for networks*

In this section, we describe a Boltzmann sampler for the class of planar networks, which plays a central role in our analysis; see Section 4. This sampler was already developed in [16] for general classes that can be decomposed into 3-connected components (see also [17] for the case of planar graphs). We repeat here the exposition, tailored to our needs, as several details are important in our proofs.

We start with the Boltzmann sampler for the class $\mathcal{D}$ of all networks. Recall (2.6), which says that $\mathcal{D}$ is the disjoint union of the classes e (single edge), $\mathcal{S}$ (series networks), $\mathcal{P}$ (parallel networks) and $\mathcal{H}$ (core networks). By applying the rules from Subsection 2.2, a Boltzmann sampler for $\mathcal{D}$ calls a sampler for a subclass with a probability proportional to the value of the generating function of this subclass. More precisely, we say that a variable X is *network-distributed* with parameters x and y, $X \sim \texttt{Net}(x,y)$, if its domain is the set of symbols $\Omega_{\texttt{Net}} = \{e,S,P,H\}$ and for any $s \in \Omega_{\texttt{Net}}$ it holds $\Pr[X=s] = s(x,y)/D(x,y)$. Then the sampler $\Gamma D(x,y)$ with parameters x,y for $\mathcal{D}$ can be described concisely as follows, where $\Gamma e, \Gamma S, \Gamma P$ and ΓH are (yet to be defined) Boltzmann samplers for the classes $e, \mathcal{S}, \mathcal{P}$ and $\mathcal{H}$.

$\Gamma D(x,y)$: $s \leftarrow \texttt{Net}(x,y)$
return $\Gamma s(x,y)$

Next we describe the sampler for $\mathcal{S}$. The combinatorial relation in (2.8) implies that $\mathcal{S} = \mathcal{A} \times \mathcal{X} \times \mathcal{D}$, where $\mathcal{A} = e + \mathcal{P} + \mathcal{H}$. Again using the rules from Subsection 2.2, we infer that a Boltzmann sampler for $\mathcal{S}$ proceeds in the following way. It first samples a network from $\mathcal{A}$, by making a 'three-way' Bernulli choice among e, $\mathcal{P}$ and $\mathcal{H}$ with the appropriate probabilities, and generates a Boltzmann distributed object D_1 from the chosen class. Then, it generates a network D_2 that is Boltzmann distributed from $\mathcal{D}$. Finally, it creates and returns a network (D_1, D_2) such that the right pole of D_1 is identified with the left pole of D_2, and in which the labels are distributed randomly. Formally, we say that a variable X is *series-distributed* with parameters x and y, $X \sim \texttt{Ser}(x,y)$, if its domain is the set of symbols $\Omega_{\texttt{Ser}} = \{e, P, H\}$ and for any $s \in \Omega_{\texttt{Ser}}$ it holds $\Pr[X = s] = s(x,y)/S(x,y)$. Then $\Gamma S(x,y)$ can be described concisely as follows:

$\Gamma S(x,y)$: $s \leftarrow \texttt{Ser}(x,y)$
$D_1 \leftarrow \Gamma s(x,y)$
$D_2 \leftarrow \Gamma D(x,y)$
return (D_1, D_2), relabeling randomly its non-pole vertices

We proceed with the class $\mathcal{P}$. The combinatorial relation (2.9) guarantees that $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, where $\mathcal{P}_1 = e \times \mathrm{Set}_{\geqslant 1}(\mathcal{S} + \mathcal{H})$ and $\mathcal{P}_2 = \mathrm{Set}_{\geqslant 2}(\mathcal{S} + \mathcal{H})$. Together with the rules for Boltzmann samplers from Subsection 2.2 for disjoint union and set, this implies that $\Gamma P(x,y)$ first makes a Bernulli choice between $\mathcal{P}_1$ and $\mathcal{P}_2$, and then samples a set (with a given lower bound on the number of elements) of networks from $\mathcal{S}$ or $\mathcal{H}$ according to the Boltzmann distribution.

Let us introduce some notation before, we describe formally the sampler. We say that a variable X is *parallel-distributed* with parameters x and y, and write $X \sim \texttt{Par}(x,y)$, if

$$X \sim 1 + \texttt{Be}\left(\frac{e^{S(x,y)+H(x,y)} - 1 - S(x,y) - H(x,y)}{P(x,y)}\right).$$

In words, X 'distinguishes' between the two possibilities in the definition a parallel network. We say that a variable is *sh-distributed* with parameters x and y, $X \sim \texttt{sh}(x,y)$, if its domain is the set of symbols $\Omega_{\texttt{sh}} = \{S, H\}$ and for $s \in \Omega_{\texttt{sh}}$

$$\mathbb{P}(X = s) = \frac{s(x,y)}{S(x,y) + H(x,y)}.$$

Using $\texttt{Po}_{\geqslant p}(\lambda)$ to denote a Poisson distributed random variable with parameter λ conditioned on being at least p, the Boltzmann sampler ΓP works as follows:

$\Gamma P(x,y)$: $p \leftarrow \texttt{Par}(x,y)$
$k \leftarrow \texttt{Po}_{\geqslant p}(S(x,y) + H(x,y))$
for $i = 1 \ldots k$
  $b_i \leftarrow \texttt{sh}(x,y)$
  $p_i \leftarrow \Gamma b_i(x,y)$
construct a network P by identifying the left and right poles of $p_1, \ldots, p_k$
relabel randomly the non-pole vertices of P
if $p = 1$ **then return** P, where the poles are joined by an edge
else return P

Finally, we describe the sampler for $\mathcal{H}$. Recall (2.11), which guarantees that an $\mathcal{H}$-network is obtained by substituting the edges of a network from $\bar{\mathcal{T}}$ by graphs from $\mathcal{D}$. Here we assume that we have an auxiliary sampler $\Gamma\bar{T}(x,y)$, which samples graphs from $\bar{\mathcal{T}}$ according to the Boltzmann distribution. Then the sampler for $\mathcal{H}$ can be described as follows:

$\Gamma H(x,y)$: $T \leftarrow \Gamma\bar{T}(x, D(x,y))$
foreach edge e of T
$\gamma_e \leftarrow \Gamma D(x,y)$
replace every e in T by γ_e
return T, relabeling randomly its non-pole vertices

This completes the description of the samplers. The next lemma was shown in [**16**], and it can be proved in the present case directly by using the asymptotic enumeration results for 2-connected planar graphs, as obtained by Bender, Gao and Wormald [**1**], or by using Lemma 2.2. The proof is included for completeness.

Lemma 2.3. *Let $x, y \geqslant 0$ be such that $D(x,y) < \infty$. Then $\Gamma D(x,y)$ is a Boltzmann sampler with parameters x and y for $\mathcal{D}$. Moreover,*

$$\Pr[\Gamma D(\rho_D, 1) \in \mathcal{D}_n] = \Theta(n^{-5/2}),$$

where $\rho_D = \rho_B$ denotes the singularity of $D(x,1)$ and ρ_B is given in Lemma 2.2.

Proof. Recall equation (2.27), which says that

$$D(x) = D_0 + D_2(1 - x/\rho_D) + D_3(1 - x/\rho_D)^{3/2} + O((1 - x/\rho_D)^2).$$

Moreover, the discussion after (2.27) guarantees that $D(x)$ is analytic in an appropriate Δ-domain. Thus, the Tranfer Lemma applies, implying that

$$|\mathcal{D}_n| = n![x^n]D(x) = \Theta(1)\cdot n^{-5/2}\rho_D^{-n} n!.$$

The definition of the Boltzmann model then implies that

$$\Pr[\Gamma D(\rho_D, 1) \in \mathcal{D}_n] = |\mathcal{D}_n| \cdot \frac{\rho_D^n}{n!\, D(\rho_D, 1)} = \Theta(n^{-5/2}).$$

In other words, if we choose $(x, y) = (\rho_B, 1)$, then $\Gamma D(x,y)$ has a polynomially small probability of generating a network of a given size n. This important fact will be used in Section 4.

3. *The upper bound*

In this section, we prove the upper bound in Theorem 1.1. The main tools are Lemmas 3.1 and 3.2, where we determine appropriate singular expansions for the generating functions of 2-connected and connected graphs, respectively. In Subsection 3.1, we describe how the upper bound follows from these lemmas. The remaining subsections are devoted to the proof of these lemmas.

3.1. *Generating functions and the first moment method*

In order to obtain an upper bound for the distribution of the maximum degree, we use the first moment method. Let $X_{n,k}$ denote the number of vertices of degree k in a 2-connected random planar graph B_n with n vertices and let $Y_{n,k} = \sum_{\ell > k} X_{n,\ell}$ denote the number of vertices of degree larger than k. If we denote by $\Delta(G)$ the maximum degree of a vertex in a graph G, then obviously we have

$$\Delta(B_n) > k \iff Y_{n,k} > 0,$$

and consequently

$$\Pr[\Delta(B_n) > k] = \Pr[Y_{n,k} > 0] \leqslant \mathbb{E}Y_{n,k}.$$

Let $d_{n,k}$ denote the probability that the root degree (in a 2-connected graph of size n) equals k. Then $\mathbb{E}\, X_{n,k} = d_{n,k}\, n$. Hence, it is sufficient to provide upper bounds for

$$d_{n,k} = \frac{[x^{n-1}w^k]\, B'(x,1,w)}{[x^{n-1}]\, B'(x)}.$$

The asymptotic expansion of $[x^n]\, B'(x) \sim c \cdot n^{-5/2}\rho_B^{-n}$ is known, where $c > 0$ and $\rho_B = 0.03672841\ldots$, see [**1**, **19**] or Subsection 2.5. Consequently, we just need upper bounds for $[x^n w^k]B'(x,1,w)$. Suppose that $w_0 > 0$ is chosen in a way that $B'(x,1,w_0)$ is a convergent power series. Then, the non-negativity of the coefficients of B implies that

$$[x^n w^k]\, B'(x,1,w) \leqslant w_0^{-k}[x^n]\, B'(x,1,w_0).$$

Actually, it will turn out that we can choose $w_0 > 1$ in an 'optimal way' so that $B'(x,1,w_0)$ has the same radius of convergence ρ_B as $B'(x)$ and also the same kind of singularity.

LEMMA 3.1. *Let $t(y)$ be given by (2.26) and set*

$$w_0 = \frac{1}{1-t(1)} \exp\left(\frac{t(1)(t(1)-1)(t(1)+6)}{6t(1)^2+20t(1)+6}\right) - 1 \approx 1.48488989. \tag{3.1}$$

Then $B'(x,1,w_0)$ has a local representation of the form

$$B'(x,1,w_0) = \bar{g}(x) + \bar{h}(x)(1-x/\rho_B)^{3/2},$$

with functions $\bar{g}(x)$, $\bar{h}(x)$ that are non-zero and analytic at ρ_B. Furthermore,

$$[x^n]\, B'(x,1,w_0) \sim \bar{c} \cdot n^{-5/2}\rho_B^{-n},$$

for some constant $\bar{c} > 0$.

The proof of this lemma is spread over the next sections. We recall that w_0 is the radius of convergence of the generating function $\sum_{k\geqslant 1} d_k w^k$ of the limiting degree distribution of 2-connected planar graphs (see [**11**]). Summing up, we have

$$\mathbb{E}\, X_{n,k} = O(nw_0^{-k}), \tag{3.2}$$

and consequently

$$\Pr[\Delta(B_n) > k] = O(nw_0^{-k}).$$

Of course, this estimate provides the upper bound in Theorem 1.1, equation (1.1), for random 2-connected planar graphs. The proof of the upper bound for connected graphs is very similar, and follows from the analogous counting estimate provided by the next lemma.

LEMMA 3.2. *Let w_0 be as in Lemma 3.1. Then $C'(x,1,w_0)$ has a local representation*

$$C'(x,1,w_0) = \bar{g}_2(x) + \bar{h}_2(x)(1-x/\rho_C)^{3/2},$$

with functions $\bar{g}_2(x)$, $\bar{h}_2(x)$ that are non-zero and analytic at ρ_C. Furthermore,

$$[x^n]\, C'(x,1,w_0) \sim \bar{c}_2 \cdot n^{-5/2}\rho_C^{-n},$$

for some constant $\bar{c}_2 > 0$.

It remains to show the claimed upper bound for random (not necessarily) connected planar graphs in Theorem 1.1. To this end, we use the following property, see [**19**, **24**]. Let P_n be a random planar graph with n vertices, and let ω_n be an arbitrary slowly growing function. Let $c(P_n)$ denote the size of the largest connected component in P_n. Then, w.h.p., $c(P_n) \geqslant n - \omega_n$. Then, conditional on any specific value of $c(P_n)$ within the given bounds, note that any connected planar graph with $c(P_n)$ vertices is equally likely to be the largest component of P_n. The upper bound for P_n in Theorem 1.1 follows immediately from Lemma 3.2, since ω_n is arbitrary.

3.2. *Singular functional equations*

Toward the proofs of Lemmas 3.1 and 3.2, we first have a closer look at equation (2.31). If we set $w = 1$, then it reduces to an equation for $D(x, y, 1)$, which is equivalent to (2.12). In order to avoid conflicts with the notation, we set $E(x, y) := D(x, y, 1)$. From (2.27), we know the analytic behavior of $E(x, y)$ around its dominant singularity:

$$E(x, y) = E_0(y) + E_2(y)X^2 + E_3(y)X^3 + O(X^4) \quad \text{where } X = \sqrt{1 - x/\rho_D(y)}. \tag{3.3}$$

Recall that the coefficient of the square-root term X vanishes. Since we are not interested in the number of edges, we will set $y = 1$ in (most of) the following calculations. A crucial step in our analysis is the discussion of the relation in (2.31). First, we rewrite it to

$$D + 1 = \exp(G(x, D, w, E, U, V) + H(x, D, E, U, V)\sqrt{J(D, E, U, V)}), \tag{3.4}$$

where

$$\begin{aligned} G &= \log(1 + w) + \frac{xDE}{1 + xE} \\ &\quad + \frac{D}{2}\left(\frac{1}{1 + D} + \frac{1}{1 + xE} - 1 + \frac{(U + 1)^2 w_1(U, V, D/E)}{2D/E(VD/E + U^2 + 2U + 1)(1 + U + V)^3}\right), \\ H &= -\frac{(U + 1)^2 D(U - D/E + 1)}{4D/E(VD/E + U^2 + 2U + 1)(1 + U + V)^3}, \end{aligned}$$

and $J = w_2(U, V, D/E)$, where U and V abbreviate $U = U(x, E(x, 1))$ and $V = V(x, E(x, 1))$. Note that this equation for $D(x, y, w)$ has no combinatorial meaning, since the right-hand side is not a power series with non-negative coefficients. Nevertheless, this equation is appropriate for the further analysis.

In the sequel, we will consider first E, U, V as *new variables*, in particular when we apply Lemma 3.4. Finally, we will substitute them by $E = E(x, 1)$, $U = U(x, E(x, 1))$, $V = V(x, E(x, 1))$. Before we come to that let us identify the relevant singularities of D. Set

$$\begin{aligned} t_0 &= t(1) \approx 0.626371 \quad \text{where } t(\cdot) \text{ is defined in (2.26)}, \\ x_0 &= \rho_D(1) = \frac{(3t_0 + 1)(1 - t_0)^3}{16t_0^3} \approx 0.038191, \\ w_0 &= \frac{1}{1 - t_0} \exp\left(\frac{t_0(t_0 - 1)(t_0 + 6)}{6t_0^2 + 20t_0 + 6}\right) - 1 \approx 1.48488989, \end{aligned}$$

and further

$$D_0 = D(x_0, 1, w_0) = \frac{t_0}{1 - t_0} \approx 1.676, \quad E_0 = E(x_0, 1) = \frac{3t_0^2}{(1 - t_0)(3t_0 + 1)} \approx 1.094$$

and

$$U_0 = U(x_0, E_0) = \frac{1}{3t_0} \approx 0.532, \quad V_0 = V(x_0, E_0) = \frac{1 + 3t_0}{3(1 - t_0)} \approx 2.568.$$

The following lemma justifies the choice of the specific point (x_0, w_0).

LEMMA 3.3. *The function $D(x,1,w)$ is analytic for $|x| < x_0$ and $|w| < w_0$. Moreover, x_0 is a singularity of $D(x,1,w_0)$ and w_0 is a singularity for $D(x_0,1,w)$.*

Proof. We already know that $D(x,1,1) = E(x,1)$ is analytic for $|x| < x_0$. We claim that the solution D of (3.4) at $x = x_0$ as a function in w, that is, $w \mapsto D(x_0,w,E_0,U_0,V_0)$, is analytic for $|w| < w_0$. This can be seen by replacing (3.4) by a positive system of four equations for D,P,S,H (similar to that described in Subsection 2.3 for the counting problem) and, for example, by checking that the Jacobian of the right-hand side has spectral radius equal to 1 at $w = w_0$. (Since it is a positive and strongly connected system, the Jacobian of the right-hand side is strictly increasing as a function in $w \in [0,w_0]$ so that the system is regular for $w \in [0,w_0)$ and, thus, for all $|w| < w_0$.) Again by positivity, it finally follows that the function $D(x,1,w) = D(x,w,E(x,1),U(x,E(x,1)),V(x,E(x,1)))$ is analytic for $|x| < x_0$ and $|w| < w_0$. □

The point (x_0,w_0) has also another crucial property that will allow us to extract the coefficients of $D(x,1,w_0)$ and $D(x_0,1,w)$. A simple computation shows that

$$H(x_0,D_0,E_0,U_0,V_0) = J(D_0,E_0,U_0,V_0) = 0;$$

this can be verified, for example, by writing $H(x_0,D_0,w_0,E_0,U_0,V_0)$ and $J(D_0,E_0,U_0,V_0)$ in terms of t_0. In order to further understand the behavior of the function D defined by (3.4), let us start with the following auxiliary statement.

LEMMA 3.4. *Let $\mathbf{v} = (v_1,\ldots,v_d)$ be a d-dimensional complex vector and let $y = f(\mathbf{v})$ be a function with $f(\mathbf{v}_0) = y_0$ that satisfies a functional equation*

$$R(y,\mathbf{v})^2 + S(y,\mathbf{v}) = 0, \tag{3.5}$$

where $R(y,\mathbf{v})$ and $S(y,\mathbf{v})$ are analytic functions at $(y_0,\mathbf{v}_0)$ such that

$$R(y_0,\mathbf{v}_0) = S(y_0,\mathbf{v}_0) = 0,$$

and, in addition, all the partial derivatives of S up to order 2 are zero at $(y_0,\mathbf{v}_0)$, and $R_y(y_0,\mathbf{v}_0) \neq 0$. Then, $f(\mathbf{v})$ has a local representation of the form

$$f(\mathbf{v}) = P(\mathbf{v}) \pm \sqrt{Q(\mathbf{v})}, \tag{3.6}$$

where P and Q are analytic at $\mathbf{v}_0$ with $P(\mathbf{v}_0) = Q(\mathbf{v}_0) = 0$, and Q and all its partial derivatives up to order 2 are zero at $\mathbf{v}_0$. Furthermore, the evaluations of the partial derivatives Q_{xxx}, Q_{xxw} and Q_{xwz} at $(\mathbf{v}_0)$ for any variables x,w,z of $\mathbf{v}$ are

$$\begin{aligned}
Q_{xxx} &= \frac{R_x^3 S_{yyy} - 3R_x^2 R_y S_{xyy} + 3R_x R_y^2 S_{xxy} - R_y^3 S_{xxx}}{R_y^5},\\
Q_{xxw} &= \frac{1}{R_y^5}(R_x^2 R_w S_{yyy} - 2R_x R_w R_y S_{xyy} + 2R_x R_y^2 S_{xwy}\\
&\quad - R_x^2 R_y S_{wyy} + R_w R_y^2 S_{xxy} - R_y^3 S_{xxw}),\\
Q_{xwz} &= \frac{1}{R_y^5}(R_x R_w R_z S_{yyy} - R_w R_z R_y S_{xyy} - R_x R_z R_y S_{wyy}\\
&\quad - R_x R_w R_y S_{zyy} + R_w R_y^2 S_{xzy} + R_z R_y^2 S_{xwy} + R_x R_y^2 S_{wzy} - R_y^3 S_{xwz}).
\end{aligned}$$

Proof. Set

$$F(y,\mathbf{v}) := R(y,\mathbf{v})^2 + S(y,\mathbf{v}). \tag{3.7}$$

By the assumptions, we have

$$F(y_0,\mathbf{v}_0)=0,\quad F_y(y_0,\mathbf{v}_0)=0,\quad F_{yy}(y_0,\mathbf{v}_0)=2R_y(y_0,\mathbf{v}_0)^2\neq 0.$$

Hence, by the Weierstrass preparation theorem (see [**15**, Appendix B]), there exist analytic functions $p=p(\mathbf{v})$, $q=q(\mathbf{v})$ and $K=K(y,\mathbf{v})$ with $p(\mathbf{v}_0)=q(\mathbf{v}_0)=0$ and $K(y_0,\mathbf{v}_0)\neq 0$ such that

$$F(y,\mathbf{v})=K(y,\mathbf{v})((y-y_0)^2+p(\mathbf{v})(y-y_0)+q(\mathbf{v})). \tag{3.8}$$

Consequently, the original equation (3.5) is equivalent to

$$(y-y_0)^2+p(\mathbf{v})(y-y_0)+q(\mathbf{v})=0,$$

and, thus, we obtain (3.6) with

$$P(\mathbf{v})=y_0-\frac{p(\mathbf{v})}{2}\quad\text{and}\quad Q(\mathbf{v})=\frac{p(\mathbf{v})^2}{4}-q(\mathbf{v}).$$

We now compute the partial derivatives of $Q(\mathbf{v})$. The basic idea is to differentiate both equations (3.7) and (3.8), and to rewrite the partial derivatives of $p(\mathbf{v})$ and $q(\mathbf{v})$ in terms of those of $R(y,\mathbf{v})$ and $S(y,\mathbf{v})$. In what follows, all functions are evaluated at $(y_0,\mathbf{v}_0)$ or $(\mathbf{v}_0)$, and the symbols x,w,z denote any three variables of $\mathbf{v}$.

First observe that, due to equation (3.7) and the fact that $R=S=0$ and the condition on S, the first partial derivatives of F vanish,

$$F_y=0,\quad F_x=0, \tag{3.9}$$

and that the second derivatives of $F(y,\mathbf{v})$ are given by

$$\begin{aligned} F_{yy}&=2R_y^2, & F_{xy}&=2R_xR_y,\\ F_{xx}&=2R_x^2, & F_{xw}&=2R_xR_w. \end{aligned} \tag{3.10}$$

Next, by using equation (3.8) and $p=q=0$, we obtain that

$$\begin{aligned} F_y&=0, & F_x&=Kq_x,\\ F_{yy}&=2K, & F_{xy}&=K_yq_x+Kp_x,\\ F_{xx}&=2K_xq_x+Kq_{xx}, & F_{xw}&=K_xq_w+K_wq_x+Kq_{xw}. \end{aligned} \tag{3.11}$$

Hence, from equations (3.9)–(3.11), we derive that $K=R_y^2$, and that

$$q_x=0,\quad p_x=2R_x/R_y,\quad q_{xx}=2R_x^2/R_y^2,\quad q_{xw}=2R_xR_w/R_y^2.$$

Consequently,

$$Q_x=\frac{pp_x}{2}-q_x=0,\quad Q_{xx}=\frac{pp_{xx}+p_x^2}{2}-q_{xx}=2\frac{R_x^2}{R_y^2}-2\frac{R_x^2}{R_y^2}=0$$

and

$$Q_{xw}=\frac{pp_{xw}+p_xp_w}{2}-q_{xw}=2\frac{R_xR_w}{R_y^2}-2\frac{R_xR_w}{R_y^2}=0,$$

as claimed. Finally, it remains to obtain the values of Q_{xxx}, Q_{xxw} and Q_{xwz} in terms of the partial derivatives of H and J. To compute $Q_{xxx}=3p_xp_{xx}/2-q_{xxx}$ observe that, on the one hand, by differentiating (3.8) we obtain

$$\begin{aligned} F_{xxx}&=3K_xq_{xx}+Kq_{xxx},\\ F_{xxy}&=K_yq_{xx}+2K_xp_x+Kp_{xx},\\ F_{xyy}&=2K_yp_x+2K_x,\\ F_{yyy}&=6K_y, \end{aligned}$$

and that, on the other hand, from equation (3.7) we obtain

$$\begin{aligned}F_{xxx} &= 6R_x R_{xx} + S_{xxx},\\ F_{xxy} &= 4R_x R_{xy} + 2R_y R_{xx} + S_{xxy},\\ F_{xyy} &= 4R_y R_{xy} + 2R_x R_{yy} + S_{xyy},\\ F_{yyy} &= 6R_y R_{yy} + S_{yyy}.\end{aligned}$$

It is just a matter of computation to derive the claimed expression for Q_{xxx}. In a completely analogous way, and by considering the other third derivatives of F, one obtains the expressions for Q_{xxw} and Q_{xwz}. □

Lemma 3.4 can be used directly to study the behavior of the generating function D. In particular, the following corollary applies to a more general class of functions.

COROLLARY 3.5. *Let $\mathbf{v} = (v_1, \ldots, v_d)$ be a d-dimensional complex vector and let $y = f(\mathbf{v})$ be a function with $f(\mathbf{v}_0) = y_0$ that satisfies a functional equation*

$$y = \exp(G(y, \mathbf{v}) + H(y, \mathbf{v})\sqrt{J(y, \mathbf{v})}), \tag{3.12}$$

where G, H and J are analytic functions at $(y_0, \mathbf{v}_0)$ such that

$$H(y_0, \mathbf{v}_0) = J(y_0, \mathbf{v}_0) = 0 \quad \text{and} \quad y_0 G_y(y_0, \mathbf{v}_0) \neq 1.$$

Then, $f(\mathbf{v})$ has a local representation of the same form as in Lemma 3.4, that is,

$$f(\mathbf{v}) = P(\mathbf{v}) \pm \sqrt{Q(\mathbf{v})},$$

where P and Q are analytic at $\mathbf{v}_0$, $P(\mathbf{v}_0) = y_0$ and Q and all its partial derivatives up to order 2 are zero at $\mathbf{v}_0$. The third derivatives can be computed explicitly. For example, for any variable x in $\mathbf{v}$

$$Q_{xxx}(\mathbf{v}_0) = \frac{6(y_0 H_y G_x - H_x(y_0 G_y - 1))^2 (y_0 J_y G_x - J_x(y_0 G_y - 1)) y_0^2}{(y_0 G_y - 1)^5}.$$

Proof. We set

$$R(y, \mathbf{v}) := \log y - G(y, \mathbf{v}), \quad S(y, \mathbf{v}) := -H(y, \mathbf{v})^2 J(y, \mathbf{v}),$$

and apply Lemma 3.4. Note that the condition $y_0 G_y(y_0, \mathbf{v}_0) \neq 1$ guarantees that $R_y(y_0, \mathbf{v}_0) \neq 0$. Of course, by rewriting the derivatives of R and S in terms of the derivatives of G, H and J we obtain the proposed representation for Q_{xxx}. □

Next we apply Corollary 3.5 to equation (3.4) with $y = D + 1$ and $\mathbf{v} = (x, w, E, U, V)$. Indeed, note that

$$(D+1)G_D = -\frac{(-1+t_0)(5\,{t_0}^2 + 16\,t_0 + 6)}{2(3\,t_0 + 1)(t_0 + 3)} \neq 1,$$

and the other conditions are verified easily. Thus, we obtain a representation of D as a function of x, w, E, U, V of the form

$$D = P(x, w, E, U, V) \pm \sqrt{Q(x, w, E, U, V)}, \tag{3.13}$$

where Q and all partial derivatives of Q up to order 2 vanish. In particular, if we substitute $E = E(x, 1)$, etc. we see that $Q(x, w, E(x), U(x, E(x)), V(x, E(x)))$ can be represented as

$$\begin{aligned}&Q(x, w, E(x,1), U(x, E(x,1)), V(x, E(x,1)))\\ &\quad = X^6 h_1(X) + X^4 W h_2(X, W) + X^2 W^2 h_3(W) + W^3 h_4(W),\end{aligned} \tag{3.14}$$

where $W = 1 - w/w_0$, $X = \sqrt{1 - x/x_0}$ and $h_1, \ldots, h_4$ are proper convergent power series. Note that we have to be careful in the computation of the expansion of $U(x, E(x,1))$ and $V(x, E(x,1))$. Recall that $Z = \sqrt{1 - z/\tau(x)}$, so that we have to substitute $z = E(x,1)$ and that $\tau(x_0) = E_0$. Hence, Z can be represented as

$$Z = c_1 X + c_2 X^2 + c_3 X^3 + \cdots,$$

for certain (computable) constants c_j. Thus, $U(x, E(x,1))$ and $V(x, E(x,1))$ are given by expansions of the form $\tilde{U}_0 + \tilde{U}_1 X + \cdots$ and $\tilde{V}_0 + \tilde{V}_1 X + \cdots$, respectively, where $\tilde{U}_1 \neq 0$ and $\tilde{V}_1 \neq 0$. However, similarly to the fact that the Z-term cancels in $(1+U)^2(1+V)^2/(1+U+V)^3$ (see the Proof of Lemma 2.2) the X-term corresponding to $U(x, E(x,1))$ and $V(x, E(x,1))$ cancels in the expansion of $Q(x, w, E(x,1), U(x, E(x,1)), V(x, E(x,1)))$ so that we actually obtain the form (3.14). Finally, a tedious computation provides

$$\begin{aligned} h_1(0) &\approx 0.009976458560, \\ h_2(0,0) &\approx -0.03944762502, \\ h_3(0) &= 0, \\ h_4(0) &\approx 0.09137050078. \end{aligned}$$

It should be remarked that $h_1(0) > 0$, $h_4(0) > 0$ and $h_3(0) = 0$. This shows that $D(x, 1, w_0)$ has a singular behavior of the form

$$D(x, 1, w_0) = \bar{g}(x) \pm \bar{h}(x) X^3, \tag{3.15}$$

with $X = \sqrt{1 - x/x_0}$ and $\bar{h}(x_0) > 0$. Moreover, since by Lemma 3.3 x_0 is the smallest singularity of $D(x, 1, w_0)$ we infer that have to choose the + sign. Finally, it is not difficult to show that $D(x, 1, w_0)$ has an analytic continuation to a Δ-region. For this purpose, we can proceed similarly as for the function $D(x, y) = D(x, y, 1)$. For technical reasons, it is preferable to work with $f(x, y, w) = S(x, y, w) + H(x, y, w)$ that satisfies a functional equation of the form $f = F(x, y, w, f)$, where F has non-negative coefficients. The point $(x_0, 1, w_0, f(x_0, 1, w_0))$ has the property that $F_f(x_0, 1, w_0, f(x_0, 1, w_0)) = 1$. So, we have $|F_f(x, 1, w_0, f(x, 1, w_0))| < 1$ for $|x| \leqslant x_0$ and $x \neq x_0$, and the implicit function theorem implies that $f(x, 1, w_0)$ can be continued analytically to a Δ-region. Consequently, the same holds for $D(x, 1, w_0)$.

REMARK 3.6. We want to note that in the expansion (3.14) we actually have $h_3(W) = 0$ which can be shown without doing any numerical calculations. If $h_3 \neq 0$, then it would follow that the dominant singularity of $D(x, 1, w)$ would have a singular behavior of the form $XW^{\ell - 1/2}$ for some integer $\ell \geqslant 0$ which would lead to an asymptotic leading term of the coefficient of $x^n w^k$ of the square-root part of the form $c\, x_0^{-n} w_0^{-k} n^{-3/2} k^{-\ell - 1/2}$. Similarly, if $P(x, w, E(x,1), U(x, E(x,1)), V(x, E(x,1)))$ has a factor X in its expansion, then the dominant behavior in n would be of the form $x_0^{-n} n^{-3/2}$. In both cases, this contradicts the asymptotic expansion for the coefficient $[x^n]\, D(x, 1, 1) \sim c_1\, x_0^{-n} n^{-5/2}$.

3.3. *Proof of Lemma* 3.1

With all the above facts at hand, it is now not very difficult to provide the proof of Lemma 3.1 We use the explicit representation (2.33) and apply the local expansion (3.3) for $E(x,1)$ and (3.15) for $D(x, 1, w_0)$ (and also those of $u = U(x, E(x,1))$ and $v = V(x, E(x,1))$). This leads directly to a singular representation of $B'(x, 1, w_0)$ of the type

$$B'(x, 1, w_0) = \bar{g}_1(x) + \bar{h}_1(x) X^3. \tag{3.16}$$

Note that we definitely have $\bar{h}_1(x_0) \neq 0$ and hence $\bar{h}_1(x_0) > 0$. Namely, if $h_1(x_0) = 0$, then we would have $[x^n]\, B'(x, 1, w_0) = O(x_0^{-n} n^{-7/2})$, which is impossible. Thus, by applying the transfer lemma of Flajolet and Odlyzko [**14**] we obtain

$$[x^n]\, B'(x, 1, w_0) \sim c_1 x_0^{-n} n^{-5/2},$$

which completes the proof of Lemma 3.1. (Note that it is easy to establish analytic continuation of $B'(x, 1, w_0)$ to a proper Δ-region. Indeed, as mentioned above this holds for $D(x, 1, w_0)$, and of course for $E(x, 1) = D(x, 1, 1)$ and $u = U(x, E(x))$ and $v = V(x, E(x))$, too. Hence, the representation (2.33) transfers the analytic continuation property to $B'(x, 1, w_0)$.)

3.4. *Proof of Lemma* 3.2

By using (2.29) and the local expansions (2.28) and (3.16), it follows that

$$C^\bullet(x, 1, w_0) = \bar{g}_2(x) + \bar{h}_2(x)(1 - x/\rho_C)^{3/2}. \tag{3.17}$$

Now we proceed as in the 2-connected case.

4. *The lower bound*

This section is structured as follows. In the next subsection, we collect some basic facts and tools that will be useful in our arguments. Then, in Subsection 4.2, we give the full proof of the lower bound in Theorem 1.1 for 2-connected graphs, that is, we show a lower bound for the maximum degree in random 2-connected planar graphs that holds w.h.p. Finally, in Subsections 4.3 and 4.4, we demonstrate that the lower bounds for (connected) graphs in Theorem 1.1 are simple corollaries of the lower bound for 2-connected graphs.

4.1. *Networks and Boltzmann sampling*

Before we investigate the maximum degree of graph that is drawn uniformly at random from the class of 2-connected planar graphs, let us mention an auxiliary result that reduces the analysis to the study of random networks. The following lemma is from [**16**].

Lemma 4.1. *Let B_n be a uniform random graph from $\mathcal{B}_n$, and D_n be a network that is drawn uniformly at random from $\mathcal{D}_n$. Suppose that* $\Pr[D_{n-2} \in \mathcal{P}] \geqslant 1 - f(n-2)$, *where $\mathcal{P}$ is any property of graphs that is closed under automorphisms. Then* $\Pr[B_n \in \mathcal{P}] \geqslant 1 - 6f(n-2)$.

It is therefore sufficient to show a lower bound for the maximum degree of a random network. Recall the decomposition of networks that is described in Subsection 2.3, see (2.6)–(2.11). In particular, (2.6) guarantees that a network is either an edge, or a series network, or a parallel network, or a core network. Except for the first case, in all other cases the classes of networks are described recursively. We will say that a network D *has a* (3-*connected*) *core of size* s, if the largest graph from $\mathcal{T}$ that was used in the decomposition of D has s vertices. Note that a network can have an empty core, in which case it consists only of series and parallel connections. However, in [**16**, **20**] it was shown that a 'typical' network has a very large core; here we present a simplified version of that result that is sufficient for our purposes.

Theorem 4.2. *There is a constant $c > \frac{1}{2}$ such that the following is true. Let $\varepsilon > 0$ and denote by $C(D_n)$ the size of the largest core in a random network D_n from $\mathcal{D}_n$. Then, with probability* $1 - o(1)$, *we have that* $C(D_n) > cn$.

The pole degree in the Boltzmann model In the sequel, we will write $ld(D)$ for the degree of the left pole of a network D. The following technical lemma is an important tool in the proof of the lower bound of the maximum degree of random networks.

LEMMA 4.3. *Let γ be a random network drawn from the Boltzmann distribution for $\mathcal{D}$ with parameters $x = \rho_D$ and $y = 1$. Then*

$$\Pr[ld(\gamma) \geqslant k] \sim ck^{-5/2}w_0^{-k},$$

for some constant $c > 0$, where w_0 is given in (3.1).

Proof. Let $\ell \geqslant 1$. The definition of the Boltzmann model implies that

$$\Pr[ld(\gamma) = \ell] = \frac{1}{D(\rho_D, 1)} \sum_{D \in \mathcal{D}:\ ld(D)=\ell} \frac{\rho_D^{v(D)}}{v(D)!} = \frac{[w^\ell]D(\rho_D, 1, w)}{D(\rho_D, 1)}.$$

By following the representation (3.13) of $D(x, 1, w)$ and by setting $x = x_0$ in (3.14), we obtain a singular representation of the form

$$D(x_0, 1, w) = a(w) + b(w)(1 - w/w_0)^{3/2},$$

for some functions $a(w), b(w)$ that are non-zero and analytic at w_0. It is also easy to see that $D(x_0, 1, w)$ has an analytic continuation to a proper Δ-domain in w. We just have to modify the arguments at the end of Subsection 3.2. Hence, we can apply the Transfer Lemma of Flajolet and Odlyzko and obtain

$$\Pr[ld(\gamma) = \ell] \sim c_1 \ell^{-5/2} w_0^{-\ell},$$

for some constant $c_1 > 0$. By adding these values up for $\ell \geqslant k$, we obtain the statement of the lemma. □

4.2. *Proof of the lower bound for* 2*-connected graphs in Theorem* 1.1

Let D_n denote a random network from $\mathcal{D}_n$, and recall the definition of w_0 in (3.1). Moreover, let $\varepsilon = \varepsilon(n) = c' \log\log n / \log_{w_0} n$, where $c' = 10/\log w_0$. By applying Lemma 4.1, we infer that if

$$\Pr[\Delta(D_{n-2}) \leqslant (1 - \varepsilon)\log_{w_0} n] = o(1),$$

then it follows also that $\Pr[\Delta(B_n) \leqslant (1 - \varepsilon)\log_{w_0} n] = o(1)$. We thus proceed with the estimation of the above probability, where we write n instead of $n - 2$ for simplicity, since it can be absorbed by a small change in ε. First of all, by applying Theorem 4.2 we obtain that

$$p = \Pr[\Delta(D_n) \leqslant (1 - \varepsilon)\log_{w_0} n] = \Pr[\Delta(D_n) \leqslant (1 - \varepsilon)\log_{w_0} n \text{ and } C(D_n) > n/2] + o(1),$$

where $C(D)$ denotes the size of the largest core in a network D. Let us write $\gamma = \Gamma D(\rho_D, 1)$, where ΓD is the Boltzmann sampler for the class of networks described in Subsection 2.7. By using the first equality in (2.2), we infer that

$$p = \Pr[\Delta(\gamma) \leqslant (1 - \varepsilon)\log_{w_0} n \text{ and } C(\gamma) > n/2 \mid \gamma \in \mathcal{D}_n] + o(1).$$

By Lemma 2.3, we obtain that $\Pr[\gamma \in \mathcal{D}_n] = \Theta(n^{-5/2})$. So,

$$p = O(n^{5/2}) \Pr[\Delta(\gamma) \leqslant (1 - \varepsilon)\log_{w_0} n \text{ and } C(\gamma) > n/2 \text{ and } \gamma \in \mathcal{D}_n] + o(1). \tag{4.1}$$

In the subsequent analysis, we will make the following modification of the Boltzmann sampler $\Gamma D(x, y)$. Let $L = (L_1, L_2, \dots)$ be an infinite list, where for all $i \geqslant 1$ we have $L_i \in \mathcal{D}$. Recall the definition of the sampler $\Gamma H(x, y)$ that generates core networks. The sampler $\Gamma H(x, y)$ first samples a network from $\bar{\mathcal{T}}$, and then replaces independently every edge by a network that is drawn from the Boltzmann distribution with parameters x and y for $\mathcal{D}$. Instead of doing this, we modify $\Gamma H(x,y)$ so that it uses graphs from L instead, provided that the network sampled from $\bar{\mathcal{T}}$ is large. In particular, the sampler $\tilde{\Gamma} H(x,y;\, n, L)$ works as follows:

```
Γ̃H(x, y; n, L) :    T ← ΓT̄(x, D(x, y))        (∗)
                     if T has more than n/2 vertices
                        i ← 1
                        foreach edge e of T
                           γ_e ← L_i
                           i ← i + 1
                     else
                        foreach edge e of T
                           γ_e ← ΓD(x, y)
                     replace every e in T by γ_e
                     return T, relabeling randomly its non-pole vertices
```

Note that if we choose the elements of L independently from the Boltzmann distribution with parameters x and y for $\mathcal{D}$, then for any $D \in \mathcal{D}$ we have for all values of n that

$$\Pr[\Gamma H(x,y) = D] = \Pr[\tilde{\Gamma} H(x,y;\, n, L) = D].$$

In other words, we can work with $\tilde{\Gamma} H$ instead of ΓH. In particular, we shall assume that ΓD, ΓS and ΓP use $\tilde{\Gamma} H$ instead of ΓH, where the elements of L are independent samples from the Boltzmann distribution with parameters x and y for $\mathcal{D}$.

With these assumptions in mind, let us proceed with the estimation of the probability on the right-hand side of (4.1). First of all, the event '$C(\gamma) > n/2$' implies that at some point in time in the construction of $\gamma = \Gamma D(x,y)$ the sampler $\tilde{\Gamma} H(x,y;\, n, L)$ is used, and the graph T (generated in the line marked with $(*)$) has $> n/2$ vertices. Since T is a 3-connected planar graph minus an edge, it has $\geqslant n/2$ edges. Thus, in the construction of γ certainly the first $\lfloor n/2 \rfloor$ graphs from L are used. Recall that every edge $e = \{u, v\}$ of T is subsequently replaced by some network L_i from L, so that the degree of, say, u is at least $ld(L_i)$. In other words, the event '$\Delta(\gamma) \leqslant (1-\varepsilon)\log_{w_0} n$ and $C(\gamma) > n/2$ and $\gamma \in \mathcal{D}_n$' implies that the first $\lfloor n/2 \rfloor$ graphs in L have the property that the root degree of their left pole is at most $(1-\varepsilon)\log_{w_0} n$. Hence, by using (4.1), the desired probability is at most

$$p \leqslant O(n^{5/2}) \Pr[\forall 1 \leqslant i \leqslant \lfloor n/2 \rfloor \,:\, ld(L_i) \leqslant (1-\varepsilon)\log_{w_0} n] + o(1).$$

Recall that the elements of L are independent samples from the Boltzmann distribution with parameters ρ_D and 1 for $\mathcal{D}$. By applying Lemma 4.3, we obtain for sufficiently large n that

$$\Pr[ld(L_i) \leqslant (1-\varepsilon)\log_{w_0} n] \leqslant 1 - (\log n)^{-3} w_0^{-(1-\varepsilon)\log_{w_0} n} = 1 - (\log n)^{-3} n^{-(1-\varepsilon)}.$$

So, since $\varepsilon = c' \log\log n / \log_{w_0} n$, by choosing, say, $c' = 10/\log(w_0)$

$$p \leqslant O(n^{5/2})(1 - (\log n)^{-3} n^{-(1-\varepsilon)})^{\lfloor n/2 \rfloor} + o(1) = o(1),$$

and the proof is completed.

4.3. *Proof of the lower bound for connected graphs in Theorem* 1.1

The proof of the lower bound for random connected planar graphs in Theorem 1.1 follows directly from the lower bound in the previous section. More precisely, in [**20**, **24**] it was shown that a random planar graph contains with probability $1 - o(1)$ a very large 2-connected subgraph.

THEOREM 4.4. *There is a constant $c > \frac{1}{2}$ such that the following is true. Let $\varepsilon > 0$ and denote by $b(C_n)$ the size of the largest 2-connected subgraph in a random graph C_n from $\mathcal{C}_n$. Then, w.h.p., $|b(C_n) - cn| \leqslant \varepsilon n$.*

Note that, conditional on any specific value of $b(C_n)$ that is within the bounds given in the above theorem, any 2-connected planar graph with $b(C_n)$ vertices is equally likely to be the largest 2-connected subgraph of C_n. Thus, for sufficiently large n, the maximum degree in C_n is w.h.p. at least the maximum degree of a random 2-connected planar graph with, say, $n/4$ vertices. By using the results of the previous section, this is w.h.p. at least $\log(n/4)/\log w_0 - O(\log\log n) = \log n/\log w_0 - O(\log\log n)$, and the proof is completed.

4.4. *Proof of the lower bound for planar graphs in Theorem* 1.1

We proceed as in Subsection 3.1: we know that w.h.p. a random planar graph contains a component of size at least $n - \omega_n$, where ω_n is any function that diverges for $n \to \infty$. Since the largest degree of a vertex outside this component is bounded by ω_n the claim follows from the results in Subsection 4.3.

5. *The expected value of the maximum degree*

In this section, we present the proof of (1.2) in Theorem 1.1. We shall restrict ourselves to the case of 2-connected graphs, since the same statement for connected and general planar graphs follows by similar arguments. First of all, if we write B_n for a random 2-connected planar graph, then note that

$$\begin{aligned}\mathbb{E}\Delta(B_n) &= \sum_{\ell \geqslant 1} \ell \Pr[\Delta(B_n) = \ell] \\ &\geqslant (c\log n - O(\log\log n)) \sum_{|\ell - c\log n| = O(\log\log n)} \Pr[\Delta(B_n) = \ell].\end{aligned}$$

Since (1.1) guarantees that $|\Delta(B_n) - c\log n| = O(\log\log n)$ w.h.p., we infer that

$$\mathbb{E}\Delta(B_n) \geqslant (1 - o(1))\, c\log n,$$

as claimed. To see the upper bound for the expectation, let us write, as in Section 3, $X_{n,k}$ for the number of vertices of degree k in B_n. Moreover, abbreviate $\ell^+ = c\log n + 2\log\log n$. Then

$$\mathbb{E}\Delta(B_n) \leqslant \ell^+ + \sum_{\ell \geqslant \ell^+} \ell \Pr[X_{n,\ell} > 0].$$

However, by applying (3.2) we obtain that

$$\Pr[X_{n,\ell} > 0] \leqslant \mathbb{E}X_{n,\ell} = O(n\, w_0^{-\ell}).$$

Thus,

$$\mathbb{E}\Delta(B_n) \leqslant \ell^+ + O(1) \sum_{\ell \geqslant \ell^+} \ell n w_0^{-\ell} \leqslant \ell^+ + O(1)\ell^+\, n\, w_0^{-\ell^+} = (1 + o(1))\ell^+,$$

and the proof is completed.

6. *Conclusion and discussion*

The main objective of this paper is to derive asymptotic bounds for the maximum degree of random planar graphs. We remark that for random planar *maps* (graphs with a fixed embedding in the plane), much more precise results are known. It is shown in [**18**] that the maximum degree Δ_n of random planar maps with n edges is asymptotically $\log n/\log(\frac{6}{5})$, and that $\Delta_n - \mathbb{E}\Delta_n$ follows asymptotically an extreme value (Gumbel) distribution. We conjecture the same results for graphs, since a random planar graph is made out of a large 3-connected map to which small graphs are substituted for edges and attached to vertices. However, the results in [**18**] require the analysis of higher moments for the number of vertices of given degree, and already the analysis of the second moment does not appear feasible for planar graphs.

In the remainder of this section, we describe two possible extensions of the present work: random planar graphs with fixed average degree, and the expected number of vertices of degree k.

6.1. *Fixed average degree*

Besides the uniform distribution on the class of all planar graphs with n vertices, it is also interesting to study random planar graphs, where the ratio of the number of vertices and the number of edges is fixed to some constant $1 < \alpha < 3$. Analytically, this means that we have to take in the generating functions also the variable y into account (and usually it will be set to a value different from 1). For example, in [**19**], using this method, the asymptotic number of planar graphs with a fixed edge density was determined.

Actually, this program could also be worked out for determining the maximum degree. More precisely, it is possible to adapt and extend several parts of our analysis to show that, w.h.p., the maximum degree $\Delta_{n,\alpha}$ of a random planar graph with n vertices and αn edges satisfies

$$|\Delta_{n,\alpha} - c(\alpha)\log n| \leqslant O(\log\log n), \tag{6.1}$$

where $c(\alpha) = 1/\log(w(t(y))/y)$, $t = t(y)$ is given by (2.26),

$$w(t) = \frac{1}{(1-t)}\exp\left\{\frac{t(t-1)(t+6)}{(3t+1)(t+3)}\right\} - 1,$$

and α and y are linked by the equation $\alpha = -y\rho_C'(y)/\rho_C(y)$. Indeed, as already shown in [**11**], $w_0 = w(t(y))/y$ is the singular point of the functions $C(\rho_C(y), y, w)$ and $B(\rho_B(y), y, w)$. A more careful analysis along the lines of Lemma 3.4, Corollary 3.5 and the argument in Subsection 4.2, where now all functions also depend on y, yields the statement in (6.1). However, the calculations are technically more involved. In particular, we need an additional (saddle point like) Cauchy integration in order to obtain the asymptotics of the coefficient of $[y^{\alpha n}]$. Figure 6.1 contains a plot of the function $c(\alpha)$.

6.2. *The expected number of vertices of degree k*

We recall that the expected number of vertices of degree k is given by $\mathbb{E}\, X_{n,k} = np_{n,k}$, where $p_{n,k}$ denotes the probability that the root vertex of a random planar graph with n vertices has degree equal to k. Moreover, note that

$$p_{n,k} = \frac{[x^n w^k]C^\bullet(x,1,w)}{[x^n]C^\bullet(x,1,1)}.$$

Lemma 2.2 implies that

$$[x^n]C^\bullet(x,1,1) = (1+o(1))cn^{-5/2}\rho_C^{-n}n!.$$

Thus, in order to obtain the asymptotic value of $p_{n,k}$ it is necessary to derive bivariate asymptotics for the coefficients of $C^\bullet(x,1,w)$ with respect to x and w. A similar task was performed in [**12**], where the authors solved this problem for the case of series parallel graphs. In the present setting, the generating functions are significantly more involved. However, since the analytic structure of the generating function enumerating planar networks, see Subsection 3.2, is

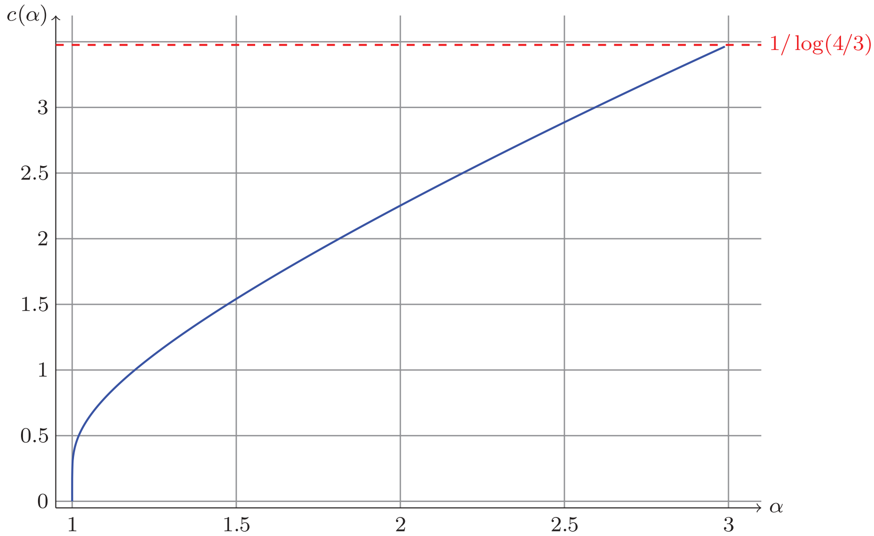

FIGURE 6.1. *The function* $c(\alpha)$. *Note that* $\lim_{\alpha=0+} c(\alpha) = 0$ *and* $\lim_{\alpha=3-} c(\alpha) = \log(\frac{4}{3})$. *This is consistent with the facts that w.h.p. the maximum degree of a random tree is* $o(\log n)$ *and the maximum degree of a random triangulation is* $(1 + o(1)) \log n / \log(\frac{4}{3})$.

analyzed already quite thoroughly in this work, it is possible to extend the methods developed in [**12**] to compute the asymptotic value of $p_{n,k}$. Again we have to apply another Cauchy integration in order to obtain the asymptotics for the coefficient of w^k for k of order $O(\log n)$.

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