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Enumeration and limit laws for series–parallel graphs

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Abstract

We show that the number g_n of labelled series–parallel graphs on n vertices is asymptotically $g_n \sim g \cdot n^{-5/2}\gamma^n n!$, where γ and g are explicit computable constants. We show that the number of edges in random series–parallel graphs is asymptotically normal with linear mean and variance, and that it is sharply concentrated around its expected value. Similar results are proved for labelled outerplanar graphs and for graphs not containing $K_{2,3}$ as a minor.

1. Introduction

A graph is series–parallel (SP for short) if it does not contain the complete graph K_4 as a minor; or, equivalently, if it does not contain a subdivision of K_4. Since both K_5 and $K_{3,3}$ contain a subdivision of K_4, by Kuratowski's theorem a SP graph is planar. Another characterization, justifying the name, is the following. A connected graph is SP if it can be obtained from a tree by means of the following two operations: subdividing an edge (series extension); and duplicating an edge (parallel extension). In addition, a 2-connected graph is SP if it can be obtained from a double edge by means of series and parallel extensions; in particular, this implies that a 2-connected SP graph has always a vertex of degree two. Although SP operations may give rise to multiple edges, in this paper all graphs considered are simple. Yet another characterization is that SP graphs are precisely the graphs with tree-width at most two. Equivalently they are subgraphs

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of 2-trees, where a 2-tree is a graph formed by, starting from a triangle, adding repeatedly a new vertex and joining it to an existing edge.

An outerplanar graph is a planar graph that can be embedded in the plane so that all vertices are incident to the outer face. They are characterized as those graphs not containing a minor isomorphic to (or a subdivision of) either K_4 or $K_{2,3}$. They constitute an important subclass of the class of SP graphs.

These are important subfamilies of planar graphs, as they are much simpler but often they already capture the essential structural properties of planar graphs. In particular, they are used as a natural first benchmark for many algorithmic problems and conjectures related to planar graphs.

In this paper we study the enumeration of labelled series–parallel and outerplanar graphs. From now on, unless stated otherwise, all graphs are labelled. Next we summarize what is known about this problem. An SP graph on n vertices has at most $2n - 3$ edges. Those having this number of edges are precisely the 2-trees; it is known [8] that the number of labelled 2-trees on n vertices is equal to $\binom{n}{2}(2n-3)^{n-4}$. On the other hand, an outerplanar graph is 2-connected if and only it has a unique Hamilton cycle. It follows that a 2-connected outerplanar graph is in fact equivalent to a dissection of a convex polygon, the boundary of the polygon being the unique Hamilton cycle. Hence counting 2-connected outerplanar graphs amounts essentially to counting dissections of a convex polygon, a classical and well-known problem that we revisit in Section 5. Finally, an outerplanar *map* (a map is a planar graph together with a particular embedding in the plane) on n vertices can be encoded with $3n$ bits [2]. Hence the number of outerplanar graphs is at most $2^{3n} = 8^n$; as shown later, the growth constant of outerplanar graphs is ≈ 7.3209.

The main goal of this paper is to give precise asymptotic estimates for the number of SP and outerplanar graphs. In Section 2 we show that the number b_n of 2-connected SP graphs is asymptotically of the form

$$b_n \sim b \cdot n^{-5/2} R^{-n} n!,$$

where b and $R \approx 0.12800$ are computable constants. All the constants that appear in this paper are given by explicit analytic expressions and can be computed to any degree of accuracy.

Then in Section 3 we show that the total number g_n of SP graphs is given by

$$g_n \sim g \cdot n^{-5/2} \rho^{-n} n!,$$

where $\rho \approx 0.11021$.

In Section 4 we analyze the distribution of the number of edges in SP graphs. If X_n is the random variable denoting the number of edges in a random SP graph on n vertices, we prove that X_n is asymptotically normal and that the mean μ_n and variance σ_n^2 of X_n satisfy

$$\mu_n \sim \kappa n, \qquad \sigma_n^2 \sim \lambda n,$$

where $\kappa \approx 1.61673$ and $\lambda \approx 0.55347$. As a consequence, the number of edges is sharply concentrated around its expected value.

In Section 5 we study the same problems for outerplanar graphs. We prove that the number of outerplanar graphs is

$$h_n \sim h \cdot n^{-3/2} \sigma^{-n} n!,$$

where $\sigma \approx 0.13659$, and that the number of edges in random outerplanar graphs is asymptotically normal with mean and variance

$$\mu_n \sim \zeta n, \qquad \sigma_n^2 \sim \eta n,$$

where $\zeta \approx 1.56251$ and $\eta \approx 0.22399$. Previously, it had been shown [6] that $\zeta \geq 7/5$.

In Section 6 we study the distribution of the number of connected components in series–parallel and outerplanar graphs. In both cases it turns out that the distribution is asymptotically a shifted Poisson law with parameter $\nu = 0.117614$ for SP graphs, and $\xi = 0.14840$ for outerplanar graphs. As a consequence the probability that a random SP graph is connected tends to $\mathrm{e}^{-\nu} = 0.889038$, and to $\mathrm{e}^{-\xi} = 0.86208$ for outerplanar graphs.

Finally, in Section 7 we study the family of graphs not containing $K_{2,3}$ as a minor. This is a subfamily of series–parallel graphs which turns out to be very close to the family of outerplanar graphs.

The proofs are based on singularity analysis of generating functions and perturbation of singularities (see [4,5]), and on several ideas developed in [1,7] for solving similar problems for the class of planar graphs. For the techniques and results of singularity analysis used in the sequel we refer the reader to the forthcoming book *Analytic Combinatorics* by Flajolet and Sedgewick [5].

2. Counting 2-connected series–parallel graphs

Let $b_{n,q}$ be the number of 2-connected SP graphs with n vertices and q edges, and let

$$B(x, y) = \sum b_{n,q} y^q \frac{x^n}{n!}$$

be the corresponding exponential generating function (EGF).

Following [9], we define a *network* as a graph with two distinguished vertices, called poles, such that the multigraph obtained by adding an edge between the two poles is 2-connected. If $D(x, y)$ is the EGF for SP networks, where again x marks vertices and y marks edges, then, as shown in [9], we have

$$\frac{\partial B(x, y)}{\partial y} = \frac{x^2}{2} \left(\frac{1 + D(x, y)}{1 + y} \right). \tag{2.1}$$

Since a 2-connected SP graph has always a vertex of degree 2, it follows that there are no 3-connected SP graphs; in the terminology of [9] there are only s-networks and p-networks and there are no h-networks. Hence Eq. (12) in [1] simplifies to

$$\log \left(\frac{1 + D}{1 + y} \right) = \frac{x D^2}{1 + x D}. \tag{2.2}$$

Our goal is to perform a complete singularity analysis of $B(x, y)$ using Eqs. (2.1) and (2.2). To this end we first determine the singularities of $D(x, y)$.

From now on y is a fixed positive value. Because of (2.2), the inverse of $D(x, y)$ as a function of x is given by

$$\psi_y(u) = \frac{\log \left(\frac{1+u}{1+y} \right)}{u \left(u - \log \left(\frac{1+u}{1+y} \right) \right)}.$$

We show that the equation $\psi_y'(u) = 0$ has a unique positive root $u = \upsilon(y)$ for every positive y. (We often write ψ and υ instead of ψ_y and $\upsilon(y)$ for brevity.) Hence $D(x, y)$, being the inverse of ψ, ceases to be analytic at $x = R(y) = \psi(\upsilon)$. By Proposition IV.4 in [5], it follows that the dominant singularity (that is, the singularity of smallest modulus) of $D(x, y)$ for fixed y is at $R(y)$. The next result gives a procedure for obtaining $R(y)$ as a function of y.

Theorem 2.1. *For fixed $y > 0$, the dominant singularity of $D(x, y)$ is at $R(y) = q(t)$, where*

$$q(t) = \frac{(1+t)(t-1)^2}{t^3}, \tag{2.3}$$

and t is the unique root of $Y(t) = y$, where

$$Y(t) = \frac{1}{1-t^2}\exp\left(\frac{-t^2}{1+t}\right) - 1.$$

Proof. Let

$$L = L(u) = \log\left(\frac{1+u}{1+y}\right). \tag{2.4}$$

A routine computation gives

$$\psi'(u) = \frac{(1+u)L^2 - 2u(1+u)L + u^2}{(1+u)u^2(L-u)^2}.$$

The numerator vanishes when the corresponding quadratic equation on L has a root, necessarily at $u = \upsilon = \upsilon(y)$. This gives

$$L(\upsilon) = \frac{2\upsilon(1+\upsilon) - \sqrt{4\upsilon^2(1+\upsilon)^2 - 4\upsilon^2(1+\upsilon)}}{2(1+\upsilon)} = \upsilon - \upsilon^{3/2}(1+\upsilon)^{-1/2}. \tag{2.5}$$

In order to simplify (2.5), we set $t = \sqrt{\upsilon}/\sqrt{1+\upsilon}$ so that $\upsilon = t^2/(1-t^2)$. Eqs. (2.4) and (2.5) then become

$$L(t) = \log\frac{1}{(1+y)(1-t^2)},$$
$$L(t) = \frac{t^2}{1+t}.$$

We solve for y and we observe that t is determined by the equation $Y(t) = y$. Since

$$Y'(t) = \frac{(t+3)t^2}{(t^2-1)^2(1+t)}\exp\left(\frac{-t^2}{1+t}\right) > 0,$$

for $t \in (0, 1)$, we deduce that $Y(t)$ is one to one in this domain. Since the limits of $Y(t)$ when t approaches 0 and 1 are, respectively, 0 and $+\infty$, it follows that every $y > 0$ has a corresponding $t \in (0, 1)$; this determines the unique root υ of $\psi'(u)$.

The dominant singularity $R(y)$ is at $\psi(\upsilon) = L/(\upsilon(\upsilon - L))$ which, in terms of t, gives (2.3). □

Since ψ' has a root υ, we know that $D(x, y)$, for fixed y, has a singularity of square-root type. In the following lemma we show that $\psi''(\upsilon) < 0$ for every y; hence the singular expansion of

$D(x, y)$ at the singularity $R(y)$ is (see Proposition VI.1 in [5])

$$D(x, y) = \upsilon(y) - \sqrt{\frac{-2R(y)}{\psi''(\upsilon)}}X + \mathcal{O}(X^2),$$

where $X = \sqrt{1 - x/R(y)}$.

Lemma 2.2. *For fixed $y > 0$, the singular expansion of $D(x, y)$ at $R(y)$ is*

$$D(x, y) = D_0(y) + D_1(y)X + D_2(y)X + \mathcal{O}(X^3), \tag{2.6}$$

where $X = \sqrt{1 - x/R(y)}$ and

$$D_0 = \frac{t^2}{1 - t^2}, \qquad D_1 = \frac{\sqrt{2t^3}}{\sqrt{t + 3}(t^2 - 1)}, \qquad D_2 = \frac{2t(t^2 + 3t + 3)}{3(1 - t)(3 + t)^2}$$

where t is the unique root of $Y(t) = y$.

Proof. The constant term is $D_0 = D(R(y), y) = \upsilon$, and the terms D_1 and D_2 are

$$D_1 = -\sqrt{-2R(y)/\psi''(\upsilon)}, \qquad D_2 = \frac{R\psi'''(\upsilon)}{3\psi''(\upsilon)^2},$$

where these expressions can be obtained by inverting the Taylor series of $\psi(u)$ at the point $u = \upsilon$ where $\psi'(\upsilon)$ vanishes.

Now we use Maple to compute $\psi''(u)$ and $\psi'''(u)$, which are rational expressions on u and L. Then we set $u = t^2/(1 - t^2)$ and $L = t^2/(1 + t)$, and simplifying we obtain

$$\psi''(t) = -\frac{(t + 3)(t - 1)^4(1 + t)^3}{t^6},$$
$$\psi'''(t) = -\frac{2(t^2 + 3t + 3)(t - 1)^5(1 + t)^5}{t^8}.$$

Hence $\psi''(t) < 0$ for $t \in (0, 1)$, and we use this expression to obtain D_1 and D_2 in terms of t. □

After completing the analysis of $D(x, y)$ we turn to that of $B(x, y)$. The first task is to express B in terms of D.

Lemma 2.3. *The following holds, where $D = D(x, y)$:*

$$B(x, y) = \frac{1}{2}\log(1 + xD) - \frac{xD(x^2D^2 + xD + 2 - 2x)}{4(1 + xD)}. \tag{2.7}$$

Proof. We follow the proof of Lemma 5 in [7]. From now on x is a fixed value. From (2.1) it follows that

$$B(x, y) = \frac{x^2}{2}\log(1 + y) + \frac{x^2}{2}\int_0^y \frac{D(x, t)}{1 + t}\mathrm{d}t.$$

Integrating by parts we get

$$\int_0^y \frac{D(x, t)}{1 + t}\mathrm{d}t = \log(1 + y)D - \int_0^y \log(1 + t)\frac{\partial D}{\partial t}\mathrm{d}t.$$

Now we notice that the inverse of D with respect to y is

$$\phi(u) = -1 + (1+u)\exp\left(-\frac{xu^2}{1+xu}\right).$$

The last integral, after the change $s = D(x,t)$, becomes

$$\int_0^{D(x,y)} \left(\log(1+s) - \frac{xs^2}{1+xs}\right) \mathrm{d}s,$$

which can be integrated in elementary terms. The rest of the computation is routine and the claim follows. □

In view of the expression in Lemma 2.3, it is clear that, for fixed y, the dominant singularity of $B(x,y)$ is the same as that of $D(x,y)$, namely $R(y)$. Using (2.7) we can find the singular expansion of $B(x,y)$.

Lemma 2.4. *For $y > 0$, the singular expansion of $B(x,y)$ at its singularity $R(y)$ is*

$$B(x,y) = B_0(y) + B_2(y)X^2 + B_3(y)X^3 + \mathcal{O}(X^4), \tag{2.8}$$

where $X = \sqrt{1 - x/R(y)}$ and $B_0(y), B_2(y), B_3(y)$ are the following analytic functions of the unique root t of $Y(t) = y$:

$$B_0(t) = \frac{t^3 + 2\ln(1/t)t^3 + 2t^2 - 5t + 2}{4t^3}$$

$$B_2(t) = \frac{(t-1)^3(t+2)}{2t^3}$$

$$B_3(t) = (1-t)^3\sqrt{\frac{2}{3(t+3)t^3}}.$$

Proof. It is enough to set $x = R(1 - X^2)$ and $D = D_0 + D_1X + D_2X^2 + D_3X^3$ in (2.7). All the calculations have been performed with Maple. In particular, we obtain that B_1 vanishes identically as a function of t, and that B_3 does not depend on the value of D_3. □

Theorem 2.5. *The number of 2-connected SP graphs b_n is asymptotically*

$$b_n \sim b \cdot n^{-5/2} \cdot R^{-n} n!$$

where $R = R(1) \approx 0.12800$ and $b \approx 0.0010131$.

Proof. By transfer theorems, the asymptotic estimate follows from the singularity expansion of $B(x,1)$ of Lemma 2.4. Solving $Y(t) = 1$ gives $t \approx 0.80703$, and from here we obtain the values of $R(1) = q(t)$ and of $b = 3B_3(t)/(4\sqrt{\pi})$. □

3. Counting series–parallel graphs

Recall that g_n, c_n and b_n denote, respectively, the number of SP graphs, connected SP graphs, and 2-connected SP graphs on n vertices. Let $G(x)$, $C(x)$ and $B(x)$ be the corresponding generating functions.

Adapting the proof of Lemma 1 in [7], we obtain that the corresponding exponential generating functions are related as follows.

Lemma 3.1. *The series $G(x)$, $C(x)$ and $B(x)$ satisfy the following equations:*

$$G(x) = \exp(C(x)), \qquad xC'(x) = x\exp(B'(xC'(x))),$$

where $C'(x) = \mathrm{d}C(x)/\mathrm{d}x$ and $B'(x) = \mathrm{d}B(x)/\mathrm{d}x$.

Let us recall that $b_{n,q}$ is the number of 2-connected planar graphs with n vertices and q edges, and that

$$B(x, y) = \sum b_{n,q} y^q \frac{x^n}{n!}$$

is the corresponding bivariate generating function. The generating functions $C(x, y)$ and $G(x, y)$ are defined analogously. Notice that $B(x, 1) = B(x)$, and analogously for $C(x)$ and $G(x)$. Since the parameter "number of edges" is additive under taking connected and 2-connected components, the previous lemma can be extended as follows.

Lemma 3.2. *The series $G(x, y)$, $C(x, y)$ and $B(x, y)$ satisfy the following equations:*

$$G(x, y) = \exp(C(x, y)), \qquad x\frac{\partial}{\partial x}C(x, y) = x\exp\left(\frac{\partial}{\partial x}B\left(x\frac{\partial}{\partial x}C(x, y), y\right)\right).$$

Let $F(x, y) = xC'(x, y)$, where the derivative is with respect to the first variable. Lemma 3.1 implies that

$$F(x, y) = x\exp(B'(F(x, y), y)).$$

It follows that, for fixed y, the functional inverse of $F(x, y)$ is

$$\Psi_y(u) = u\mathrm{e}^{-B'(u,y)}.$$

The function Ψ_y should not be confused with ψ_y in Section 2, although it plays a similar role. Our goal is to prove that for each $y > 0$, $\Psi'_y(u) = 0$ has a root $\tau(y)$. As in the previous section we often omit the fact that Ψ and τ depend on a fixed y.

In order to determine the dominant singularity of $F(x, y)$, which is the same as that of $C(x, y)$ and $G(x, y)$, we need a technical lemma.

Lemma 3.3. *The equation*

$$u^4D^6 + u^3D^5 + 2u^2D^3 + 4uD^2 - 2 = 0 \tag{3.1}$$

where $D = D(u, y)$ and y is a fixed positive value, has a unique solution $u = \tau(y)$ in $(0, R(y))$.

Proof. Let $T(u, D) = T(u, y)$ be the left hand side of (3.3), which is an increasing function of u since $D(u, y)$ has non-negative coefficients. Since $T(0, y) = -2$, it follows that τ exists and is unique if and only if $T(R(y), y) > 0$. We use the expressions in terms of t for $R(y)$ and $D(R(y), y)$ given in Theorem 2.1 and Lemma 2.2 and, after simplification, we obtain that $T(R(y), y)$ written as a function of t is

$$\frac{1-t}{(1+t)^2}.$$

This is a positive value when $t \in (0, 1)$, so the claim follows. □

Theorem 3.4. *Let y be a fixed positive value. The unique root of* $\Psi'(u) = 0$ *is given by* $\tau(y)$ *in Lemma 3.3. The dominant singularity of* $F(x, y)$ *is at* $\rho(y)$*, where* ρ*, as a function of* τ*, is*

$$\rho(\tau) = \tau \exp\left(\frac{\tau D(\tau D^2 - 2)}{2(1 + \tau D)}\right), \tag{3.2}$$

where $D = D(\tau(y), y)$.

The singular expansion of $F(x, y)$ *at its dominant singularity* $\rho(y)$ *is*

$$F(x, y) = F_0(y) + F_1(y)X + \mathcal{O}(X^2), \tag{3.3}$$

where $X = \sqrt{1 - x/\rho(y)}$ *and*

$$F_0(\tau) = \tau$$
$$F_1(\tau) = 2\frac{1 - 2\tau D^2 - \tau^2 D^3}{D}\sqrt{\frac{\tau(1 + \tau D)}{S}}$$
$$S = -4\tau^5 D^7 - 5\tau^4 D^6 + (6\tau^4 - \tau^3)D^5 + 5\tau^3 D^4 - 3\tau^2 D^3 + 6\tau^2 D^2 + 12\tau D + 4.$$

Proof. We start by differentiating $\Psi(u)$:

$$\Psi'(u) = \exp(-B'(u, y))(1 - uB''(u, y)).$$

By Lemma 2.3, the functions $\Psi(u)$ and $\Psi'(u)$ can be written in terms of $D = D(u, y)$,

$$\Psi(u) = u \exp\left(\frac{uD(uD^2 - 2)}{2(1 + uD)}\right)$$
$$\Psi'(u) = \frac{u^4 D^6 + u^3 D^5 + 2u^2 D^3 + 4uD^2 - 2}{(2u^2 D^3 + 4uD^2 - 2)(1 + uD)} \exp\left(\frac{uD(uD^2 - 2)}{2(1 + uD)}\right),$$

where $D = D(u, y)$. To obtain the previous expressions we have used the relation

$$D' = \frac{D^2(1 + D)}{1 - 2uD^2 - u^2 D^3},$$

which follows directly from (2.2).

Clearly $\Psi'(u)$ vanishes at the roots of the polynomial

$$T(u, D) = u^4 D^6 + u^3 D^5 + 2u^2 D^3 + 4uD^2 - 2,$$

and hence the root $u = \tau(y)$ of $\Psi'(u)$ is the one given by Lemma 3.3.

As for the remaining expressions, $\rho(y)$ is $\Psi(\tau)$, $F_0(y)$ is just τ, and $F_1(y)$ is given by $-\sqrt{-2\Psi(\tau)/\Psi''(\tau)}$, if we can show that $\Psi''(\tau) < 0$. To obtain $\Psi''(\tau)$ we differentiate $\Psi'(u)$ with respect to u. Note that, when evaluating at $u = \tau$, the polynomial $T(u, D)$ vanishes, and so

$$\Psi''(\tau) = \frac{\frac{\partial T}{\partial u}(\tau, D) + \frac{\partial T}{\partial D}(\tau, D)D'}{(2\tau^2 D^3 + 4\tau D^2 - 2)(1 + \tau D)} \exp\left(\frac{\tau D(\tau D^2 - 2)}{2(1 + \tau D)}\right).$$

All factors in this expression are positive but for $2\tau^2 D^3 + 4\tau D^2 - 2 < T(\tau, D) = 0$; hence we have shown that $\Psi''(\tau) < 0$. The expression for $F_1(\tau)$ follows by straightforward simplification. □

In order to find the singular expansion of $C(x, y)$, we start with a simple lemma.

Lemma 3.5. *The series $C(x, y)$, $F(x, y)$ and $B(x, y)$ satisfy the following equation:*

$$C(x, y) = F(x, y)(1 + \log x - \log F(x, y)) + B(F(x, y), y). \tag{3.4}$$

Proof. This result is given in the proof of Theorem 1 in [7]. It is analogous to that of Lemma 2.3, but simpler. Since $F(x, y) = xC'(x, y)$, we have that

$$C(x, y) = \int_0^x \frac{F(s, y)}{s} \mathrm{d}s = F(x) \log x - \int_0^x F'(s, y) \log s \mathrm{d}s.$$

Now we change variables $t = F(s)$, so that $s = \Psi(t) = t \exp(-B'(t, y))$. Then the last integral becomes

$$\int_0^{F(x,y)} \log \Psi(t) \mathrm{d}t = \int_0^{F(x,y)} (\log t - B'(t, y)) \mathrm{d}t.$$

Hence

$$C(x, y) = F(x, y)(1 + \log x - \log F(x, y)) + B(F(x, y), y). \quad \square$$

Theorem 3.6. *Let y be a fixed positive value. The dominant singularities of $C(x, y)$ and $G(x, y)$ are at $\rho(y)$, where $\rho(y)$ is as in Theorem 3.4. The singular expansions of $C(x, y)$ and $G(x, y)$ at their singularities are*

$$C(x, y) = C_0(y) + C_2(y)X^2 + C_3(y)X^3 + \mathcal{O}(X^4),$$
$$G(x, y) = G_0(y) + G_2(y)X^2 + G_3(y)X^3 + \mathcal{O}(X^4),$$

where $X = \sqrt{1 - x/\rho(y)}$ and

$$C_0 = \tau(\log \rho - \log(\tau) + 1) + B(\tau, y), \qquad C_2 = -F_0, \qquad C_3 = -\frac{3}{2} F_1,$$
$$G_0 = \exp(C_0), \qquad G_2 = \exp(C_0)C_2, \qquad G_3 = \exp(C_0)C_3.$$

Proof. It is clear that G and C have the same singularities as F. The singular expansion of $F(x, y) = xC'(x, y)$ can be obtained from that of $C(x, y)$ by differentiating (see Theorem VI.5 in [5]) and multiplying by $x = \rho(y)(1 - X^2)$, so that

$$F(x, y) = \left(-C_2(y) - \frac{3}{2} C_3(y)X\right)(1 - X^2) + \mathcal{O}(X^2).$$

By equating coefficients, the expressions for C_2 and C_3 follow. To obtain C_0 we evaluate $C(x, y)$ at its singularity $x = \rho(y)$ using Lemma 3.5, and notice that $F(\rho(y), y) = \tau(y)$.

Finally, the singular expansion of $G(x, y)$ follows from $G(x, y) = \exp(C(x, y))$, since

$$\begin{aligned} G(x, y) &= \exp(C_0) \exp(C_2 X^2 + C_3 X^3) + \mathcal{O}(X^4) \\ &= \exp(C_0)(1 + C_2 X^2 + C_3 X^3) + \mathcal{O}(X^4). \quad \square \end{aligned}$$

Theorem 3.7. *The number of connected SP graphs c_n and all SP graphs g_n are asymptotically*

$$c_n \sim c \cdot n^{-5/2} \cdot \rho^{-n} n!$$
$$g_n \sim g \cdot n^{-5/2} \cdot \rho^{-n} n!$$

where $\rho = \rho(1) \approx 0.11021$, $c \approx 0.0067912$ and $g \approx 0.0076388$.

Proof. The asymptotic estimates follow again from transfer theorems on the generating functions $C(x, 1)$ and $G(x, 1)$. As for the constants, solving Eq. (3.1) in Lemma 3.3 for $y = 1$ yields $\tau = \tau(1) \approx 0.1279695$ and $D = D(\tau, 1) \approx 1.84351$, and from here follow the values of $\rho = \Psi(\tau)$, $c = 3C_3/(4\sqrt{\pi})$ and $g = \exp(C_0)c$. Recall that C_3 and C_0 are explicit functions of τ and D. □

From the proof of the last theorem, it follows that the constants ρ, c and g are given by explicit analytic expressions and can be computed to any degree of accuracy.

4. The number of edges in series–parallel graphs

The main tool in this section is the so-called Quasi-Powers theorem [5], which allows to deduce a normal limit law for a combinatorial parameter from the bivariate singular expansion of the corresponding generating function. The proof scheme is as for Theorem 2 from [7]. The exact form we need of the Quasi-Powers theorem is that of Proposition 2 in [7].

We work out in some detail the case of 2-connected SP graphs; the remaining cases follow the same pattern. We know that for fixed y we have a singular expansion

$$B(x, y) = B_0(y) + B_2(y)X^2 + B_3(y)X^3 + \mathcal{O}(X^4),$$

where $X = \sqrt{1 - x/R(y)}$ and the $B_i(y)$ are analytic functions. We deduce that the number of edges in 2-connected graphs is normally distributed and that the expected number of edges is asymptotically αn, where

$$\alpha = -\frac{R'(1)}{R(1)} \approx 1.71891.$$

The derivative $R'(1)$ is computed using the explicit form of $R(y)$ given in Theorem 2.1; indeed $R'(y) = q'(t)/Y'(t)$, where t is the unique solution of $Y(t) = y$. The relevant values are $R(1) \approx 0.12800$ and $R'(1) \approx -0.22002$.

The variance is asymptotically βn, where

$$\beta = -\frac{R''(1)}{R(1)} - \frac{R'(1)}{R(1)} + \left(\frac{R'(1)}{R(1)}\right)^2.$$

We compute $R''(1) \approx 0.57667$, so that $\beta \approx 0.16846$. Hence we have proved:

Theorem 4.1. *Let X_n denote the number of edges in a random 2-connected series–parallel graph with n vertices. Then X_n is asymptotically normal and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \kappa_0 n, \qquad \sigma_n^2 \sim \lambda_0 n, \tag{4.1}$$

where $\kappa_0 \approx 1.71891$ and $\lambda_0 \approx 0.16846$.

For connected SP graphs and arbitrary SP graphs the same result holds, but in this case the dominant singularity is at $\rho(y)$, which is given in Theorem 3.3.

Theorem 4.2. *Let X_n denote the number of edges in a random series–parallel graph with n vertices. Then X_n is asymptotically normal and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \kappa n, \qquad \sigma_n^2 \sim \lambda n, \tag{4.2}$$

where $\kappa \approx 1.61673$ *and* $\lambda \approx 0.21125$. *The same is true, with the same constants, for connected random SP graphs.*

Since $\sigma_n = o(\mu_n)$ it follows that the number of edges in random SP is concentrated around it expected value, in the sense that for every $\epsilon > 0$ we have

$$\text{Prob}\{|X_n - \kappa n| > \epsilon n\} \to 0, \quad \text{as } n \to \infty.$$

This comment also applies to Theorem 4.1 and to the Gaussian limit laws presented in the following sections.

Proof. Since $\rho(y) = \Psi(\tau(y), y)$ it follows that

$$\rho'(y) = \frac{\partial \Psi}{\partial x}(\tau(y), y)\tau'(y) + \frac{\partial \Psi}{\partial y}(\tau(y), y) = \frac{\partial \Psi}{\partial y}(\tau(y), y),$$

where the first summand vanishes by definition of $\tau(y)$. We can compute $\partial \Psi / \partial y$ explicitly by differentiating $\Psi(u, y)$ with respect to y, and using that

$$\frac{\partial D}{\partial y}(x, y) = -\frac{(1 + xD(x, y))^2(1 + D(x, y))}{(-1 + 2xD(x, y)^2 + x^2 D(x, y)^3)(1 + y)}.$$

We obtain that $\rho'(1) \approx -0.17818$.

To compute $\rho''(1)$ we proceed in a similar way,

$$\rho''(y) = \frac{\partial^2 \Psi}{\partial x \partial y}(\tau(y), y)\tau'(y) + \frac{\partial^2 \Psi}{\partial y^2}(\tau(y), y).$$

Computing the partial derivatives of Ψ poses no problem; to obtain $\tau'(y)$ we differentiate with respect to y the equation

$$T(\tau(y), D(\tau(y), y)) = 0,$$

where $T(u, D)$ is the polynomial of Lemma 3.3. This gives a linear equation in $\tau'(y)$, from where it follows that $\tau'(1) \approx -0.21992$ and then $\rho''(1) \approx 0.44298$. Finally, the constants κ and λ are computed using

$$\kappa = -\frac{\rho'(1)}{\rho(1)}, \qquad \lambda = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2. \quad \square$$

5. Outerplanar graphs

We keep the notation of previous sections but applied to outerplanar graphs instead of series–parallel graphs. Thus g_n is the number of (labelled) outerplanar graphs on n vertices; similarly for c_n and b_n, and for the corresponding generating functions. Our exposition will be brief since the necessary machinery has been introduced in the previous section and the generating functions in this case are much simpler. Moreover, we restrict our computations to the most relevant issues, namely, the asymptotic expressions for the number of outerplanar graphs, and the mean and variance of the expected number of edges in random outerplanar graphs.

As mentioned in the introduction, a 2-connected outerplanar can be seen as a dissection of a convex polygon. The ordinary GF $A(x, y)$ for polygon dissections, where x marks vertices and y edges, can be easily obtained with the method introduced in [3] for counting polygon dissections with respect to the number of faces. Indeed, let K be a convex polygon with n vertices and fix an edge e of K. A dissection of K is either a single edge or an ordered sequence of $k \geq 2$ dissections along a face containing e, where $k-1$ pairs of vertices are identified. Thus we have

$$A(x, y) = yx^2 + y \sum_{k \geq 2} \frac{A(x, y)^k}{x^{k-1}} = yx^2 + \frac{yA^2}{x - A}.$$

The solution to the previous equation with non-negative terms is

$$A(x, y) = \sum a_{n,k} y^k x^n = \frac{x(1 + yx - \sqrt{1 - 2yx - 4y^2x + y^2x^2})}{2 + 2y}.$$

Returning to outerplanar graphs, each dissection of K gives rise to $(n-1)!/2$ (the number of labellings of a non-oriented cycle) 2-connected outerplanar graphs, except for the special case $n = 2$. Hence

$$b_{n,q} = a_{n,q}(n-1)!/2, \quad n \geq 3, \quad \text{and} \quad b_{2,1} = 1.$$

In terms of the corresponding generating functions (recall that $B(x, y)$ is an exponential GF), the former relations translate into

$$B'(x, y) = \frac{\frac{1}{x}A(x, y) + yx}{2} = \frac{1 + xy(3 + 2y) - \sqrt{1 - 2xy - 4y^2x + x^2y^2}}{4(1 + y)}.$$

For $y = 1$, the smallest positive root of the radicand $1 - 6x + x^2$ is

$$R = 3 - 2\sqrt{2} \approx 0.17157,$$

which is then the radius of convergence of $B(x) = B(x, 1)$.

Since a graph is outerplanar if and only if its connected components are outerplanar, and the blocks in the components are also outerplanar, the relations we had in the previous section between B, C and G also hold, that is

$$G(x) = \exp(C(x)), \qquad xC'(x) = x \exp(B'(xC'(x))).$$

It follows that, for fixed y, the functional inverse of $F(x, y) = xC'(x, y)$ is

$$\Psi_y(u) = u\mathrm{e}^{-B'(u,y)}.$$

Given the explicit form we have for B', it is easy to check that $\Psi_1'(u)$ has a root $\tau \approx 0.17076$ in $[0, R]$; in fact, τ is the smallest positive root of the equation

$$3u^4 - 28u^3 + 70u^2 - 58u + 8 = 0.$$

Consequently, the radius of convergence of $F(x, 1)$ and $C(x)$ is equal to

$$\rho = \Psi(\tau) \approx 0.13659.$$

Theorem 5.1. *The number h_n of outerplanar graphs is asymptotically*

$$h_n \sim h \cdot n^{-5/2} \cdot \rho^{-n} n!$$

where $\rho \approx 0.13659$ and $h \approx 0.018216$.

Proof. The value of the dominant singularity ρ has been determined previously. Since Ψ' has a root in its domain of definition, the inverse function $F(x) = xC'(x)$ has a singular expansion of square-root type in $X = \sqrt{1 - x/\rho}$; hence $C(x)$ has a singular expansion whose dominant term is $X^{3/2}$ and from this follows the subexponential term $n^{-5/2}$. Finally, the constant h is computed as in the previous section using the evaluation of $\Psi''(\tau)$. □

The proof of the next theorem is omitted, since it follows exactly the same pattern as the proof of Theorem 4.2. Computations in this case are simpler due to the explicit expression for Ψ.

Theorem 5.2. *Let X_n denote the number of edges in a random outerplanar graph with n vertices. Then X_n is asymptotically normal and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \zeta n, \qquad \sigma_n^2 \sim \eta n, \tag{5.1}$$

where $\zeta \approx 1.56251$ *and* $\eta \approx 0.22399$. *The same is true, with the same constants, for connected random outerplanar graphs.*

6. The number of connected components

In this section we determine limit laws for the number of connected components. A sequence X_n of discrete random variables converges to a discrete random variable X if, for every integer k,

$$\text{Prob}\{X_n = k\} \to \text{Prob}\{X = k\}, \quad \text{as } n \to \infty.$$

In the next statement, convergence is to a *shifted* Poisson law because the number of components is always strictly positive.

Theorem 6.1. *The distribution of the number of connected components in random series–parallel graphs is asymptotically a shifted Poisson law* $1 + P(\nu)$ *with parameter* $\nu \approx 0.11761$. *The same result holds for outerplanar graphs, in this case the parameter of the Poisson law being* $\xi \approx 0.14889$. *As a consequence the probability that a random series–parallel graph is connected tends to* $\mathrm{e}^{-\nu} \approx 0.88904$, *and to* $\mathrm{e}^{-\xi} \approx 0.86166$ *for outerplanar graphs.*

Proof. We follow the same approach as the proof of Theorem 6 in [7]. We present the proof for SP graphs; the case of outerplanar graphs is analogous.

For fixed k, the generating function of SP graphs with exactly k connected components is

$$\frac{1}{k!}C(x)^k.$$

For fixed k we have

$$[x^k]C(x)^k \sim kC_0^{k-1}[x^n]C(x).$$

Hence the probability that a random planar SP has exactly k components is asymptotically

$$\frac{[x^n]C(x)/k!}{[x^n]G(x)} \sim \frac{kC_0^{k-1}}{k!}\mathrm{e}^{-C_0} = \frac{\nu^{k-1}}{(k-1)!}\mathrm{e}^{-\nu}.$$

If we let $\nu = C(\rho) = C_0$, the evaluation of $C(x)$ at its dominant singularity, then the previous expression implies convergence to a shifted Poisson law of parameter ν. Since we know the local developments around the dominant singularity, we can compute C_0 exactly. □

7. Graphs without a $K_{2,3}$ minor

In this section we analyze briefly the class of graphs that do not contain $K_{2,3}$ as a minor; or equivalently, since $K_{2,3}$ has maximum degree 3, graphs that do not contain $K_{2,3}$ as a subdivision. They form a class strictly larger than the class of outerplanar graphs; as we are going to see, they are not far from this class.

Let G be a 2-connected graph not containing $K_{2,3}$ as a minor. Then either G is outerplanar (no K_4 minor) or else G contains K_4 as a minor, and hence also as a subdivision. But if we subdivide just one edge of K_4, a $K_{2,3}$ minor shows up. Hence this subdivision can be only K_4. If G contains an additional vertex x then, by 2-connectivity, there exist an edge yz of the given K_4 and two internally disjoint paths from x to y and z; hence again we have a $K_{2,3}$ minor.

In conclusion, if G is 2-connected and does not contain $K_{2,3}$ as a minor, then either G is outerplanar or $G = K_4$. If we apply the notation of the previous section to the new class, then the generating function $B(x, y)$ is the same as for the class of outerplanar graphs, plus the addition of a single monomial $y^6x^4/4!$ corresponding to the exceptional graph K_4. Hence

$$B'(x, y) = \frac{y^6x^3}{6} + \frac{1 + xy(3 + 2y) - \sqrt{1 - 2xy - 4y^2x + x^2y^2}}{4(1 + y)}.$$

From this, following the same steps as in the previous section, we obtain the following. Details are omitted in order to avoid repetition.

Theorem 7.1. *The number s_n of graphs not containing $K_{2,3}$ as a minor is asymptotically*

$$s_n \sim s \cdot n^{-5/2} \cdot \rho^{-n} n!$$

where $\rho \approx 0.13648$ and $s \approx 0.018288$.

Theorem 7.2. *Let X_n denote the number of edges in a random graph not containing $K_{2,3}$ as a minor with n vertices. Then X_n is asymptotically normal and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \zeta n, \qquad \sigma_n^2 \sim \eta n, \tag{7.1}$$

where $\zeta \approx 1.56325$ and $\eta \approx 0.224206$. The same is true, with the same constants, for connected random graphs not containing $K_{2,3}$ as a minor.

Theorem 7.3. *The distribution of the number of connected components in random graphs that do not contain $K_{2,3}$ as a minor is asymptotically a shifted Poisson law $1 + P(\xi_0)$ with parameter $\xi_0 \approx 0.14879$. As a consequence the probability of connectedness tends to $e^{-\xi_0} \approx 0.86175$.*

8. Concluding remarks

We conclude with a table showing the values of the main parameters for the classes studied in this paper, together with the class of planar graphs studied in [7]. For a given class of labelled graphs $\mathcal{G}$, the growth constant γ is ρ^{-1}, where ρ is the dominant singularity of the associated generating function $G(x)$. In other words, $\gamma = \lim_{n\to\infty}(g_n/n!)^{1/n}$. In the second column we display the constant κ such that the expected number of edges is asymptotically κn; the third column contains the parameter ν of the shifted Poisson law for the number of connected components; finally the last column shows the asymptotic probability p of connectedness.

Class of graphs	γ	κ	ν	p
Planar	27.2268	2.2132	0.0374	0.9632
Series–parallel	9.0733	1.6167	0.1176	0.8890
Outerplanar	7.3209	1.5625	0.1489	0.8616
No $K_{2,3}$ minor	7.3270	1.5632	0.1488	0.8617

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