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Degree distribution in random planar graphs ☆

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ABSTRACT

We prove that for each $k \geqslant 0$, the probability that a root vertex in a random planar graph has degree k tends to a computable constant d_k, so that the expected number of vertices of degree k is asymptotically $d_k n$, and moreover that $\sum_k d_k = 1$. The proof uses the tools developed by Giménez and Noy in their solution to the problem of the asymptotic enumeration of planar graphs, and is based on a detailed analysis of the generating functions involved in counting planar graphs. However, in order to keep track of the degree of the root, new technical difficulties arise. We obtain explicit, although quite involved expressions, for the coefficients in the singular expansions of the generating functions of interest, which allow us to use transfer theorems in order to get an explicit expression for the probability generating function $p(w) = \sum_k d_k w^k$. From this we can compute the d_k to any degree of accuracy, and derive the asymptotic estimate $d_k \sim c \cdot k^{-1/2} q^k$ for large values of k, where $q \approx 0.67$ is a constant defined analytically.

1. Introduction

In this paper all graphs are simple and labelled with labels $\{1, 2, \dots, n\}$. As usual, a graph is planar if it can be embedded in the plane without edge crossings. A planar graph together with a particular embedding in the plane is called a map. There is a rich theory of counting maps, and part of it is needed later. However, in this paper we consider planar graphs as combinatorial objects, regardless of how many non-equivalent topological embeddings they may have.

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Random planar graphs were introduced by Denise, Vasconcellos and Welsh [9], and since then they have been widely studied. Let us first recall the probability model. Let $\mathcal{G}_n$ be the family of (labelled) planar graphs with n vertices. A random planar graph $\mathcal{R}_n$ is a graph drawn from $\mathcal{G}_n$ with the uniform distribution, that is, all planar graphs with n vertices have the same probability of being chosen. As opposed to the classical Erdős–Rényi model, we cannot produce a random planar graph by drawing edges independently. In fact, our analysis of random planar graphs relies on exact and asymptotic counting.

Several natural parameters defined on $\mathcal{R}_n$ have been studied, starting with the number of edges, which is probably the most basic one. Partial results where obtained in [9,16,26,7], until it was shown by Giménez and Noy [17] that the number of edges in random planar graphs obeys asymptotically a normal limit law with linear expectation and variance. The expectation is asymptotically κn, where $\kappa \approx 2.21326$ is a well-defined analytic constant. This implies that the average degree of the vertices is $2\kappa \approx 4.42652$.

McDiarmid, Steger and Welsh [23] showed that with high probability a planar graph has a linear number of vertices of degree k, for each $k \geqslant 1$. Our main result is that for each $k \geqslant 1$, the expected number of vertices of degree k is asymptotically $d_k n$, for computable constants $d_k \geqslant 0$. This is equivalent to saying that the probability that a fixed vertex, say vertex 1, has degree k tends to a limit d_k as n goes to infinity. In Theorem 6.13 we show that this limit exists and we give an explicit expression for the probability generating function

$$p(w) = \sum_{k\geqslant 1} d_k w^k,$$

from which the coefficients d_k can be computed to any degree of accuracy. Moreover, we show that $p(w)$ is indeed a probability generating function, that is, $\sum d_k = 1$.

The proof is based on a detailed analysis of the generating functions involved in counting planar graphs, as developed in [17], where the long standing problem of estimating the number of planar graphs was solved. However, in this case we need to keep track of the degree of a root vertex, and this makes the analysis considerably more difficult.

Here is a sketch of the paper. We start with some preliminaries, including the fact that for the degree distribution it is enough to consider connected planar graphs, and that $d_0 = 0$. Then we obtain the degree distribution in simpler families of planar graphs: outerplanar graphs (Section 3) and series-parallel graphs (Section 4). We recall that a graph is series-parallel if it does not contain the complete graph K_4 as a minor; equivalently, if it does not contain a subdivision of K_4. Since both K_5 and $K_{3,3}$ contain a subdivision of K_4, by Kuratowski's theorem, or directly, a series-parallel graph is planar. An outerplanar graph is a planar graph that can be embedded in the plane so that all vertices are incident to the outer face. They are characterised as those graphs not containing a minor isomorphic to (or a subdivision of) either K_4 or $K_{2,3}$. These results are interesting on their own and pave the way to the more complex analysis of general planar graphs. We remark that the degree distribution in these simpler cases has been obtained independently in [5] using different techniques.

In Section 5 we compute the generating function of 3-connected maps taking into account the degree of the root, which is an essential piece in proving the main result. We rely on a classical bijection between rooted maps and rooted quadrangulations [8,24], and again the main difficulty is to keep track of the root degree.

The task is completed in Section 6, which contains the analysis for planar graphs. First we have to obtain a closed form for the generating function $B^{\bullet}(x, y, w)$ of rooted 2-connected planar graphs, where x marks vertices, y edges, and w the degree of the root: the main problem we encounter here is solving a differential equation involving algebraic functions and other functions defined implicitly. The second step is to obtain singular expansions of the various generating functions near their dominant singularities; this is particularly demanding, as the coefficients of the singular expansions are rather complex expressions. Finally, using a technical lemma on singularity analysis and composition of singular expansions, we are able to work out the asymptotics for the generating function $C^{\bullet}(x, y, w)$ of rooted connected planar graphs, and from this the probability generating function can be computed exactly. We also compute the degree distribution for 3-connected and 2-connected

Table 1
Degree distribution for small degrees.

	d_1	d_2	d_3	d_4	d_5	d_6
Outerplanar	0.1365937	0.2875331	0.2428739	0.1550795	0.0874382	0.0460030
Series-parallel	0.1102133	0.3563715	0.2233570	0.1257639	0.0717254	0.0421514
Planar	0.0367284	0.1625794	0.2354360	0.1867737	0.1295023	0.0861805
Planar 2-connected	0	0.1728434	0.2481213	0.1925340	0.1325252	0.0879779
Planar 3-connected	0	0	0.3274859	0.2432187	0.1594160	0.1010441

Table 2
Asymptotic estimates of d_k for large k. The constants c resp. c' and q in each case are defined analytically. The two approximate values in the last row are exactly the same constant.

	Connected	q	2-connected	q	3-connected	q
Outerplanar	$c \cdot k^{1/4} e^{c'\sqrt{k}} q^k$	0.3808138	$c \cdot k q^k$	$\sqrt{2}-1$		
Series-parallel	$c \cdot k^{-3/2} q^k$	0.7504161	$c \cdot k^{-3/2} q^k$	0.7620402		
Planar	$c \cdot k^{-1/2} q^k$	0.6734506	$c \cdot k^{-1/2} q^k$	0.6734506	$c \cdot k^{-1/2} q^k$	$\sqrt{7}-2$

planar graphs. Finally in Section 7 we show that there exists a computable degree distribution for planar graphs with a given edge density or, equivalently, given average degree.

For each of the three families studied we obtain an explicit expression, of increasing complexity, for the probability generating function $p(w) = \sum_{k \geqslant 1} d_k w^k$. Theorems 3.3, 4.5 and 6.13 give the exact expressions in each case. The expression we obtain for $p(w)$ in the planar case is quite involved and needs several pages to write it down. However, the functions involved are elementary and computations can be performed with the help of MAPLE.

Remark. Due to lack of space, several of the expressions needed to compute the probability generating functions for the degree distribution in planar graphs are not included in this paper, but are fully accessible in the appendix of [12]. We will refer to this appendix when necessary.

Table 1 shows the approximate values of the probabilities d_k for small values of k, which are obtained by extracting coefficients in the power series $p(w)$, and can be computed to any degree of accuracy.

We also determine the asymptotic behaviour for large k, and the result we obtain in each case is a geometric distribution modified by a suitable subexponential term. We perform the analysis for connected and 2-connected graphs, and also for 3-connected graphs in the planar case. Table 2 contains a summary of the main results from Sections 3, 4 and 6. It is worth noticing that the shape of the asymptotic estimates for planar graphs agrees with the general pattern for the degree distribution in several classes of maps, where maps are counted according to the number of edges [21].

As a final remark, let us mention that in a companion paper [11], we prove a central limit theorem for the number of vertices of degree k in outerplanar and series-parallel graphs, together with strong concentration results. It remains an open problem to show that this is also the case for planar graphs. Our results in the present paper show that the degree distribution exists and moreover can be computed explicitly.

2. Preliminaries

For background on generating functions associated to planar graphs, we refer to [17] and [6], and to [25] for a less technical description. For background on singularity analysis of generating functions, we refer to [13] and [14].

For each class of graphs under consideration, c_n and b_n denote, respectively, the number of connected and 2-connected graphs on n vertices. For the three graphs classes under consideration, outerplanar, series-parallel, and planar, we have both for c_n and b_n estimates of the form

$$c \cdot n^{-\alpha} \rho^{-n} n!, \tag{2.1}$$

where c, α and ρ are suitable constants [6,17]. For outerplanar and series-parallel graphs we have $\alpha = -5/2$, whereas for planar graphs $\alpha = -7/2$. A general methodology for graph enumeration explaining these critical exponents has been developed in [19].

We introduce the exponential generating functions $C(x) = \sum c_n x^n/n!$ and $B(x) = \sum b_n x^n/n!$. Let C_k be the exponential generating function (GF for short) for rooted connected graphs, where the root bears no label and has degree k; that is, the coefficient $[x^n/n!]C_k(x)$ equals the number of rooted connected graphs with $n+1$ vertices, in which the root has no label and has degree k. Analogously we define B_k for 2-connected graphs. Also, let

$$B^\bullet(x, w) = \sum_{k\geqslant 2} B_k(x) w^k, \qquad C^\bullet(x, w) = \sum_{k\geqslant 0} C_k(x) w^k.$$

A basic property shared by the classes of outerplanar, series-parallel and planar graphs is that a connected graph G is in the class if and only if the 2-connected components of G are also in the class. As shown in [17], this implies the basic equation

$$C'(x) = e^{B'(xC'(x))}$$

between univariate GFs. If we introduce the degree of the root, then the equation becomes

$$C^\bullet(x, w) = e^{B^\bullet(xC'(x), w)}. \tag{2.2}$$

The reason is that only the 2-connected components containing the root vertex contribute to its degree.

Our goal in each case is to estimate $[x^n]C_k(x)$, since the limiting probability that a given fixed vertex has degree k is equal to

$$d_k = \lim_{n\to\infty} \frac{[x^n]C_k(x)}{[x^n]C'(x)}. \tag{2.3}$$

Notice that in the denominator we have the coefficient of $C'(x)$, corresponding to vertex rooted graphs in which the root bears no label, in agreement with the definition of $C_k(x)$.

A first observation is that the asymptotic degree distribution is the same for connected members of a class as for all members in the class. Let $G(x)$ be the GF for all members in the class, and let $G_k(x)$ be the GF of all rooted graphs in the class, where the root has degree k. Then we have

$$G(x) = e^{C(x)}, \qquad G_k(x) = C_k(x)e^{C(x)}.$$

The first equation is standard, and in the second equation the factor $C_k(x)$ corresponds to the connected component containing the root, and the second factor to the remaining components. The functions $G(x)$ and $C(x)$ have the same dominant singularity. Given the singular expansions of $G(x)$ and $C(x)$ at the dominant singularity in each of the cases under consideration (see [17,18]), it follows that

$$\lim_{n\to\infty} \frac{[x^n]G_k(x)}{[x^n]G'(x)} = \lim_{n\to\infty} \frac{[x^n]C_k(x)}{[x^n]C'(x)}.$$

Hence, in each case we only need to determine the degree distribution for connected graphs. A more intuitive explanation is that the largest component in random planar graphs eats up almost everything: the expected number of vertices not in the largest component is constant [22].

Another observation is that $d_0 = 0$ and $d_1 = \rho$, where ρ is the constant appearing in the estimate (2.1) for c_n; as we are going to see, ρ is the radius of convergence of $C(x)$. Indeed, there are no vertices of degree zero in a connected graph, and the proportion of connected graphs in which the root has degree one is $n(n-1)c_{n-1}/nc_n \sim \rho$.

The general approach we use for computing the d_k is the following. Let $f(x) = xC'(x)$ and let $H(z) = e^{B^\bullet(z,w)}$, where w is considered as a parameter. Let also ρ be the radius of convergence of $C(x)$, which is the same as that of $f(x)$. According to (2.2) we have to estimate $[x^n]H(f(x))$, and this will depend on the behaviour of $H(z)$ at $z = f(\rho)$. In the outerplanar and series-parallel cases, $H(z)$ turns out be analytic at $f(\rho)$, whereas in the planar case we have a critical composition scheme, that is, the dominant singularity of $H(z)$ is precisely $f(\rho)$. This is a fundamental difference and we have to use different tools accordingly. Another difference is that $B^\bullet$ is much more difficult to determine for planar graphs.

Finally we comment on an asymptotic method that we apply several times. Suppose that $f(z) = \sum_{n\geqslant 0} a_n z^n$ is the power series representation of an analytic function and $\rho > 0$ is the radius of convergence of $f(z)$. We say that $f(z)$ is analytic in a Δ-region if $f(z)$ can be analytically continued to a region of the form

$$\Delta = \{z \in \mathbb{C}:\ |z| < \rho + \eta,\ \left|\arg(z-\rho)\right| > \theta\}, \tag{2.4}$$

for some $\eta > 0$ and $0 < \theta < \frac{\pi}{2}$.

If we know that $|f(z)| \leqslant C \cdot |1 - z/\rho|^{-\alpha}$ for $z \in \Delta$, then it follows that $|a_n| \leqslant C' \cdot \rho^{-n} n^{\alpha-1}$ for some $C' > 0$ that depends on C, α, η, and θ; see [13]. In particular, if we know that $f(z)$ is analytic in a Δ-region and has a local representation of the form

$$f(z) = A_0 + A_2 Z^2 + A_3 Z^3 + O(Z^4), \tag{2.5}$$

where $Z = \sqrt{1 - z/\rho}$, then it follows that $|f(z) - A_0 - A_2 Z^2 - A_3 Z^3| \leqslant C \cdot |1 - z/\rho|^2$ for $z \in \Delta$. As a consequence

$$a_n = \frac{3A_3}{4\sqrt{\pi}} \rho^{-n} n^{-5/2} + O(\rho^{-n} n^{-3}).$$

In fact, we focus mainly on the derivation of local expansions of the form (2.5). The analytic continuation to a Δ-region is usually easy to establish. We either have explicit equations in known functions or implicit equations where we can continue analytically with the help of the implicit function theorem.

As a key situation, we consider a function $y = y(z)$ that has a power series representation at $z_0 = 0$ and that satisfies an analytic functional equation $\Phi(y, z) = 0$. Suppose that we have $y(z_0) = y_0$, so that $\Phi(y_0, z_0) = 0$, and $\Phi_y(y_0, z_0) \neq 0$. Then the implicit function theorem implies that $y(z)$ can be extended analytically to a neighbourhood of $z = z_0$. In particular it follows that $y(z)$ cannot be singular at $z = z_0$. On the other hand if we know that there exists z_0 and $y_0 = y(z_0)$ with

$$\Phi(y_0, z_0) = 0 \quad \text{and} \quad \Phi_y(y_0, z_0) = 0$$

and the conditions

$$\Phi_z(y_0, z_0) \neq 0 \quad \text{and} \quad \Phi_{yy}(y_0, z_0) \neq 0,$$

then z_0 is a singularity of $y(z)$ and there is a local expansion of the form

$$y(z) = Y_0 + Y_1 Z + Y_1 Z^2 + \cdots,$$

where $Z = \sqrt{1 - z/z_0}$, $Y_0 = y_0$ and $Y_1 = -\sqrt{2z_0 \Phi_z(y_0, z_0)/\Phi_{yy}(y_0, z_0)}$; see [10]. In our applications it is usually easy to show that $\Phi_y(y(z), z) \neq 0$ for $|z| \leqslant z_0$ and $z \neq z_0$. Hence, in this situation $z = z_0$ is the only singularity on the boundary of the circle $|z| \leqslant z_0$ and $y(z)$ can be analytically continued to a Δ-region.

3. Outerplanar graphs

In this section $C(x)$ and $B(x)$ now denote the GFs of connected and 2-connected, respectively, outerplanar graphs. We start by recalling some results from [6]. From the equivalence between rooted 2-connected outerplanar graphs and polygon dissections where the vertices are labelled $1, 2, \ldots, n$ in clockwise order (see Section 5 in [6] for details), we have the explicit expression

$$B'(x) = \frac{1 + 5x - \sqrt{1 - 6x + x^2}}{8}.$$

The radius of convergence of $B(x)$ is $3-2\sqrt{2}$, the smallest positive root of $1-6x+x^2=0$. The radius of convergence of $C(x)$ is $\rho=\psi(\tau)$, where $\psi(u)=ue^{-B'(u)}$, and τ is the unique positive root of $\psi'(u)=0$. Notice that τ satisfies $\tau B''(\tau)=1$. The approximate values are $\tau\approx 0.17076$ and $\rho\approx 0.13659$. We also need the fact that ψ is the functional inverse of $xC'(x)$, so that $\tau=\rho C'(\rho)$.

Let

$$D(x)=\frac{1+x-\sqrt{1-6x+x^2}}{4} \tag{3.1}$$

and let $D_k(x)=x(2D(x)-x)^{k-1}$ (D_k is the GF for polygon dissections in which the root vertex has degree k). Then we have

$$B_k=\frac{1}{2}D_k,\quad k\geqslant 2,\qquad B_1=x.$$

By summing a geometric series we have an explicit expression for $B^\bullet$, namely

$$B^\bullet(x,w)=xw+\sum_{k=2}^{\infty}\frac{x}{2}\bigl(2D(x)-x\bigr)^{k-1}w^k=xw+\frac{xw^2}{2}\frac{2D(x)-x}{1-(2D(x)-x)w}. \tag{3.2}$$

Our goal is to analyse $B^\bullet(x,w)$ and $C^\bullet(x,w)=\exp(B^\bullet(xC'(x),w))$. For this we need the following technical lemma.

Lemma 3.1. *Let $f(x)=\sum_{n\geqslant 0}a_nx^n/n!$ denote the exponential generating function of a sequence a_n of non-negative real numbers and assume that $f(x)$ has exactly one dominating square-root singularity at $x=\rho$ of the form*

$$f(x)=g(x)-h(x)\sqrt{1-x/\rho},$$

where $g(x)$ and $h(x)$ are analytic at $x=\rho$ and $f(x)$ has an analytic continuation to the region $\{x\in\mathbb{C}\colon |x|<\rho+\varepsilon\}\setminus\{x\in\mathbb{R}\colon x\geqslant\rho\}$ for some $\varepsilon>0$. Further, let $H(x,z)$ denote a function that is analytic for $|x|<\rho+\varepsilon$ and $|z|<f(\rho)+\varepsilon$ such that $H_z(\rho,f(\rho))\neq 0$. Then the function

$$f_H(x)=H\bigl(x,f(x)\bigr)$$

has a power series expansion $f_H(x)=\sum_{n\geqslant 0}b_nx^n/n!$ and the coefficients b_n satisfy

$$\lim_{n\to\infty}\frac{b_n}{a_n}=H_z\bigl(\rho,f(\rho)\bigr). \tag{3.3}$$

Proof. From $f(x)=g(x)-h(x)\sqrt{1-x/\rho}$ it follows from singularity analysis [14] that the sequence a_n is given asymptotically by

$$\frac{a_n}{n!}\sim\frac{h(\rho)}{2\sqrt{\pi}}\rho^{-n}n^{-3/2}.$$

Since H is analytic at $f(\rho)$, it has a Taylor series

$$H(x,z)=H\bigl(\rho,f(\rho)\bigr)+H_z\bigl(\rho,f(\rho)\bigr)\bigl(z-f(\rho)\bigr)+H_x\bigl(\rho,f(\rho)\bigr)(x-\rho)+\cdots.$$

The function $f_H(x)$ has also a square-root singularity at $x=\rho$ with a singular expansion, obtained by composing the analytic expansion of $H(x,z)$ with the singular expansion of $f(x)$, namely

$$f_H(x)=H\bigl(\rho,f(\rho)\bigr)-H_z\bigl(\rho,f(\rho)\bigr)h(\rho)\sqrt{1-\frac{x}{\rho}}+O\bigl(|1-x/\rho|\bigr).$$

Consequently, the coefficients b_n can be estimated as

$$\frac{b_n}{n!} \sim \frac{H_z(\rho, f(\rho))h(\rho)}{2\sqrt{\pi}}\rho^{-n}n^{-3/2},$$

and (3.3) follows. □

We are ready for obtaining the degree distribution of two-connected outerplanar graphs and connected outerplanar graphs. Both results have been obtained independently in [5], and our respective results agree.

Theorem 3.2. *Let d_k be the limiting probability that a vertex of a random two-connected outerplanar graph has degree k. Then*

$$p(w) = \sum_{k\geqslant 1} d_k w^k = \frac{2(3-2\sqrt{2})w^2}{(1-(\sqrt{2}-1)w)^2} = \sum_{k\geqslant 2} 2(k-1)(\sqrt{2}-1)^k w^k.$$

Moreover $p(1) = 1$, so that the d_k are indeed a probability distribution.

Proof. Since $B^\bullet(x,1) = B'(x)$ and $D(x) = 2B'(x) - x$ we can represent $B^\bullet(x,w)$ as

$$B^\bullet(x,w) = xw + \frac{xw^2}{2}\frac{4B'(x)-3x}{1-(4B'(x)-3x)w}.$$

Hence, by applying Lemma 3.1 with $f(x) = B'(x)$ and

$$H(x,z) = xw + \frac{xw^2}{2}\frac{4z-3x}{1-(4z-3x)w}$$

we obtain

$$p(w) = \lim_{n\to\infty}\frac{[x^n]B^\bullet(x,w)}{[x^n]B'(x)} = \frac{2(3-2\sqrt{2})w^2}{(1-(\sqrt{2}-1)w)^2}.$$

Note that $\rho = 3-2\sqrt{2}$ and that w is considered here as an additional (complex) parameter. □

Theorem 3.3. *Let d_k be the limiting probability that a vertex of a random connected outerplanar graph has degree k. Then*

$$p(w) = \sum_{k\geqslant 1} d_k w^k = \rho\cdot\frac{\partial}{\partial x}e^{B^\bullet(x,w)}\bigg|_{x=\rho C'(\rho)},$$

where $B^\bullet$ is given by Eqs. (3.1) *and* (3.2)*.*

Moreover $p(1) = 1$, so that the d_k are indeed a probability distribution and we have asymptotically, as $k\to\infty$

$$d_k \sim c_1 k^{1/4} e^{c_2\sqrt{k}} q^k,$$

where $c_1 \approx 0.667187$, $c_2 \approx 0.947130$, and $q = 2D(\tau) - \tau \approx 0.3808138$.

Proof. We have

$$\sum_k C_k(x)w^k = C^\bullet(x,w) = e^{B^\bullet(xC'(x),w)}.$$

The radius of convergence $3-2\sqrt{2}$ of $B(x)$ is larger than $\rho C'(\rho) = \tau \approx 0.17076$. Hence we can apply the previous lemma with $f(x) = xC'(x)$ and $H(z) = e^{B^\bullet(z,w)}$, where w is considered as a parameter. Then we have

$$\frac{\partial}{\partial x} e^{B^\bullet(x,w)}\bigg|_{x=\rho C'(\rho)} = \lim_{n\to\infty} \frac{[x^n]C^\bullet(x,w)}{[x^n]xC'(x)} = \lim_{n\to\infty} \sum_{k\geqslant 1} \rho^{-1} \frac{[x^n]C_k(x)}{[x^n]C'(x)} w^k = \rho^{-1} \sum_{k\geqslant 1} d_k w^k,$$

and the result follows.

For the second assertion let us note that $B^\bullet(x,1) = B'(x)$. If we recall that $\rho C'(\rho) = \tau$ and $\tau B''(\tau) = 1$, then

$$p(1) = \rho e^{B'(\tau)} B''(\tau) = \rho C'(\rho)\tau^{-1} = 1.$$

In order to get an asymptotic expansion for d_k we have to compute $p(w)$ explicitly:

$$p(w) = \rho \frac{\tau(2D(\tau)-\tau)(2D'(\tau)-1)w^2}{2(1-(2D(\tau)-\tau)w)^2} \exp\left(\tau w + \frac{\tau(2D(\tau)-\tau)w^2}{2(1-(2D(\tau)-\tau)w)}\right).$$

This is a function that is admissible in the sense of Hayman [20]. Hence, it follows that

$$d_k \sim \frac{p(r_k) r_k^{-k}}{\sqrt{2\pi b(r_k)}},$$

where r_k is given by the equation $r_k p'(r_k)/p(r_k) = k$ and $b(w) = w^2 p''(w)/p(w) + wp'(w)/p(w) - (wp'(w)/p(w))^2$. A standard calculation gives the asymptotic expansion for the coefficients d_k. □

With the help of the explicit expression for $p(w)$ we obtain the values for small k shown in Table 1.

4. Series-parallel graphs

In this section $C(x)$ and $B(x)$ now denote the GFs of connected and 2-connected, respectively, series-parallel graphs. First we recall the necessary results from [6]. The radius of convergence of $B(x)$ is $R \approx 0.1280038$. The radius of convergence of $C(x)$ is, as for outerplanar graphs, $\rho = \psi(\tau)$, where $\psi(u) = ue^{-B'(u)}$, and τ is the unique positive root of $\psi'(u) = 0$. Again we have that ψ is the functional inverse of $xC'(x)$, so that $\tau = \rho C'(\rho)$, and τ satisfies $\tau B''(\tau) = 0$. The approximate values are $\tau \approx 0.1279695$ and $\rho \approx 0.1102133$.

In order to study 2-connected series-parallel graphs, we need to consider series-parallel networks, as in [6]. We recall that a network is a graph with two distinguished vertices, called poles, such that the graph obtained by adding an edge between the two poles is 2-connected. Let $D(x, y, w)$ be the exponential GF of series-parallel networks, where x, y, w mark, respectively, vertices, edges, and the degree of the first pole. Define $S(x, y, w)$ analogously for series networks. Then we have

$$D(x,y,w) = (1+yw)e^{S(x,y,w)} - 1,$$
$$S(x,y,w) = \big(D(x,y,w) - S(x,y,w)\big)xD(x,y,1).$$

The first equation reflects the fact that a network is a parallel composition of series networks, and the second one the fact that a series network is obtained by connecting a non-series network with an arbitrary network (see [27] for details); the factor $D(x, y, 1)$ appears because we only keep track of the degree of the first pole.

Remark. For the results of the present section, we do not need to take into account the number of edges and we could set $y = 1$ everywhere. However, in the case of planar graphs we do need the GF according to all three variables and it is convenient to present already here the full development. In the proof of the main result of this section, Theorem 4.5, we just set $y = 1$.

Set $E(x, y) = D(x, y, 1)$, the GF for series-parallel networks without marking the degree of the root, which satisfies (see [6]) the equation

$$\log\left(\frac{1+E(x,y)}{1+y}\right)=\frac{xE(x,y)^2}{1+xE(x,y)}. \tag{4.1}$$

From the previous equations it follows that

$$\log\left(\frac{1+D(x,y,w)}{1+yw}\right)=\frac{xE(x,y)D(x,y,w)}{1+xE(x,y)}. \tag{4.2}$$

Let now $B_k^\bullet(x,y)$ be the GF for 2-connected series-parallel graphs, where the root bears no label and has degree k, and where y marks edges. Then we have the following relation.

Lemma 4.1.

$$w\frac{\partial B^\bullet(x,y,w)}{\partial w}=xywe^{S(x,y,w)}.$$

Proof. We have $w\partial B^\bullet(x,y,w)/\partial w=\sum_{k\geqslant 1}kB_k^\bullet(x,y)w^k$. The last summation enumerates rooted 2-connected graphs with a distinguished edge incident to the root, and of these there as many as networks containing the edge between the poles (this corresponds to the term $e^{S(x,y,w)}$). The degree of the root in a 2-connected graph corresponds to the degree of the first pole in the corresponding network, hence the equation follows. □

From the previous equation it follows that

$$B^\bullet(x,y,w)=xy\int e^{S(x,y,w)}\,dw. \tag{4.3}$$

Our next task is to get rid of the integral and to express $B^\bullet$ in terms of D. Recall that $E(x,y)=D(x,y,1)$.

Lemma 4.2. *The generating function of rooted* 2*-connected series-parallel graphs is equal to*

$$B^\bullet(x,y,w)=x\left(D(x,y,w)-\frac{xE(x,y)}{1+xE(x,y)}D(x,y,w)\left(1+\frac{D(x,y,w)}{2}\right)\right).$$

Proof. We use the techniques developed in [17,6] in order to integrate (4.3) in closed-form.

$$\int e^S\,dw=\int\frac{1+D}{1+yw}\,dw=y^{-1}\log(1+yw)+\int\frac{D}{1+yw}\,dw.$$

Now we integrate by parts and

$$\int\frac{D}{1+yw}\,dw=y^{-1}\log(1+yw)D-\int y^{-1}\log(1+yw)\frac{\partial D}{\partial w}\,dw.$$

For the last integral we change variables $t=D(x,y,w)$ and use the fact that $\log(1+yw)=\log(1+t)-xEt/(1+xE)$. We obtain

$$\int\log(1+yw)\frac{\partial D}{\partial w}\,dw=\int_0^D\log(1+t)\,dt-\frac{xE}{1+xE}\int_0^D t\,dt.$$

Now everything can be integrated in closed form and, after a simple manipulation, we obtain the result as claimed. □

In order to prove the main results in this section we need the singular expansions of $D(x,y)$ and $B(x,y)$, for a fixed value of y, near the dominant singularity $R(y)$.

Lemma 4.3. *For $|w| \leqslant 1$ and for fixed y (sufficiently close to 1) the dominant singularity of the functions $E(x,y)$, $D(x,y,w)$, and $B^\bullet(x,y,w)$ (considered as functions in x) is given by $x = R(y)$, where $R(y)$ is an analytic function in y with $R = R(1) \approx 0.1280038$. Furthermore, we have the following local expansion:*

$$E(x,y) = E_0(y) + E_1(y)X + E_2(y)X^2 + \cdots,$$
$$D(x,y,w) = D_0(y,w) + D_1(y,w)X + D_2(y,w)X^2 + \cdots,$$
$$B^\bullet(x,y,w) = B_0(y,w) + B_1(y,w)X + B_2(y,w)X^2 + \cdots,$$

where $X = \sqrt{1 - x/R(y)}$.

The functions $R(y)$, $E_j(y)$, $D_j(y,w)$, and $B_j(y,w)$ are analytic in y resp. in w and satisfy the relations

$$\frac{E_0(y)^3}{E_0(y)-1} = \left(\log\frac{1+E_0(y)}{1+R(y)} - E_0(y)\right)^2,$$
$$R(y) = \frac{\sqrt{1-1/E_0(y)}-1}{E_0(y)},$$
$$E_1(y) = -\left(\frac{2R(y)E_0(y)^2(1+R(y)E_0(y))^2}{(2R(y)E_0(y)+R(y)^2E_0(y)^2)^2 + 2R(y)(1+R(y)E_0(y))}\right)^{1/2},$$
$$D_0(y,w) = (1+yw)\exp\left(\frac{R(y)E_0(y)}{1+R(y)E_0(y)}D_0(y,w)\right) - 1,$$
$$D_1(y,w) = -\frac{D_0(y,w)E_1(y)R(y)(D_0(y,w)+1)}{(R(y)E_0(y)D_0(y,w)-1)(1+R(y)E_0(y))},$$
$$B_0(y,w) = -\frac{R(y)D_0(y,w)(R(y)E_0(y)D_0(y,w)-2)}{2(1+R(y)E_0(y))},$$
$$B_1(y,w) = \frac{E_1(y)R(y)^2D_0(y,w)^2}{2(1+R(y)E_0(y))^2}.$$

Proof. Since $E(x,y)$ satisfies Eq. (4.1) it follows that the dominant singularity of $E(x,y)$ is of square-root type and there is an expansion of the form $E(x,y) = E_0(y) + E_1(y)X + O(X^2)$, with $X = \sqrt{1-x/R(y)}$, and where $R(y)$ and $E_j(y)$ are analytic in y; compare with [1,10]. Furthermore, if we set

$$\Phi(x,y,z) = (1+y)\exp\left(\frac{xz^2}{1+xz}\right) - z - 1$$

then $R(y)$ and $E_0(y)$ satisfy the two equations

$$\Phi\bigl(R(y), y, E_0(y)\bigr) = 0 \quad\text{and}\quad \Phi_z\bigl(R(y), y, E_0(y)\bigr) = 0$$

and $E_1(y)$ is then given by

$$E_1(y) = -\left(\frac{2R(y)\Phi_x(R(y), y, E_0(y))}{\Phi_{zz}(R(y), y, E_0(y))}\right)^{1/2}.$$

Next observe that for $|w| \leqslant 1$ the radius of convergence of the function $x \mapsto D(x,y,w)$ is surely $\geqslant |R(y)|$. However, $D(x,y,w)$ satisfies Eq. (4.2), which implies that the dominant singularity of $E(x,y)$ carries over to that of $D(x,y,w)$. Thus, the mapping $x \mapsto D(x,y,w)$ has dominant singularity $R(y)$ and it also follows that $D(x,y,w)$ has a singular expansion of the form $D(x,y,w) = D_0(y,w) + D_1(y,w)X + O(X^2)$. Hence, by Lemma 4.2 we also get an expansion for $B^\bullet(x,y,w)$ of that form. Finally the relations for D_0, D_1 and B_0, B_1 follow by comparing coefficients in the corresponding expansions. □

Theorem 4.4. *Let d_k be the limiting probability that a vertex of a random two-connected series-parallel graph has degree k. Then*

$$p(w)=\sum_{k\geqslant 1} d_k w^k=\frac{B_1(1,w)}{B_1(1,1)}.$$

Obviously, $p(1)=1$, so that the d_k are indeed a probability distribution. We have asymptotically, as $k\to\infty$,

$$d_k\sim c\cdot k^{-3/2}q^k,$$

where $c\approx 3.7340799$ is a computable constant and

$$q=\left(\left(1+1/\left(R(1)E_0(1)\right)\right)e^{-1/(1+R(1)E_0(1))}-1\right)^{-1}\approx 0.7620402.$$

Proof. First observe that

$$p(w)=\lim_{n\to\infty}\frac{[x^n]B^\bullet(x,1,w)}{[x^n]B^\bullet(x,1,1)}.$$

However, from the local expansion of $B^\bullet(x,1,w)$ that is given in Lemma 4.3 (and by the fact that $B^\bullet(x,1,w)$ can be analytically continued to a Δ-region; see Section 2) it follows that

$$\left[x^n\right]B^\bullet(x,1,w)=-\frac{B_1(1,w)}{2\sqrt{\pi}}n^{-3/2}R(1)^{-n}\left(1+O\left(\frac{1}{n}\right)\right).$$

Hence, $p(w)=B_1(1,w)/B_1(1,1)$.

Next observe that Lemma 4.3 provides $B_1(1,w)$ only for $|w|\leqslant 1$. However, it is easy to continue $B_1(1,w)$ analytically to a larger region and it is also possible to determine the dominant singularity of $B_1(1,w)$, from which we deduce an asymptotic relation for the coefficients of $p(w)=B_1(1,w)/B_1(1,1)$.

For this purpose first observe from Lemma 4.3 that $D_0(y,w)$ satisfies a functional equation which provides an analytic continuation of the mapping $w\mapsto D_0(y,w)$ to a region including the unit disc. In addition, it follows that there exists a dominant singularity $w_0(y)$ and a local expansion of the form

$$D_0(y,w)=D_{00}(y)+D_{01}(y)W+D_{02}(y)W^2+\cdots,$$

where $W=\sqrt{1-w/w_0(y)}$. Furthermore, if we set

$$\Psi(y,w,z)=(1+yw)\exp\left(\frac{R(y)E_0(y)}{1+R(y)E_0(y)}z\right)-z-1$$

then $w_0(y)$ and $D_{00}(y)$ satisfy the equations

$$\Psi\left(y,w_0(y),D_{00}(y)\right)=0\quad\text{and}\quad\Psi_z\left(y,w_0(y),D_{00}(y)\right)=0.$$

Hence

$$D_{00}(y)=\frac{1}{R(y)E_0(y)}\quad\text{and}\quad w_0(y)=\frac{1}{y}\left(1+\frac{1}{R(y)E_0(y)}\right)\exp\left(-\frac{1}{1+R(y)E_0(y)}\right)-\frac{1}{y}.$$

Finally, with the help of Lemma 4.3 it also follows that this local representation of $D_0(y,w)$ provides similar local representations for D_1, B_0, and B_1:

$$D_1(y,w)=D_{1,-1}(y)W^{-1}+D_{10}(y)+D_{11}(y)W+\cdots,$$
$$B_0(y,w)=B_{00}(y)+B_{02}(y)W^2+B_{03}(y)W^3+\cdots,$$
$$B_1(y,w)=B_{10}(y)+B_{11}(y)W+B_{12}(y)W^2+\cdots,$$

where $W = \sqrt{1 - w/w_0(y)}$ is as above. Hence, all functions of interest D_0, D_1, B_0, B_1 can be analytically continued to a Δ-region, and the asymptotic relation for d_k follows immediately. Since $w_0(1)$ is the dominant singularity, we have $q = 1/w_0(1)$. □

The next theorem provides the degree distribution in series-parallel graphs. This result has been obtained independently in [5], and again our respective results agree.

Theorem 4.5. *Let d_k be the limiting probability that a vertex of a random connected series-parallel graph has degree k. Then*

$$p(w) = \sum_{k\geqslant 1} d_k w^k = \rho \cdot \frac{\partial}{\partial x} e^{B^\bullet(x,1,w)}\bigg|_{x=\rho C'(\rho)},$$

where $B^\bullet$ is given by Lemma 4.2 and Eqs. (4.2) and (4.1).

Moreover $p(1) = 1$, so that the d_k are indeed a probability distribution. We have asymptotically, as $k \to \infty$,

$$d_k \sim c \cdot k^{-3/2} q^k,$$

where $c \approx 3.5952391$ is a computable constant and

$$q = \left(\left(1 + 1/\left(\tau E(\tau,1)\right)\right) e^{-1/\tau E(\tau,1)} - 1\right)^{-1} \approx 0.7504161.$$

Proof. The proof of the first statement is exactly the same as that of Theorem 3.3. Again, we know that $\rho C'(\rho) = \tau \approx 0.127$ is larger than the radius of convergence $\rho \approx 0.110$ of $C(x)$, so that Lemma 3.1 applies. The proof that $p(1) = 1$ is also the same.

Recall that $\tau < R(1)$. Hence the dominant singularity $x = R(1)$ of the mapping $x \mapsto B^\bullet(x,1,w)$ will have no influence to the analysis of $p(w)$. Nevertheless, since

$$\frac{\partial}{\partial x} e^{B^\bullet(x,1,w)} = e^{B^\bullet(x,1,w)} \frac{\partial B^\bullet(x,1,w)}{\partial x}$$

we have to get some information on $D(x,1,w)$ and its derivative $\partial D(x,1,w)/\partial x$ with $x = \tau$.

Let us start with the analysis of the mapping $w \mapsto D(\tau,1,w)$. Since $D(x,y,w)$ satisfies Eq. (4.2) it follows that $D(\tau,1,w)$ satisfies

$$D(\tau,1,w) = (1+w)\exp\left(\frac{\tau E(\tau,1) D(\tau,1,w)}{1+\tau E(\tau,1)}\right) - 1.$$

Hence there exists a dominant singularity w_1 and a singular expansion of the form

$$D(\tau,1,w) = \widetilde{D}_0 + \widetilde{D}_1 \widetilde{W} + \widetilde{D}_2 \widetilde{W}^2 + \cdots,$$

where $\widetilde{W} = \sqrt{1 - w/w_1}$. Furthermore, if we set

$$\Xi(w,z) = (1+w)\exp\left(\frac{\tau E(\tau,1) z}{1+\tau E(\tau,1)}\right) - z - 1$$

then w_1 and $\widetilde{D}_0$ satisfy the equations

$$\Xi(w_1, \widetilde{D}_0) = 0 \quad \text{and} \quad \Xi_z(w_1, \widetilde{D}_0) = 0.$$

Consequently,

$$\widetilde{D}_0 = \frac{1}{\tau E(\tau,1)} \quad \text{and} \quad w_1 = \left(1 + \frac{1}{\tau E(\tau,1)}\right)\exp\left(-\frac{1}{1+\tau E(\tau,1)}\right) - 1.$$

Next, by taking derivatives with respect to x in (4.2) we obtain the relation

$$\frac{\partial D(x,1,w)}{\partial x} = \frac{(1+D(x,1,w))D(x,1,w)(E(x,1)+xE_x(x,1))}{(xE(x,1)D(x,1,w)-1)(1+xE(x,1))}.$$

Thus if we set $x=\tau$ and insert the singular representation of $D(\tau,1,w)$, it follows that $\frac{\partial D(x,1,w)}{\partial x}|_{x=\tau}$ has a corresponding singular representation too. By Lemma 4.2 we get the same property for $\frac{\partial B^\bullet(x,1,w)}{\partial x}|_{x=\tau}$ and finally for

$$\rho\cdot\frac{\partial}{\partial x}e^{B^\bullet(x,1,w)}\bigg|_{x=\tau}=\widetilde{C}_0+\widetilde{C}_1\widetilde{W}+\widetilde{C}_2\widetilde{W}^2+\cdots.$$

This implies the asymptotic relation for d_k with $q=1/w_1$. □

In this case, we obtain an expression for $p(w)$ in terms of the functions $E(x,1)$ and $D(x,1,w)$ and their derivatives. The derivatives can be computed using Eqs. (4.1) and (4.2) as in the previous proof. Expanding $p(w)$ in powers of w we obtain the approximate values for small k shown in Table 1.

Remark. We have $d_1=\rho$. Also, there is an easy relation between d_1 and d_2, namely

$$d_2=d_1(2\kappa),$$

where κn is asymptotically the expected number of edges in series-parallel graphs. This is shown in [23, Theorem 4.10] for planar graphs, phrased in terms of the average degree; the only property required is that subdividing an edge preserves planarity, which is also true in the case of series-parallel graphs (but not for outerplanar graphs). The value of $\kappa\approx 1.61673$ was determined in [6] and one can check that the relation holds.

5. Quadrangulations and 3-connected planar graphs

From now on and for the rest of the paper, all generating functions are associated to planar graphs. The goal of this section is to find the generating function of 3-connected planar graphs according to the degree of the root. This is an essential ingredient in the next section.

First we work out the problem for simple quadrangulations, which are in bijection with 3-connected maps. In order to do that we must revisit the classical work of Brown and Tutte [8] on 2-connected (non-separable) maps. Finally, using the fact that a 3-connected planar graph has a unique embedding in the sphere, we finish the job.

5.1. Simple quadrangulations

A *rooted quadrangulation* is a planar map where every face is a quadrangle, and with a distinguished directed edge of the external face, which is called the *root edge* of the quadrangulation. The *root vertex* of the quadrangulation is the tail of the root edge. A *diagonal* is an internal path of length two joining two opposite vertices of the external face. A quadrangulation is *simple* if it has no diagonal, every cycle of length 4 other than the external one defines a face, and it is not the trivial map reduced to a single quadrangle. In Section 5 of [24] it is shown how to count simple quadrangulations. Here we extend this result to count them also according to the degree of the root vertex.

A quadrangulation is bipartite and connected, so if we fix the colour of the root vertex there is a unique way of 2-colouring the vertices. We call the two colours black and white, and we assume that the root is black. Diagonals are called black or white according to the colour of the external vertices they join.

Let $F(x,y,w)$ be the GF of rooted quadrangulations, where the variables x, y and w mark, respectively, the number of black vertices minus one, the number of white vertices minus one, and the degree of the root vertex minus one. Generating functions for maps are always ordinary, since maps are unlabelled objects.

The generating functions F_N, F_B and F_W are associated, respectively, to quadrangulations with no diagonal, to those with at least one black diagonal (at the root vertex), and to those with at least one white diagonal (not at the root vertex). By planarity only one of the two kinds of diagonals can appear in a quadrangulation; it follows that

$$F(x,y,w) = F_N(x,y,w) + F_B(x,y,w) + F_W(x,y,w).$$

A quadrangulation with a diagonal can be decomposed into two quadrangulations, by considering the maps to the left and to the right of this diagonal. This gives rise to the equations

$$F_B(x,y,w) = \big(F_N(x,y,w) + F_W(x,y,w)\big)\frac{F(x,y,w)}{x},$$
$$F_W(x,y,w) = \big(F_N(x,y,w) + F_B(x,y,w)\big)\frac{F(x,y,1)}{y}.$$

In the second case, only one of the two quadrangulations contributes to the degree of the root vertex; this is the reason why the term $F(x,y,1)$ appears. The x and the y in the denominators appear because the three vertices of the diagonal are common to the two quadrangulations. Since we are considering vertices minus one, we only need to correct the colour that appears twice at the diagonal. Incidentally, no term w appears in the equations for the same reason.

Let us write $F = F(x,y,w)$ and $F(1) = F(x,y,1)$. From the previous equations we deduce that

$$F = F_N + F_B + F_W = (F_N + F_B)\left(1 + \frac{F}{x}\right),$$
$$F = F_N + F_B + F_W = (F_N + F_W)\left(1 + \frac{F(1)}{y}\right),$$

so that

$$F + F_N = (F_N + F_B) + (F_N + F_W) = F\left(\frac{1}{1 + F/x} + \frac{1}{1 + F(1)/y}\right),$$

and finally

$$F_N = F\left(\frac{1}{1 + F/x} + \frac{1}{1 + F(1)/y} - 1\right). \tag{5.1}$$

Now we proceed to count simple quadrangulations. We use the following combinatorial decomposition of quadrangulations with no diagonals in terms of simple quadrangulations: all quadrangulations with no diagonals, with the only exception of the trivial one, can be decomposed uniquely into a simple quadrangulation q and as many quadrangulations as internal faces q has (replace every internal face of q by its corresponding quadrangulation).

Let

$$Q(x,y,w) = \sum_{i,j,k} q_{i,j,k} x^i y^j w^k$$

be the GF of simple quadrangulations, where x, y and w have the same meaning as for F. We notice that this GF is called Q_N^* in [24]. We translate the combinatorial decomposition of simple quadrangulations into generating functions as follows.

$$\begin{aligned} F_N(x,y,w) - xyw &= \sum_{i,j,k} q_{i,j,k} x^i y^j \left(\frac{F}{xy}\right)^k \left(\frac{F(1)}{xy}\right)^{i+j-1-k} \\ &= \sum_{i,j,k} q_{i,j,k} \frac{xy}{F(1)} \left(\frac{F(1)}{y}\right)^i \left(\frac{F(1)}{x}\right)^j \left(\frac{F}{F(1)}\right)^k \\ &= \frac{xy}{F(1)} Q\left(\frac{F(1)}{y}, \frac{F(1)}{x}, \frac{F}{F(1)}\right), \end{aligned} \tag{5.2}$$

where we are using the fact that a quadrangulation counted by $q_{i,j,k}$ has $i+j+2$ vertices, $i+j-1$ internal faces, and k of them are incident to the root vertex.

At this point we change variables as $X = F(1)/y$, $Y = F(1)/x$ and $W = F/F(1)$. Then Eqs. (5.1) and (5.2) can be rewritten as

$$\frac{xy}{F(1)} Q(X,Y,W) = F_N - xyw = F\left(\frac{1}{1+F/x} + \frac{1}{1+F(1)/y} - 1\right) - xyw,$$

$$Q(X,Y,W) = XYW\left(\frac{1}{1+WY} + \frac{1}{1+X} - 1\right) - F(1)w. \tag{5.3}$$

The last equation would be an explicit expression of Q in terms of X, Y, W if it were not for the term $F(1)w = F(x,y,1)w$. In [24] it is shown that

$$F(1) = \frac{RS}{(1+R+S)^3}, \tag{5.4}$$

where $R = R(X,Y)$ and $S(X,Y)$ are algebraic functions defined by

$$R = X(S+1)^2, \qquad S = Y(R+1)^2. \tag{5.5}$$

Hence it remains only to obtain an expression for $w = w(X,Y,W)$ in order to obtain an explicit expression for Q. This is done in the next subsection.

5.2. Rooted non-separable planar maps

In [8] the authors studied the generating function $h(x,y,w)$ of rooted non-separable planar maps where x, y and w count, respectively the number of vertices minus one, the number of faces minus one, and the valency (number of edges) of the external face. We notice that the variable z is used instead of w in [8]. There is a bijection between rooted quadrangulations and non-separable rooted planar maps: black and white vertices in the quadrangulation correspond, respectively, to faces and vertices of the map; quadrangles become edges; and the root vertex becomes the external face, and its degree becomes its valency. As a consequence $h(y,x,w) = wF(x,y,w)$, where the extra factor w appears because in F we are counting the degree of the root vertex minus one. It follows that Eq. (3.9) from [8] becomes

$$(1-w)(1-yw)wF = -w^2F^2 + \bigl(-xw + wF(1)\bigr)wF + xw^2\bigl(x(1-w) + F(1)\bigr).$$

By dividing both sides by $F(1)^2$ and rewriting in terms of $X = F(1)/y$, $Y = F(1)/x$ and $W = F/F(1)$, we obtain

$$(1-w)\left(\frac{1}{F(1)} - \frac{w}{X}\right)wW = -w^2W^2 + \left(1 - \frac{1}{Y}\right)w^2W + \frac{w^2}{Y}\left(\frac{1}{X}(1-w) + 1\right),$$

$$\frac{(1-w)(X - wF(1))wW}{XF(1)} = \frac{w^2(-XYW^2 + XYW - XW + 1 - w + X)}{XY},$$

$$Y(1-w)\bigl(X - wF(1)\bigr)W = wF(1)\bigl(-XYW^2 + XYW - XW + 1 - w + X\bigr). \tag{5.6}$$

Observe that this is a quadratic equation in w. Solving for w in (5.6) and using (5.4) and (5.5) we get (the plus sign is because $T^\bullet$ has positive coefficients in coming Theorem 5.1)

$$w = \frac{-w_1(R,S,W) + (R - W + 1)\sqrt{w_2(R,S,W)}}{2(S+1)^2(SW + R^2 + 2R + 1)}, \tag{5.7}$$

where $w_1(R,S,W)$ and $w_2(R,S,W)$ are polynomials given by

$$\begin{aligned} w_1 = {} & -RSW^2 + W\bigl(1 + 4S + 3RS^2 + 5S^2 + R^2 + 2R + 2S^3 + 3R^2S + 7RS\bigr) \\ & + (R+1)^2\bigl(R + 2S + 1 + S^2\bigr), \end{aligned} \tag{5.8}$$

$$\begin{aligned} w_2 = {} & R^2S^2W^2 - 2WRS\bigl(2R^2S + 6RS + 2S^3 + 3RS^2 + 5S^2 + R^2 + 2R + 4S + 1\bigr) \\ & + (R+1)^2\bigl(R + 2S + 1 + S^2\bigr)^2. \end{aligned} \tag{5.9}$$

The reason we choose to write w as a function of (R, S, W) instead of (X, Y, W) will become clear later on.

Thus, together with Eqs. (5.3) and (5.4), we have finally obtained an explicit expression for the generating function $Q(X, Y, W)$ of simple quadrangulations in terms of W and algebraic functions $R(X, Y)$ and $S(X, Y)$.

5.3. 3-*connected planar graphs*

Let $T^\bullet(x, z, w)$ be the GF of 3-connected planar graphs, where one edge is taken as the root and given a direction, and where x counts vertices, z counts edges, and w counts the degree of the tail of the root edge. (Although these are edge-rooted graphs, we use the notation $T^\bullet$ for the sake of uniformity.) Now we relate $T^\bullet$ to the GF $Q(X, Y, W)$ of simple quadrangulations.

By the bijection between simple quadrangulations and 3-connected planar maps, and using Euler's relation, the GF $xwQ(xz, z, w)$ counts rooted 3-connected planar maps, where z marks edges (we have added an extra term w to correct the 'minus one' in the definition of Q).

According to Whitney's theorem 3-connected planar graphs have a unique embedding in the sphere. As noticed in [3], the are two ways of rooting an embedding of a directed edge-rooted graph in order to get a rooted map, since there are two ways of choosing the root face adjacent to the root edge. It follows that

$$T^\bullet(x, z, w) = \frac{xw}{2} Q(xz, z, w). \tag{5.10}$$

Theorem 5.1. *The generating function of directed edge-rooted* 3*-connected planar graphs, where x, z, w mark, respectively, vertices, edges, and the degree of the root vertex, is equal to*

$$T^\bullet = \frac{x^2z^2w^2}{2}\left(\frac{1}{1+wz} + \frac{1}{1+xz} - 1 - \frac{(u+1)^2(-w_1(u, v, w) + (u-w+1)\sqrt{w_2(u, v, w)})}{2w(vw+u^2+2u+1)(1+u+v)^3}\right), \tag{5.11}$$

where u and v are algebraic functions defined by

$$u = xz(1+v)^2, \qquad v = z(1+u)^2, \tag{5.12}$$

and $w_1(u, v, w)$ and $w_2(u, v, w)$ are given by (5.8) *and* (5.9) *replacing R, S, W by u, v, w, respectively.*

Proof. Combine Eq. (5.10), together with Eqs. (5.3), (5.4), and (5.7). □

When we set $w = 1$ in Eq. (5.11) we recover the GF of edge-rooted 3-connected planar graphs without taking into account the degree of the root vertex.

6. Planar graphs

This section is divided into three parts. First we obtain an explicit expression for $B^\bullet(x, y, w)$, the generating function of rooted 2-connected planar graphs taking into account the degree of the root. Secondly, we compute singular expansions at dominant singularities for several generating functions. And finally we obtain the asymptotic degree distribution in random planar graphs.

6.1. 2-*connected planar graphs*

Let $B^\bullet(x, y, w)$ be the generating function of rooted 2-connected planar graphs taking into account the degree of the root. As for series-parallel graphs we have to work with networks.

Let $T^\bullet(x, z, w)$ be the GF for directed edge-rooted 3-connected planar maps as in the previous section. As in Section 4, we denote by $D(x, y, w)$ and $S(x, y, w)$, respectively, the GFs of (planar) networks and series networks, with the same meaning for the variables x, y and w.

Lemma 6.1. *We have*

$$D(x,y,w)=(1+yw)\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}T^\bullet\left(x,E(x,y),\frac{D(x,y,w)}{E(x,y)}\right)\right)-1,$$
$$S(x,y,w)=xE(x,y)\big(D(x,y,w)-S(x,y,w)\big),$$

where $E(x,y)=D(x,y,1)$ is the GF for planar networks (without marking the degree of the root).

Proof. The proof is a variant of the equations developed by Walsh [27], taking into account the degree of the first pole in a network. The main point is the substitution of variables in $T^\bullet$: an edge is substituted by an ordinary planar network (this accounts for the term $E(x,y)$), except if it is incident with the first pole, in which case it is substituted by a planar network marking the degree, hence the term $D(x,y,w)$ (it is divided by $E(x,y)$ in order to avoid overcounting of ordinary edges). □

As in Lemma 4.1, and for the same reason, we have

$$w\frac{\partial B^\bullet(x,y,w)}{\partial w}=\sum_{k\geqslant 1}kB_k^\bullet(x,y)w^k$$
$$=xyw\exp\left(S(x,y,w)+\frac{1}{x^2D(x,y,w)}T^\bullet\left(x,E(x,y),\frac{D(x,y,w)}{E(x,y)}\right)\right).$$

Lemma 6.2. *The generating function of rooted 2-connected planar graphs is equal to*

$$B^\bullet(x,y,w)=x\left(D-\frac{xED}{1+xE}\left(1+\frac{D}{2}\right)\right)-\frac{1+D}{xD}T^\bullet(x,E,D/E)+\frac{1}{x}\int_0^D\frac{T^\bullet(x,E,t/E)}{t}\,dt, \tag{6.1}$$

where for simplicity we let $D=D(x,y,w)$ and $E=E(x,y)$.

Proof. We start as in the proof of Lemma 4.2.

$$\int\frac{1+D}{1+yw}\,dw=y^{-1}\log(1+yw)+y^{-1}\log(1+yw)D-\int y^{-1}\log(1+yw)\frac{\partial D}{\partial w}\,dw.$$

For the last integral we change variables $t=D(x,y,w)$ and use the fact that

$$\log(1+yw)=\log(1+D)-\frac{xED}{(1+xE)}-\frac{1}{x^2D}T^\bullet(x,E,D/E).$$

We obtain

$$\int\log(1+yw)\frac{\partial D}{\partial w}\,dw=\int_0^D\log(1+t)\,dt-\frac{xE}{1+xE}\int_0^D t\,dt+\frac{1}{x^2}\int_0^D\frac{T^\bullet(x,E,t/E)}{t}\,dt.$$

On the right-hand side, all the integrals except the last one are elementary. Now we use

$$B^\bullet(x,y,w)=xy\int\frac{1+D}{1+yw}\,dw$$

and after simple manipulations the result follows. □

In order to get a full expression for $B^\bullet(x,y,w)$, it remains to compute the integral in the formula of the previous lemma.

Lemma 6.3. *Let $T^\bullet(x,z,w)$ be the GF of* 3*-connected planar graphs as before. Then*

$$\begin{aligned}
\int_0^w \frac{T^\bullet(x,z,t)}{t}\,dt
&= -\frac{x^2(z^3xw^2-2wz-2xz^2w+(2+2xz)\log(1+wz))}{4(1+xz)}\\
&\quad -\frac{uvx}{2(1+u+v)^3}\\
&\quad \times\Bigg(\frac{w(2u^3+(6v+6)u^2+(6v^2-vw+14v+6)u+4v^3+10v^2+8v+2)}{4v(v+1)^2}\\
&\quad +\frac{(1+u)(1+u+2v+v^2)(2u^3+(4v+5)u^2+(3v^2+8v+4)u+2v^3+5v^2+4v+1)}{4uv^2(v+1)^2}\\
&\quad -\frac{\sqrt{Q}(2u^3+(4v+5)u^2+(3v^2-vw+8v+4)u+5v^2+2v^3+4v+1)}{4uv^2(v+1)^2}\\
&\quad +\frac{(1+u)^2(1+u+v)^3\log(Q_1)}{2v^2(1+v)^2}\\
&\quad +\frac{(u^3+2u^2+u-2v^3-4v^2-2v)(1+u+v)^3\log(Q_2)}{2v^2(1+v)^2u}\Bigg),
\end{aligned}$$

where the expressions Q, Q_1 and Q_2 are given by

$$\begin{aligned}
Q &= u^2v^2w^2-2uvw\big(u^2(2v+1)+u\big(3v^2+6v+2\big)+2v^3+5v^2+4v+1\big)\\
&\quad +(1+u)^2\big(u+(v+1)^2\big)^2,\\
Q_1 &= \frac{1}{2(wv+(u+1)^2)^2(v+1)(u^2+u(v+2)+(v+1)^2)}\\
&\quad \times\big(-uvw\big(u^2+u(v+2)+2v^2+3v+1\big)\\
&\quad +(u+1)(u+v+1)\sqrt{Q}+(u+1)^2\big(2u^2(v+1)\\
&\quad +u\big(v^2+3v+2\big)+v^3+3v^2+3v+1\big)\big),\\
Q_2 &= \frac{-wuv+u^2(2v-1)+u(3v^2+6v+2)+2v^3+5v^2+4v+1-\sqrt{Q}}{2v(u^2+u(v+2)+(v+1)^2)}.
\end{aligned}$$

Proof. We use Eq. (5.11) to integrate $T^\bullet(x,z,w)/w$. Notice that neither u nor v have any dependence on w. We have used Maple to help us in the burdensome process of computing a primitive, to which we have added the appropriate constant $c(x,z)$ to ensure that the resulting expression evaluates to 0 when $w=0$. We have checked the correctness by differentiating the expression above, again with the help of Maple.

A key point in the previous derivation is that, by expressing $w(X,Y,W)$ (see Eq. (5.6)) in terms of R, S instead of X, Y (see Eq. (5.7)), we obtain a quadratic polynomial $w_2(R,S,W)$ in terms of W inside the square root of $T^\bullet(x,z,w)$ in Eq. (5.11). Otherwise, we would have obtained a cubic polynomial inside the square root, and the integration would have been much harder. □

Combining Lemmas 6.2 and 6.3 we can produce an explicit (although quite long) expression for $B^\bullet(x,y,w)$ in terms of $D(x,y,w)$, $E(x,y)$, and the algebraic functions $u(x,y)$, $v(x,y)$. This is needed in the next section for computing the singular expansion of $B^\bullet(x,y,w)$ at its dominant singularity.

6.2. Singular expansions

In this section we find singular expansions of $T^\bullet(x,z,w)$, $D(x,y,w)$ and $B^\bullet(x,y,w)$ at their dominant singularities. As we show here, these singularities do not depend on w and were found in [3] and [17]. But the coefficients of the singular expansions do depend on w, and we need to compute them exactly in each case. In the next section we need the singular expansion for $B^\bullet$, but to compute it we first need the singular expansions of u, v, $T^\bullet$ and D (for u and v, see also [4,3]).

Lemma 6.4. *Let $u=u(x,z)$ and $v=v(x,z)$ be the solutions of the system of equations $u=xz(1+v)^2$ and $v=z(1+u)^2$. Let $r(z)$ be given explicitly by*

$$r(z)=\frac{\widetilde{u}_0(z)}{z(1+z(1+\widetilde{u}_0(z))^2)^2}, \tag{6.2}$$

where

$$\widetilde{u}_0(z)=-\frac{1}{3}+\sqrt{\frac{4}{9}+\frac{1}{3z}}.$$

Furthermore, let $\tau(x)$ be the inverse function of $r(z)$ and let $u_0(x)=\widetilde{u}_0(\tau(x))$ which is also the solution of the equation

$$x=\frac{(1+u)(3u-1)^3}{16u}.$$

Then, for given x sufficiently close to the positive reals the functions $u(x,z)$ and $v(x,z)$ have a dominant singularity at $z=\tau(x)$ and have local expansions of the form

$$u(x,z)=u_0(x)+u_1(x)Z+u_2(x)Z^2+u_3(x)Z^3+O\big(Z^4\big),$$
$$v(x,z)=v_0(x)+v_1(x)Z+v_2(x)Z^2+v_3(x)Z^3+O\big(Z^4\big),$$

where $Z=\sqrt{1-z/\tau(x)}$. The functions $u_j(x)$ and $v_j(x)$ are also analytic and can be given explicitly in terms of $u=u_0(x)$. In particular we have

$$u_0(x)=u, \qquad v_0(x)=\frac{1+u}{3u-1},$$
$$u_1(x)=-\sqrt{2u(u+1)}, \qquad v_1(x)=-\frac{2\sqrt{2u(u+1)}}{3u-1},$$
$$u_2(x)=\frac{(1+u)(7u+1)}{2(1+3u)}, \qquad v_2(x)=\frac{2u(3+5u)}{(3u-1)(1+3u)},$$
$$u_3(x)=-\frac{(1+u)(67u^2+50u+11)u}{4(1+3u)^2\sqrt{2u^2+2u}}, \qquad v_3(x)=-\frac{\sqrt{2}u(1+u)(79u^2+42u+7)}{4(1+3u)^2(3u-1)\sqrt{u(1+u)}}.$$

Similarly, for given z sufficiently close to the reals the functions $u=u(x,z)$ and $v=v(x,z)$ have a dominant singularity $x=r(z)$ and there is also a local expansion of the form

$$u(x,z)=\widetilde{u}_0(z)+\widetilde{u}_1(z)X+\widetilde{u}_2(z)X^2+O\big(X^3\big),$$
$$v(x,z)=\widetilde{v}_0(z)+\widetilde{v}_1(z)X+\widetilde{v}_2(z)X^2+O\big(X^3\big),$$

where now $X=\sqrt{1-x/r(z)}$. The functions $\widetilde{u}_j(z)$ and $\widetilde{v}_j(z)$ are analytic and can be given explicitly in terms of $\widetilde{u}=\widetilde{u}_0(z)$. In particular, we have

$$\widetilde{u}_0(z) = \widetilde{u}, \qquad \widetilde{v}_0(z) = \frac{1+\widetilde{u}}{3\widetilde{u}-1},$$

$$\widetilde{u}_1(z) = -\frac{2\widetilde{u}\sqrt{1+\widetilde{u}}}{\sqrt{1+3\widetilde{u}}}, \qquad \widetilde{v}_1(z) = -\frac{4\widetilde{u}\sqrt{1+\widetilde{u}}}{(3\widetilde{u}-1)\sqrt{1+3\widetilde{u}}},$$

$$\widetilde{u}_2(z) = \frac{2(1+\widetilde{u})\widetilde{u}(2\widetilde{u}+1)}{(1+3\widetilde{u})^2}, \qquad \widetilde{v}_2(z) = \frac{4\widetilde{u}(5\widetilde{u}^2+4\widetilde{u}+1)}{(3\widetilde{u}-1)(1+3\widetilde{u})^2},$$

$$\widetilde{u}_3(z) = -\frac{2\widetilde{u}(10\widetilde{u}^3+11\widetilde{u}^2+5\widetilde{u}+1)\sqrt{1+\widetilde{u}}}{(1+3\widetilde{u})^{7/2}}, \qquad \widetilde{v}_3(z) = -\frac{4\widetilde{u}(2\widetilde{u}+1)(11\widetilde{u}^2+5\widetilde{u}+1)\sqrt{1+\widetilde{u}}}{(3\widetilde{u}-1)(1+3\widetilde{u})^{7/2}}.$$

Proof. Since $u(x,z)$ satisfies the functional equation $u = xz(1+z(1+u)^2)^2$, it follows that for any fixed real and positive z the function $x \mapsto u(x,z)$ has a square-root singularity at $r(z)$ that satisfies the equations

$$\Phi\big(r(z), z, u\big) = 0 \quad \text{and} \quad \Phi_u\big(r(z), z, u\big) = 0,$$

where $\Phi(x,z,u) = u - xz(1+z(1+u)^2)^2$. Now a short calculation gives the explicit formula (6.2) for $r(z)$. By continuity we obtain the same kind of representation if z is complex but sufficiently close to the positive reals.

We proceed in the same way if x is fixed and z is considered as the variable. Then $\tau(x)$, the functional inverse of $r(z)$, is the singularity of the mapping $z \mapsto u(x,z)$. Furthermore, the coefficients $u_1(x)$ etc. can be easily calculated. The derivations for $v(x,z)$ are completely of the same kind. □

Lemma 6.5. *Suppose that x and w are sufficiently close to the positive reals and that $|w| \leqslant 1$. Then the dominant singularity $z = \tau(x)$ of $T^\bullet(x,z,w)$ does not depend on w. The singular expansion at $\tau(x)$ is*

$$T^\bullet(x,z,w) = T_0(x,w) + T_2(x,w)Z^2 + T_3(x,w)Z^3 + O\big(Z^4\big), \tag{6.3}$$

where $Z = \sqrt{1 - z/\tau(x)}$, and the expressions for the T_i are given explicitly in [12, Appendix].

Proof. Suppose for a moment that all variables x, z, w are non-negative real numbers and let us look at the expression (5.11) for $T^\bullet$. The algebraic functions $u(x,z)$ and $v(x,z)$ are always non-negative and, since the factor w in the denominator cancels with a corresponding factor in the numerator, the only possible source of singularities are: a) those coming from u and v, or b) the vanishing of $w_2(u,v,w)$ inside the square root.

We can discard source b) as follows. For fixed $u, v > 0$, let $w_2(w) = w_2(u,v,w)$. We can check that

$$w_2(1) = \big(1 + 2u + u^2 + 2v + v^2 + uv - uv^2\big)^2,$$
$$w_2'(w) = -2uv\big((6-w)uv + 1 + 2u + u^2 + 4v + 5v^2 + 2v^3 + 3uv^2 + 2u^2v\big).$$

In particular $w_2(1) > 0$ and $w_2'(w) < 0$ for $w \in [0,1]$, thus it follows that $w_2(w) > 0$ in $w \in [0,1]$. Hence the singularities come from source a) and do not depend on w.

Following Lemma 6.4 (see also [4] and [3]), we have that $z = \tau(x)$ is the radius of convergence of $u(x,z)$, as a function of z. Now by using the expansions of $u(x,z)$ and $v(x,z)$ from Lemma 6.4 we obtain (6.3).

Finally, by continuity all properties are also valid if x and w are sufficiently close to the reals, thus completing the proof. □

Similarly we get an alternate representation expanding in the variable x.

Lemma 6.6. *Suppose that z and w are sufficiently close to the positive reals and that $|w| \leqslant 1$. Then the dominant singularity $x = r(z)$ of $T^\bullet(x,z,w)$ does not depend on w. The singular expansion at $r(z)$ is*

$$T^{\bullet}(x,z,w) = \widetilde{T}_0(z,w) + \widetilde{T}_2(z,w)X^2 + \widetilde{T}_3(z,w)X^3 + O\left(X^4\right), \tag{6.4}$$

where $X = \sqrt{1 - x/r(z)}$. Furthermore we have

$$\begin{aligned}
\widetilde{T}_0(z,w) &= T_0\big(r(z),w\big),\\
\widetilde{T}_2(z,w) &= T_2\big(r(z),w\big)H\big(r(z),z\big) - T_{0,x}\big(r(z),w\big)r(z),\\
\widetilde{T}_3(z,w) &= T_3\big(r(z),w\big)H\big(r(z),z\big)^{3/2},
\end{aligned}$$

where $H(x,z)$ is a non-zero analytic function with $1 - z/\tau(x) = H(x,z)X^2$.

Proof. We could repeat the proof of Lemma 6.5. However, we present an alternate approach that uses the results of Lemma 6.5 and a kind of singularity transfer.

By applying the Weierstrass preparation theorem (which is in a refined version of the implicit function theorem in the present case) it follows that there is a (locally) non-zero analytic function $H(x,z)$ with $1 - z/\tau(x) = H(x,z)(1 - x/r(z)) = H(x,z)X^2$. Furthermore, by using the representation $x = r(z)(1 - X^2)$ and the Taylor expansion we have

$$\begin{aligned}
H(x,z) &= H\big(r(z),z\big) - H_x\big(r(z),z\big)r(z)X^2 + O\left(X^4\right),\\
T_j(x,w) &= T_j\big(r(z),w\big) - T_{j,x}\big(r(z),w\big)r(z)X^2 + O\left(X^4\right).
\end{aligned}$$

Hence, Lemma 6.5 directly gives the result. In fact, H can be computed explicitly and satisfies $H(x,\tau(x)) = (1+3u)/2u$. □

The next result is Proposition 6.3 from [11], and is needed in order to guarantee that singular expansions of the desired kind exist.

Theorem 6.7. *Suppose that $F(x,y,u)$ has a local representation of the form*

$$F(x,y,u) = g(x,y,u) + h(x,y,u)\left(1 - \frac{y}{r(x,u)}\right)^{3/2} \tag{6.5}$$

with functions $g(x,y,u)$, $h(x,y,u)$, $r(x,u)$ that are analytic around (x_0,y_0,u_0) and satisfy

$$g_y(x_0,y_0,u_0) \neq 1, \qquad h(x_0,y_0,u_0) \neq 0, \qquad r(x_0,u_0) \neq 0, \qquad r_x(x_0,u_0) \neq g_x(x_0,y_0,u_0).$$

Furthermore, suppose that $y = y(x,u)$ is a solution of the functional equation

$$y = F(x,y,u)$$

with $y(x_0,u_0) = y_0$. Then $y(x,u)$ has a local representation of the form

$$y(x,u) = g_1(x,u) + h_1(x,u)\left(1 - \frac{x}{\rho(u)}\right)^{3/2}, \tag{6.6}$$

where $g_1(x,u)$, $h_1(x,u)$ and $\rho(u)$ are analytic at (x_0,u_0) and satisfy $h_1(x_0,u_0) \neq 0$ and $\rho(u_0) = x_0$.

Lemma 6.8. *Suppose that y and w are sufficiently close to the positive reals and that $|w| \leqslant 1$. Then the dominant singularity $x = R(y)$ of $D(x,y,w)$ does not depend on w. The singular expansion at $R(y)$ is*

$$D(x,y,w) = D_0(y,w) + D_2(y,w)X^2 + D_3(y,w)X^3 + O\left(X^4\right), \tag{6.7}$$

where $X = \sqrt{1 - x/R(y)}$, and the expressions for the D_i are given explicitly in [12, Appendix].

Proof. We use the system of equations from Lemma 6.1. If we set $w = 1$ and substitute $S(x, y, 1) = xE(x, y)^2/(1 + xE(x, y))$ then we get a single equation for $E(x, y) = D(x, y, 1)$:

$$E(x, y) = (1 + y)\exp\left(\frac{xE(x, y)^2}{1 + xE(x, y)} + \frac{1}{x^2 E(x, y)} T^{\bullet}\bigl(x, E(x, y), 1\bigr)\right) - 1. \tag{6.8}$$

Now we use the singular expansion from Lemma 6.6 and Theorem 6.7 (which is Proposition 6.3 of [11]) to conclude that $E(x, y)$ has an expansion of the form

$$E(x, y) = E_0(y) + E_2(y)X^2 + E_3(y)X^3 + O\bigl(X^4\bigr). \tag{6.9}$$

Finally we reconsider Lemma 6.1 and after substituting $S(x, y, w) = xE(x, y)D(x, y, w)/(1 + xE(x, y))$ we get a corresponding (single) equation for $D(x, y, w)$:

$$D(x, y, w) = (1 + yw)\exp\left(\frac{xE(x, y)D(x, y, w)}{1 + xE(x, y)} + \frac{1}{x^2 D(x, y, w)} T^{\bullet}\left(x, E(x, y), \frac{D(x, y, w)}{E(x, y)}\right)\right) - 1. \tag{6.10}$$

Now a second application of Lemma 6.6 and Theorem 6.7 yields the result. Note that the singularity does not depend on w. □

Remark. For later use we give an equation for $D_0(y, w)$ (see [12, Appendix]). Let $t = t(y)$ (for $y \in (0, \infty)$) be the unique solution in $(0, 1)$ of

$$y = \frac{(1 - 2t)}{(1 + 3t)(1 - t)} \exp\left(-\frac{t^2(1 - t)(18 + 36t + 5t^2)}{2(3 + t)(1 + 2t)(1 + 3t)^2}\right) - 1.$$

Then $D_0 = D_0(y, w)$ is the solution of

$$1 + D_0 = (1 + yw)\exp\left(\frac{\sqrt{S}(D_0(t - 1) + t)}{4(3t + 1)(D_0 + 1)} - \frac{D_0^2(t^4 - 12t^2 + 20t - 9) + D_0(2t^4 + 6t^3 - 6t^2 + 10t - 12) + t^4 + 6t^3 + 9t^2}{4(t + 3)(D_0 + 1)(3t + 1)}\right), \tag{6.11}$$

where S abbreviates $S = (D_0 t - D_0 + t)(D_0(t - 1)^3 + t(t + 3)^2)$.

Lemma 6.9. *Suppose that y and w are sufficiently close to the positive reals and that $|w| \leqslant 1$. Then the dominant singularity $x = R(y)$ of $B^{\bullet}(x, y, w)$ does not depend on w, and is the same as for $D(x, y, w)$. The singular expansion at $R(y)$ is*

$$B^{\bullet}(x, y, w) = B_0(y, w) + B_2(y, w)X^2 + B_3(y, w)X^3 + O\bigl(X^4\bigr), \tag{6.12}$$

where $X = \sqrt{1 - x/R(y)}$, and the expressions for the B_i are given explicitly in [12, Appendix].

Proof. We just have to use the representation of $B^{\bullet}(x, y, w)$ that is given in Lemma 6.2 and Lemma 6.3 and the singular expansion of $D(x, y, w)$ from Lemma 6.8. □

6.3. Degree distribution for planar graphs

We start with the degree distribution in 3-connected graphs, both for edge-rooted and vertex-rooted graphs.

Theorem 6.10. *Let d_k be the limiting probability that a vertex of a random three-connected planar graph has degree k, and let e_k be the limiting probability that the tail root vertex of a random edge-rooted (where the edge is oriented) three-connected planar graph has degree k. Then*

$$\sum e_k w^k = \frac{T_3(r, w)}{T_3(r, 1)},$$

where $T_3(x, w)$ is given explicitly in [12, Appendix] and $r = r(1) = (7\sqrt{7} - 17)/32$ is explicitly given by (6.2). Obviously the e_k are indeed a probability distribution. We have asymptotically, as $k \to \infty$,

$$e_k \sim c \cdot k^{1/2} q^k,$$

where $c \approx 0.9313492$ is a computable constant and $q = 1/(u_0 + 1) = \sqrt{7} - 2$, and where $u_0 = u(r) = (\sqrt{7} - 1)/3$.

Moreover, we have

$$d_k = \alpha \frac{e_k}{k} \sim c\alpha \cdot k^{-1/2} q^k,$$

where

$$\alpha = \frac{(3u_0 - 1)(3u_0 + 1)(u_0 + 1)}{u_0} = \frac{\sqrt{7} + 7}{2}$$

is the asymptotic value of the expected average degree in 3-connected planar graphs.

We remark that the degree distribution in 3-connected planar maps counted according to the number of edges was obtained in [2]. The asymptotic estimates have the same shape as our d_k, but the corresponding value of q is equal to $1/2$.

Proof. The proof uses first the singular expansion (6.4). The representation

$$\sum_{k \geqslant 1} e_k w^k = \frac{T_3(r, w)}{T_3(r, 1)}$$

follows in completely the same way as the proof of Theorem 4.4, using now Lemma 6.5, with the difference that now the dominant term is the coefficient of Z^3.

In order to characterise the dominant singularity of $T_3(1, w)$ and to determine the singular behaviour we analyse the explicit representation from [12, Appendix]:

$$T_3(x, w) = -\frac{(3u - 1)^6 w \sqrt{2u(u + 1)}(3u + 1)}{373248(u + 1)^3 u^6} \times \left((3u - 1)^2 w - 9u^2 - 10u - 1 + \frac{P_3}{(u + 1 - w)^{3/2} \sqrt{P_2}} \right),$$

where u abbreviates the solution of $x = (1+u)(3u-1)^3/(16u)$ and where $P_2 = P_2(u, w)$ and $P_3(u, w)$ are polynomials with the property that $P_2(u, u+1) = 24u(u+1)(3u+1)$ and $P_2(u, u+1) = 216u^3(u+1)^3$. Hence, $T_3(u, w)$ contains in the denominator a (dominating) singular term of the form $(-w + u + 1)^{3/2}$ and it follows that $u_0 + 1$ is the dominant singularity of the mapping $w \mapsto T_3(r, w)$.[1] We also get the proposed asymptotic relation for e_k.

For the proof of the second part of the statement, let $t_{n,k}$ be the number of vertex-rooted graphs with n vertices and with degree of the root equal to k, and let $s_{n,k}$ be the analogous quantity for edge-rooted graphs. Let also $t_n = \sum_k t_{n,k}$ and $s_n = \sum_k s_{n,k}$. Since a vertex-rooted graph with a root of

[1] We note that the singularity $w = u + 1$ is of the same form for all $u > 0$. This fact will be used in the proof of Theorem 7.1, where the edge density is fixed.

degree k is counted k times as an oriented-edge-rooted graph, we have $s_{n,k} = kt_{n,k}$ (a similar argument is used in [21]). Notice that $e_k = \lim s_{n,k}/s_n$ and $d_k = \lim t_{n,k}/t_n$.

Using the quasi-powers theorems as in [17], one shows that the expected number of edges μ_n in 3-connected planar graphs is asymptotically $\mu_n \sim \kappa n$, where $\kappa = -\tau'(1)/\tau(1)$, and $\tau(x)$ is as in Lemma 6.4. Clearly $s_n = 2\mu_n t_n/n$. Finally, $2\mu_n/n$ is asymptotic to the expected average degree $\alpha = 2\kappa$. Summing up, we obtain

$$kd_k = \alpha e_k.$$

A simple calculation gives the value of α as claimed. □

Theorem 6.11. *Let d_k be the limiting probability that a vertex of a random two-connected planar graph has degree k. Then*

$$p(w) = \frac{B_3(1, w)}{B_3(1, 1)},$$

where $B_3(y, w)$ is given explicitly in [12, Appendix].

Obviously, $p(1) = 1$, so that the d_k are indeed a probability distribution and we have asymptotically, as $k \to \infty$,

$$d_k \sim ck^{-1/2}q^k,$$

where $c \approx 3.0826285$ is a computable constant and

$$q = \left(\frac{1}{1-t_0}\exp\left(\frac{(t_0-1)(t_0+6)}{6t_0^2+20t_0+6}\right)-1\right)^{-1} \approx 0.6734506,$$

and $t_0 = t(1) \approx 0.6263717$ is a computable constant, too.

Proof. The representation of $p(w)$ follows from (6.12).

Now, in order to characterise the dominant singularity of $B_3(1, w)$ and to determine the singular behaviour we first observe that the right-hand side of Eq. (6.11) for D_0 contains a singular term of the form $(D_0(t-1)+t)^{3/2}$ that dominates the right-hand side. Hence, by applying Theorem 6.7 it follows that $D_0(1, w)$ has dominant singularity w_3, where $D_0(1, w_3) = t/(1-t)$ and we have a local singular representation of the form

$$D_0(1, w) = \widetilde{D}_{00} + \widetilde{D}_{02}\widetilde{W}^2 + \widetilde{D}_{03}\widetilde{W}^3 + \cdots,$$

where $\widetilde{W} = \sqrt{1-w/w_3}$ and $\widetilde{D}_{00} = t/(1-t)$. The fact that the coefficient of $\widetilde{W}$ vanishes is due to the shape of Eq. (6.11).

We now insert this expansion into the representation for D_2 (see [12, Appendix]). Observe that $S = (D_0t - D_0 + t)(D_0(t-1)^3 + t(t+3)^2)$ has an expansion of the form

$$S = S_2\widetilde{W}^2 + S_3\widetilde{W}^3 + \cdots$$

with $S_2 \neq 0$. Thus we have $\sqrt{S} = \sqrt{S_2}\widetilde{W} + O(\widetilde{W}^2)$. Furthermore, we get expansions for $S_{2,1}$, $S_{2,2}$, $S_{2,3}$, and $S_{2,4}$. However, we observe that $S_{2,3}(1, w_3) = 0$ whereas $S_{2,1}(1, w_3) \neq 0$, $S_{2,2}(1, w_3) \neq 0$, and $S_{2,4}(1, w_3) \neq 0$. Consequently we can represent $D_2(1, w)$ as

$$D_2(1, w) = \widetilde{D}_{2,-1}\frac{1}{\widetilde{W}} + \widetilde{D}_{2,0} + \widetilde{D}_{2,1}\widetilde{W} + +\widetilde{D}_{22}\widetilde{W}^2 + \cdots,$$

where $\widetilde{D}_{2,-1} \neq 0$.

In completely the same way it follows that $D_3(1, w)$ has a local expansion of the form

$$D_3(1, w) = \widetilde{D}_{3,-3}\frac{1}{\widetilde{W}^3} + \widetilde{D}_{3,-1}\frac{1}{\widetilde{W}} + \widetilde{D}_{3,0} + \widetilde{D}_{3,1}\widetilde{W} + \cdots,$$

where $\widetilde{D}_{3,-3} \neq 0$, and the coefficient of $\widetilde{W}^{-2}$ vanishes identically.

These types of singular expansions carry over to $B_3(1,w)$, and we get

$$B_3(1,w) = \widetilde{B}_{3,-1}\frac{1}{\widetilde{W}} + \widetilde{B}_{3,0} + \widetilde{B}_{3,1}\widetilde{W} + \cdots, \tag{6.13}$$

where $\widetilde{B}_{3,-1} \neq 0$. We stress the fact that the coefficients of $\widetilde{W}^{-3}$ and $\widetilde{W}^{-2}$ vanish as a consequence of non-trivial cancellations. Hence, we obtain the proposed asymptotic relation for the d_k. □

The following is the analogous of Lemma 3.1. The difference now is that we are composing two singular expansions and, moreover, they are of type 3/2.

Lemma 6.12. *Let $f(x) = \sum_{n\geqslant 0} a_n x^n/n!$ denote the exponential generating function of a sequence a_n of non-negative real numbers and suppose that $f(x)$ has exactly one dominating singularity at $x=\rho$ of the form*

$$f(x) = f_0 + f_2X^2 + f_3X^3 + \mathcal{O}(X^4),$$

where $X = \sqrt{1-x/\rho}$, and has an analytic continuation to the region $\{x \in \mathbb{C}\colon |x| < \rho+\varepsilon\} \setminus \{x \in \mathbb{R}\colon x \geqslant \rho\}$ for some $\varepsilon > 0$. Further, let $H(x,z,w)$ denote a function that has a dominant singularity at $z = f(\rho) > 0$ of the form

$$H(x,z,w) = h_0(x,w) + h_2(x,w)Z^2 + h_3(x,w)Z^3 + \mathcal{O}(Z^4),$$

where w is considered as a parameter, $Z = \sqrt{1-z/f(\rho)}$, the functions $h_j(x,w)$ are analytic in x, and $H(x,z,w)$ has an analytic continuation in a suitable region.

Then the function

$$f_H(x) = H\big(x, f(x), w\big)$$

has a power series expansion $f_H(x) = \sum_{n\geqslant 0} b_n x^n/n!$ and the coefficients b_n satisfy

$$\lim_{n\to\infty}\frac{b_n}{a_n} = -\frac{h_2(\rho,w)}{f_0} + \frac{h_3(\rho,w)}{f_3}\left(-\frac{f_2}{f_0}\right)^{3/2}. \tag{6.14}$$

Proof. The proof is similar to that of Lemma 3.3 and is based on composing the singular expansion of $H(x,z,w)$ with that of $f(x)$. Indeed, near $x=\rho$, and taking into account that $f(\rho) = f_0$, we have

$$f_H(x) = h_0(x,w) + h_2(x,w)\left(-\frac{f_2X^2+f_3X^3}{f_0}\right) + h_3(x,w)\left(-\frac{f_2X^2+f_3X^3}{f_0}\right)^{3/2} + \cdots.$$

Now note that $x = \rho - \rho X^2$. Thus, if we expand and extract the coefficient of X^3 and apply transfer theorems, we have

$$\frac{a_n}{n!} \sim \frac{f_3}{\Gamma(-3/2)} n^{-5/2}\rho^{-n}$$

and

$$\frac{b_n}{n!} \sim \frac{1}{\Gamma(-3/2)}\left(-\frac{h_2(\rho,w)f_3}{f_0} + h_3(\rho,w)\left(-\frac{f_2}{f_0}\right)^{3/2}\right) n^{-5/2}\rho^{-n},$$

so that the result follows. □

Theorem 6.13. *Let d_k be the limiting probability that a vertex of a random connected planar graph has degree k. Then*

$$p(w) = \sum_{k\geqslant 1} d_k w^k = -e^{B_0(1,w)-B_0(1,1)}B_2(1,w) + e^{B_0(1,w)-B_0(1,1)}\frac{1+B_2(1,1)}{B_3(1,1)}B_3(1,w),$$

where $B_j(y,w)$, $j=0,2,3$, are given explicitly in [12, Appendix]. *Moreover,* $p(1)=1$, *so that the* d_k *are indeed a probability distribution and we have asymptotically, as* $k\to\infty$,

$$d_k \sim ck^{-1/2}q^k,$$

where $c\approx 3.0175067$ *is a computable constant and* q *is as in Theorem* 6.11.

Proof. The degree distribution is encoded in the function

$$C^\bullet(x,w)=\sum_{k\geqslant 1} C_k(x,1)w^k = e^{B^\bullet(xC'(x),1,w)},$$

where the generating function $xC'(x)$ of connected rooted planar graphs satisfies the equation

$$xC'(x)=xe^{B^\bullet(xC'(x),1,1)}.$$

From Lemma 6.9 we get the local expansions

$$e^{B^\bullet(x,1,w)}=e^{B_0(1,w)}\bigl(1+B_2(1,w)X^2+B_3(1,w)X^3+O\bigl(X^4\bigr)\bigr),$$

where $X=\sqrt{1-x/R}$. Thus, we first get an expansion for $xC'(x)$

$$xC'(x)=R-\frac{R}{1+B_2(1,1)}\widetilde{X}^2+\frac{RB_3(1,1)}{(1+B_2(1,1))^{5/2}}\widetilde{X}^3+O\bigl(\widetilde{X}^4\bigr),$$

where $\widetilde{X}=\sqrt{1-x/\rho}$ and ρ is the radius of convergence of $C'(x)$ (compare with [17]). Note also that $R=\rho e^{B_0(\rho,1)}$. Thus, we can apply Lemma 6.12 with $H(x,z,w)=xe^{B^\bullet(z,1,w)}$ and $f(x)=xC'(x)$. We have

$$f_0=R,\qquad f_2=-\frac{R}{1+B_2(1,1)},\qquad f_3=\frac{RB_3(1,1)}{(1+B_2(1,1))^{5/2}}$$

and

$$h_0(\rho,w)=\rho e^{B_0(1,w)},\qquad h_2(\rho,w)=\rho e^{B_0(1,w)}B_2(1,w),$$
$$h_3(\rho,w)=\rho e^{B_0(1,w)}B_3(1,w).$$

We can express the probability generating function $p(w)$ as

$$\lim_{n\to\infty}\frac{[x^n]xC^\bullet(x,w)}{[x^n]xC'(x)}.$$

Consequently, we have

$$\begin{aligned}p(w)&=-\frac{h_2(\rho,w)}{f_0}+\frac{h_3(\rho,w)}{f_3}\left(-\frac{f_2}{f_0}\right)^{3/2}\\&=-e^{B_0(1,w)-B_0(1,1)}B_2(1,w)+e^{B_0(1,w)-B_0(1,1)}\frac{1+B_2(1,1)}{B_3(1,1)}B_3(1,w).\end{aligned}$$

The singular expansion for $B_2(1,w)$ turns out to be of the form

$$B_2(1,w)=\widetilde{B}_{2,0}+\widetilde{B}_{2,1}\widetilde{W}+\cdots.$$

Hence the expansion (6.13) for $B_3(1,w)$ gives the leading part in the asymptotic expansion for $p(w)$. It follows that $p(w)$ has the same dominant singularity as for 2-connected graphs and we obtain the asymptotic estimate for the d_k as claimed, with a different multiplicative constant. This concludes the proof of the main result. □

7. Degree distribution according to edge density

In this section we show that there exists a computable degree distribution for planar graphs with a given edge density or, equivalently, given average degree.

In [17] we showed that, for μ in the open interval $(1,3)$, the number of planar graphs with n vertices and $\lfloor \mu n \rfloor$ edges is asymptotically

$$g_{n,\lfloor \mu n \rfloor} \sim c(\mu) n^{-4} \gamma(\mu)^n n!, \tag{7.1}$$

where $c(\mu)$ and $\gamma(\mu)$ are computable analytic functions of μ. The proof was based on the fact that for each $\mu \in (1,3)$, there exists a value $y = y(\mu)$ such that the generating function $G(x,y)$ captures the asymptotic behaviour of $g_{n,\lfloor \mu n \rfloor}$. The exact equation connecting μ and y is

$$-y\frac{\rho'(y)}{\rho(y)} = \mu, \tag{7.2}$$

where $\rho(y)$ is the radius of convergence $G(x,y)$ as a function of x. More precisely, the idea behind the proof is to weight a planar graph with m edges by y^m. If g_{nm} denotes the number of planar graphs with n vertices and m edges, then the bivariate generating function

$$G(x,y) = \sum_{m,n} g_{nm} \frac{x^n}{n!} y^m = \sum_{n \geqslant 1} g_n(y) \frac{x^n}{n!}$$

can be considered as the generating function of the weighted numbers $g_n(y) = \sum_m g_{nm} y^m$ of planar graphs of size n. In addition this weighted model induces a modified probability model. Instead of the uniform distribution the probability of a planar graph of size n is now given by $y^m/g_n(y)$, where m denotes the number of edges. It follows from the singular expansion [17]

$$G(x,y) = G_0(y) + G_2(y)Y^2 + G_4(y)Y^4 + G_5(y)Y^5 + O\big(Y^6\big),$$

where $Y = \sqrt{1 - x/\rho(y)}$, that $g_n(y)$ is asymptotically given by

$$g_n(y) \sim g(y) n^{-7/2} \rho(y)^{-n} n!$$

with $g(y) = G_5(y)/\Gamma(-5/2)$. If $y = 1$ we recover the asymptotic formula for the number of planar graphs of size n. However, since $g_n(y)$ is a power series in x with coefficients g_{nm} it is possible to compute these numbers by a Cauchy integral:

$$g_{nm} = \frac{1}{2\pi i} \int_{|y|=r} \frac{g_n(y)}{y^{m+1}}\, dy.$$

Suppose that $m = \mu n$. Then due to the asymptotic structure of $g_n(y)$ the essential part of the integrand behaves like a power:

$$g_n(y) y^{-m} \sim g(y) n^{-7/2} n! \big(y^{\mu} \rho(y)\big)^{-n} = g(y) n^{7/2} n! \exp\big(-n \log\big(y^{\mu} \rho(y)\big)\big).$$

Hence, the integral can be approximated with the help of a saddle point method, where the saddle point equation $\frac{d}{dy} \log(y^{\mu}\rho(y)) = 0$ is precisely (7.2). This leads directly to (7.1); see [17] for details. Informally, one has to append just a saddle point integral at the very end of the calculations.

We can use exactly the same approach for the degree distribution. We consider the weighted number of rooted planar graphs of size n, where the root has degree k. For fixed y, the corresponding generating function $G^{\bullet}(x,y,w)$ has the property that the radius of convergence of $G^{\bullet}(x,y,w)$ does not depend on w and is, thus, given by $\rho(y)$. The asymptotic equivalent for $g_n^{\bullet}(y,w)$ contains the factor $\rho(y)^n$ and consequently it is possible to read off the coefficient of y^m with the help of a saddle point method as above. Note that we have computed in [12, Appendix] the coefficients of the singular expansion of $B(x,y,w)$ as a function not only of w but also of y. Hence, Theorem 6.13 extends directly to the following.

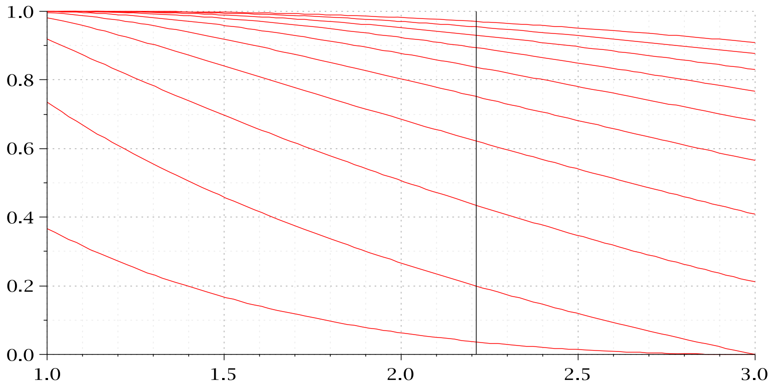

Fig. 1. Cumulative degree distribution for connected planar graphs with μn edges, $\mu \in (1,3)$ and $k = 1,\ldots,10$.

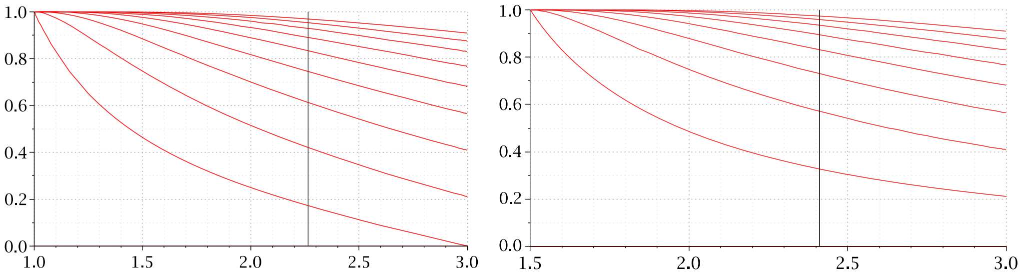

Fig. 2. Cumulative degree distribution for 2-connected planar graphs with μn edges, $\mu \in (1,3)$ and $k = 2,\ldots,10$ (left); and 3-connected planar graphs with μn edges, $\mu \in (3/2,3)$ and $k = 3,\ldots,10$ (right).

Theorem 7.1. *Let $\mu \in (1,3)$ and let $d_{\mu,k}$ be the limiting probability that a vertex of a random connected planar graph with edge density μ has degree k. Let y be the unique positive solution of* (7.2). *Then*

$$\sum_{k \geqslant 1} d_{\mu,k} w^k = -e^{B_0(y,w)-B_0(y,1)} B_2(y,w) + e^{B_0(y,w)-B_0(y,1)} \frac{1+B_2(y,1)}{B_3(y,1)} B_3(y,w),$$

where $B_j(y,w)$, $j = 0,2,3$ is given explicitly in [12, Appendix].

The probabilities $d_{\mu,k}$ can be computed explicitly using the expressions for the $B_i(y,w)$. As an illustration in Fig. 1 we present a cumulative plot for $k = 1,\ldots,10$. Each curve gives the probability that a vertex has degree at most k. The abscissa is the value of $\mu \in (1,3)$, while the ordinate gives the probability. The bottom curve corresponds to $k = 1$ and the top curve to $k \leqslant 10$. The vertical line is the value $\kappa \approx 2.21$ such that κn is the asymptotic expected number of edges in planar graphs [17]; since the number of edges is strongly concentrated around κn, the probabilities at $\mu = k$ correspond to the cumulated values from the third line of Table 1, namely 0.0367284, 0.1993078, 0.4347438, 0.6215175, 0.7510198, 0.8372003, ...

Fig. 2 shows the data for 2-connected and 3-connected planar graphs. Notice that in a 2-connected graph the degrees are at least 2, in a 3-connected graph they are at least 3, and a 3-connected graph has at least $3n/2$ edges. The main abscissa for 2-connected graphs is equal to 2.26 (see [3]), and for 3-connected graphs is equal to $(7+\sqrt{7})/4 \approx 2.41$ (see Theorem 6.10).

8. Concluding remarks

Fusy [15] has designed a very efficient algorithm for generating random planar graphs uniformly at random. He has performed extensive experiments on the degree distribution on planar graphs with

10,000 vertices and his experimental results, which he has very kindly shared with us, fit very well with the constants in Table 1.

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