Graphs and Combinatorics (2012) 28:265–275
DOI 10.1007/s00373-011-1039-9

ORIGINAL PAPER

On the Maximum Number of Cycles in Outerplanar and Series–Parallel Graphs

Anna de Mier · Marc Noy

Received: 19 February 2010 / Revised: 14 February 2011 / Published online: 8 March 2011

Abstract Let $c(n)$ be the maximum number of cycles in an outerplanar graph with n vertices. We show that $\lim c(n)^{1/n}$ exists and equals $\beta = 1.502837\ldots$, where β is a constant related to the recurrence $x_{n+1} = 1 + x_n^2,\ x_0 = 1$. The same result holds for the larger class of series–parallel graphs.

Keywords Cycles · Outerplanar graph · Series–parallel graph

Mathematics Subject Classification (2000) 05C38 · 05C35

1 Main Results

It is shown in [3] that the maximum number of cycles in a planar graph with n vertices is between 2.28^n and 3.37^n. These bounds have been recently improved in [4] to 2.42^n and 2.90^n, respectively. The problem of maximizing the number of cycles has been also investigated with respect to the cyclotomic number instead of the number of vertices [2].

Here we restrict our attention to particular classes of planar graphs and are able to obtain more precise results. A graph is outerplanar if it is planar and can be drawn in the plane so that all the vertices are in the outer face. A graph if series–parallel if it is a subgraph of a 2-tree; a 2-tree is obtained, starting from a triangle, by repeatedly

A. de Mier (✉) · M. Noy
Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya,
Jordi Girona 1-3, 08034 Barcelona, Spain
e-mail: anna.de.mier@upc.edu

M. Noy
e-mail: marc.noy@upc.edu

adding new vertices adjacent to two previously adjacent vertices. It is well known that outerplanar graphs are series–parallel, but not conversely. Series–parallel graphs are those not containing K_4 as a minor, and outerplanar graphs are those containing neither K_4 nor $K_{2,3}$ as a minor.

Let $c(n)$ be the maximum number of cycles in an outerplanar graph with $n+1$ vertices (nothing changes asymptotically if we let $n+1$ be the number of vertices instead of n, and this is technically convenient). Since adding edges among existing vertices will never decrease the number of cycles, we can restrict the analysis of $c(n)$ to maximal outerplanar graphs, which are isomorphic to polygon triangulations. Given such a triangulation P, let c be a vertex of degree two with neighbours a and b. Let P' be the result of removing c from P. There are at most $c(n-1)$ cycles in P that do not contain c, and the ones that do contain c are in bijection with the paths from a to b in P'. Let $p(n)$ be the maximum number of paths that join the ends of an exterior edge in a triangulation with $n+1$ vertices (we say that an edge of a triangulation, or more generally of a 2-tree, is exterior if it belongs to only one triangle). It follows that

$$p(n) - 1 \leq c(n) \leq c\,(n-1) + p\,(n-1) \qquad (1)$$

It turns out that $p(n)$ is easier to analyse since it satisfies the recurrence

$$p(n) = 1 + \max_{1\leq i\leq n-1} (p\,(i)\,p\,(n-i)). \qquad (2)$$

This follows easily by considering the triangle that contains a given exterior edge. The first values of $p(n)$ are given in the following table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$p(n)$	1	2	3	5	7	11	16	26	36	56	81	131	183	287	417	677

After some inspection one is naturally led to the conjecture that the maximum on the RHS of Equation (2) is attained when i is the power of two that is closest to $n/2$ (the sequence $p(n)$ and this conjecture appear as A091980 in [5]).

Theorem 1 *If r is the power of two closest to $n/2$ (either in case of a tie), then*

$$p\,(n) = 1 + p\,(r)\,p\,(n-r).$$

In particular,

$$p\left(2^k\right) = 1 + p\left(2^{k-1}\right)^2.$$

The proof is somewhat tedious, so we delay it until the following section and continue now with the analysis of $p(n)$ and $c(n)$. To do so we need the following results from [1] about the sequence $\{p(2^k)\}_{k\geq 0}$.

Proposition 1 *The solution to the quadratic recurrence* $x_{k+1} = 1 + x_k^2$, $x_0 = 1$, *is* $x_k = \lfloor \beta^{2^k} \rfloor$, *where*

$$\beta = \exp\left(\sum_{i\geq 1} \frac{1}{2^i} \log\left(1 + \frac{1}{x_{i-1}^2}\right)\right) = 1.502836801\ldots$$

Moreover, $x_k \geq \beta^{2^k} - \frac{2}{x_k}$ *for all* $k \geq 1$. *In particular, if* $k \geq 4$ *then* $x_k \geq \beta^{2^k} - \frac{1}{338}$.

Theorem 2 *For all* $n \geq 10$,

$$\beta^{n-\frac{1}{4}} < p(n) < \beta^n.$$

Proof For the lower bound we actually prove that $p(n) > \beta^{n-1/4} + 1$. The proof is by induction on n, the base case being all values of n with $10 \leq n \leq 30$, which can be easily checked computationally.

Now let $n > 30$. If n is a power of two, we have the bound of Proposition 1. Now assume that n is not a power of two and moreover that it is not of the form $2^k + j$ for j with $1 \leq j \leq 9$. Let 2^k be the first power of two that is larger than $n/2$ (note that $k \geq 4$). From the definition of $p(n)$, Proposition 1, and the induction hypothesis, we obtain

$$\begin{aligned} p(n) &\geq 1 + p\left(2^k\right) p\left(n - 2^k\right) \geq 1 + \left(\beta^{2^k} - \frac{1}{338}\right)\left(\beta^{n-2^k-\frac{1}{4}} + 1\right) \\ &= 1 + \beta^{n-\frac{1}{4}} + \beta^{2^k} - \frac{\beta^{n-2^k-\frac{1}{4}}}{338} - \frac{1}{338}. \end{aligned}$$

Since $n - 2^k - 1/4 < 2^k$, the desired (lower bound) inequality follows.

Now assume that $n = 2^k + j$ for some j with $1 \leq j \leq 9$. Theorem 1, Proposition 1, and the induction hypothesis give

$$\begin{aligned} p(n) &= 1 + p\left(2^{k-1}\right) p\left(2^{k-1} + j\right) \geq 1 + \left(\beta^{2^{k-1}} - \frac{1}{338}\right)\left(\beta^{2^{k-1}+j-\frac{1}{4}} + 1\right) \\ &= 1 + \beta^{n-1/4} + \beta^{2^{k-1}}\left(1 - \frac{\beta^{j-1/4}}{338}\right) - \frac{1}{338}. \end{aligned}$$

Since $\beta^{j-1/4} < 338$ for all $j \leq 9$, the bound follows.

As for the upper bound, the case where n is a power of two follows from Proposition 1. If n is not a power of two, we claim that $p(n) < \beta^n - 1/50$. For $n \leq 25$, this inequality is easily checked computationally.

Next we prove the claim for all n of the form $2^k + 2^{k-1}$, for $k \geq 5$. Set $a = 2^{k-1}$, $p(a) = \beta^a - \varepsilon$; we know from Proposition 1 that $\varepsilon < 1/338$. From Theorem 1 it follows that

$$p\left(2^k + 2^{k-1}\right) = p\,(3a) = 1 + p\,(a)\,p\,(2a) = 1 + p\,(a)\left(1 + p(a)^2\right)$$
$$= 1 + p\,(a) + p\,(a)^3.$$

Since $p(a) = \beta^a - \varepsilon$, we have

$$p(3a) = \beta^{3a} + 1 - \varepsilon - \varepsilon^3 + \beta^a\left(1 + 3\varepsilon^2 - 3\varepsilon\beta^a\right).$$

To prove the claim it is enough to show that $\beta^a(3\varepsilon\beta^a - 1 - 3\varepsilon^2) > 1 + 1/50$. Since we know that $\beta^{2a} - p(2a) > 0$ and $p(2a) = 1 + (\beta^a - \varepsilon)^2$, it follows that $\varepsilon\beta^a > 1/2$; this and the fact that $\beta^a > 3$ give the desired bound.

Now we prove the claim for the remaining values of n by induction. By Theorem 1 there is some r such that $p(n) = 1 + p(2^r)p(n - 2^r)$. Since n is not a sum of two consecutive powers of two, $n - 2^r$ is not a power of two, hence either from the induction hypothesis or from the particular case proved above, it follows that

$$p\,(n) < 1 + \beta^{2^r}\left(\beta^{n-2^r} - \frac{1}{50}\right) = \beta^n + 1 - \frac{\beta^{2^r}}{50}.$$

So we only need to show that

$$1 - \frac{\beta^{2^r}}{50} \leq -\frac{1}{50},$$

that is, that $\beta^{2^r} \geq 51$, and this holds for $r \geq 4$. Since $n \geq 25$, this is the case and the claim is proved. □

The lower bound in the previous theorem is of the form $c \cdot \beta^n$. We remark that the constant c cannot be arbitrarily close to 1; for instance, it can be shown by induction that $p(2^k + 11) < \beta^{2^k+11-1/5}$ for all $k \geq 3$. Although we needed the restriction $n \geq 10$ for the proof of the lower bound, the upper bound in Theorem 2 holds for all $n \geq 1$ and the lower bound for $n \geq 4$.

The bounds on $p(n)$ give almost directly the growth constant of $c(n)$.

Corollary 1 *Let $c(n)$ be the maximum number of cycles in an outerplanar graph with n vertices. Then there exist positive constants c_1 and c_2 such that*

$$c_1\,\beta^n \leq c\,(n) \leq c_2\,\beta^n.$$

As a consequence, $\lim c(n)^{1/n} = \beta$.

Proof The lower bound follows from the previous theorem and relation (1). On the other hand, $c(n) \leq p(n-1) + p(n-2) + \cdots + p(1) \leq (1-\beta)^{-1}\beta^n$. □

The previous results can be extended to arbitrary series–parallel graphs. The edge-maximal series–parallel graphs are 2-trees, so let $z(n)$ be the maximum number of cycles in a 2-tree with $n + 1$ vertices. Since a 2-tree must have a vertex of degree two, the analogue of relation (1) holds, namely

$$q(n) - 1 \leq z(n) \leq z(n-1) + q(n-1), \tag{3}$$

where $q(n)$ is the maximum number of paths joining the two ends of an edge of a 2-tree with $n + 1$ vertices.

We next show that both the numbers of paths and cycles are maximized in outerplanar graphs; that is, $q(n) = p(n)$ and $z(n) = c(n)$. We start by proving the first identity.

Proposition 2 *For all* n, $q(n) = p(n)$.

Proof Clearly $q(n) \geq p(n)$. We prove the other inequality by induction on n, the cases $n = 1, 2$ being trivial. Pick an edge of a 2-tree with $n + 1$ vertices. If this edge belongs to only one triangle, then the number of paths joining the two ends of this edge is at most $1 + q(r)q(n - r)$ for some r, and by the induction hypothesis this is at most $p(n)$. If the edge belongs to at least two triangles, the number of paths is at most $q(r) + q(n + 1 - r) - 1$ for some r with $2 \leq r \leq n + 1 - r$; by the induction hypothesis this is at most $p(r) + p(n + 1 - r) - 1$. Claim D from next section states that $p(n) - p(n - m) \geq p(n - 1) - p(n - 1 - m)$ for $m < n$ and $n \geq 2$; setting $m = r - 1$ and applying the claim $n - r$ times successively, we arrive at $p(n) - p(n + 1 - r) \geq p(r) - 1$, which implies the bound needed. □

Hence $\lim z(n)^{1/n} = \beta$, since on one side $c(n) \leq z(n)$ and on the other the same upper bound as before holds.

Next we show that among 2-trees the number of cycles is also maximized in outerplanar graphs, i.e., that $z(n) \leq c(n)$. We start by proving that for a fixed 2-tree the number of paths between the ends of an edge is maximized in an exterior edge (recall that an edge is exterior if it belongs to only one triangle). Let $p(T, e)$ be the number of paths in T joining the ends of e. In Fig. 1 below, we have $15 = p(T, e) < p(T, f) = p(T, g) = 24$.

Let G_1 and G_2 be two graphs on disjoint vertex sets and suppose in each of them there is an edge labelled e. Let the ends of e in G_1 and G_2 be x_1, y_1 and x_2, y_2, respectively. The parallel connection of G_1 and G_2 along the edge e is the graph obtained by identifying x_1 and x_2, and y_1 and y_2, and keeping just one copy of e. This definition

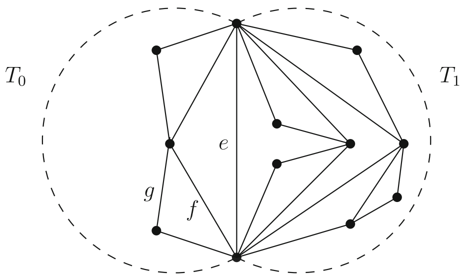

Fig. 1 Illustrating the proof of Lemma 1

extends naturally to the parallel connection of several graphs. Note that there is some ambiguity in the definition due to the choice of the labels of the end vertices of e; in case this could lead to non-isomorphic graphs, we just pick one of the possibilities arbitrarily.

Lemma 1 *For a given 2-tree, the number of paths between the ends of an edge is always maximized in an exterior edge.*

Proof Let e be an interior edge of a 2-tree T and suppose that e belongs to k triangles. We can view T as the parallel connection along the edge e of k 2-trees. Of those k 2-trees, pick the one, say T_0, such that $p(T_0, e)$ is the smallest possible, and let T_1 be the parallel connection along e of the remaining $k - 1$ 2-trees. (See Fig. 1.) Now consider the triangle in T_0 that contains e and let f be any of the other two edges of that triangle. We claim that $p(T, e) \leq p(T, f)$. This is enough to prove the lemma since by repeatedly applying this move we would get to an edge that only belongs to one triangle. Let $r = p(T_1, e)$, let s be the number of paths in T_0 between the ends of f that do not contain e and let t be the number of the ones that contain e. We have that $p(T, e) = r + st$ and $p(T, f) = s + rt$. Since s is less than r, we have $s + rt \geq r + st$, as needed. □

With this lemma we can prove that indeed the number of cycles is always maximized in outerplanar graphs. (Note though that the maximum can also be attained by graphs that are not outerplanar, as in the case of 5 or 7 vertices.)

Theorem 3 *For all n, $z(n) = c(n)$.*

Proof Given an edge e of a 2-tree T, let $\tau(e)$ be the number of triangles containing e, and define $\tau(T) = \sum_{e:\tau(e)\geq 3} \tau(e)$. Now let T be a 2-tree that has the maximum number of cycles and which minimizes $\tau(T)$. If $\tau(T) = 0$ then T is outerplanar, so assume for a contradiction that this is not the case and that e is an edge of T that belongs to at least three triangles. Express T as the parallel connection along e of T_1 and T_2, where e is an exterior edge of T_1. The number of cycles of T is the sum of the number of cycles of T_1, the number of cycles of T_2 and $(p(T_1, e) - 1)(p(T_2, e) - 1)$. By Lemma 1 above there is an exterior edge f of T_2 such that $p(T_2, f) \geq p(T_2, e)$. Let T' be the 2-tree obtained by taking the parallel connection of T_1 and T_2 along e and f, respectively. The graph T' has as many cycles as T, but $\tau(T') < \tau(T)$, which is a contradiction. □

From the previous result and Corollary 1 we get immediately the following result.

Corollary 2 *Let $z(n)$ be the maximum number of cycles in a series–parallel graph with n vertices. Then there exist positive constants c_1 and c_2 such that*

$$c_1\, \beta^n \leq z\,(n) \leq c_2\, \beta^n.$$

As a consequence, $\lim z(n)^{1/n} = \beta$.

2 Proof of Theorem 1

The proof of Theorem 1 is by induction on n. Actually, the induction hypothesis is made of the following two claims. Let $\mu(n)$ denote the power of two that is closest to $n/2$ (in case of a tie, take the largest one).

Claim M For $n \geq 2$ and $i \notin \{\mu(n), n - \mu(n)\}$,

$$p(n) = 1 + p(\mu(n))\, p(n - \mu(n)) > 1 + p(i)\, p(n - i).$$

Claim D For $n \geq 2$ and $m < n$,

$$p(n) - p(n-m) \geq p(n-1) - p(n-1-m),$$

with equality holding only when $m = 1$ and $n = 2^r + 1$ for some $r \geq 1$.

We will need the following easy lemma.

Lemma 2 *For all* $n \geq 2$, $\mu(n) = 2^k$ *if and only if* $3 \cdot 2^{k-1} \leq n < 3 \cdot 2^k$.

Both claims hold clearly for small values of n. We first prove Claim D by induction on n assuming that Claim M holds for all values less than or equal to n (this is no problem since the proof of Claim M only uses Claim D for values strictly less than n).
Proof of Claim D.

We start proving that for $n = 2^r + 1$ and $m = 1$ there is equality. The proof is by induction on r. For $r = 1$ the equality is trivial. For $r \geq 2$ clearly $\mu(2^r + 1) = \mu(2^r) = \mu(2^r - 1) = 2^{r-1}$, hence

$$p\left(2^r + 1\right) - 2p\left(2^r\right) + p\left(2^r - 1\right) = p\left(2^{r-1}\right)\left(p\left(2^{r-1} + 1\right) - 2p\left(2^{r-1}\right) + p\left(2^{r-1} - 1\right)\right) = 0,$$

where the first and second equalities follow by the induction hypotheses on M and D, respectively.

Now, to show that in all other cases the inequality is strict, note first that by an easy induction argument it is enough to prove it for $m = 1$. We distinguish three cases.
Case D.1: $\mu(n) = \mu(n-1) = \mu(n-2) = 2^k$
By the induction hypothesis on M we need to prove

$$p\left(n - 2^k\right) - p\left(n - 1 - 2^k\right) > p\left(n - 1 - 2^k\right) - p\left(n - 2 - 2^k\right),$$

which holds by the induction hypothesis on D (by Lemma 2, $3 \cdot 2^{k-1} \leq n - 2 < 3 \cdot 2^k$, and this implies that if $n - 2^k$ is of the form $2^r + 1$ then $n = 2^{k+1} + 1$, contrary to the hypothesis).
Case D.2: $\mu(n) = 2^k$ and $\mu(n-1) = \mu(n-2) = 2^{k-1}$
This follows by an argument analogous to that of Case D.1 noting that n must be $3 \cdot 2^{k-1}$.

Case D.3: $\mu(n) = \mu(n-1) = 2^k$ *and* $\mu(n-2) = 2^{k-1}$
In this case we have that $n - 1 = 3 \cdot 2^{k-1}$ and the inequality to prove becomes, after using M,

$$p\left(2^k\right)\left(p\left(2^{k-1}+1\right) - p\left(2^{k-1}\right)\right) > p\left(2^{k-1}\right)\left(p\left(2^k\right) - p\left(2^k - 1\right)\right).$$

By the induction hypothesis on D, the LHS equals $p(2^k)(p(2^{k-1}) - p(2^{k-1}-1))$, and hence the inequality is equivalent to $p(2^{k-1})p(2^k - 1) > p(2^k)p(2^{k-1}-1)$, and this holds by M since $2^k + 2^{k-1} - 1 = n - 2$ and $\mu(n-2) = 2^{k-1}$. □

Proof of Claim M.

From now on we set $\mu(n) = 2^k$ and $l = n - 2^k$. We need to show that for all i with $1 \le i \le n/2$, $i \notin \{2^k, l\}$, it holds that $p(i)p(n-i) < p(2^k)p(l)$. We distinguish two cases according to whether $n/2$ is larger or smaller than 2^k.

Case M.1: $2^k \le n/2$
The assumption implies that $l = n - 2^k \ge 2^k$, hence by Lemma 2 the value of $\mu(l)$ is either 2^k or 2^{k-1}.

Let us start by taking $i < 2^k$. By the induction hypothesis on M, the inequality to prove is equivalent to

$$p(i)(1+p(\mu(n-i))\,p(n-i-\mu(n-i))) < p\left(2^k\right)(1+p(\mu(l))\,p(l-\mu(l))).$$

Since $p(i) < p(2^k)$, it is enough to show that

$$p(i)\,p(\mu(n-i))\,p(n-i-\mu(n-i)) \le p\left(2^k\right)p(\mu(l))\,p(l-\mu(l)). \quad (4)$$

Note that $\mu(n-i)$ lies between $\mu(n)$ and $\mu(l)$, so this leads us to consider three subcases.

- *M.1.1* If $\mu(l) = \mu(n-i) = 2^k$, then Equation (4) becomes $p(i)p(n-i-2^k) \le p(2^k)p(l-2^k)$, which holds by the induction hypothesis since the RHS equals $p(l) - 1$.
- *M.1.2* If $\mu(l) = 2^{k-1}$ and $\mu(n-i) = 2^k$, the proof follows an analogous argument.
- *M.1.3* If $\mu(l) = \mu(n-i) = 2^{k-1}$, the equation to prove is $p(i)p(n-i-2^{k-1}) \le p(2^k)p(l-2^{k-1})$. This will hold by the induction hypothesis if we show that $\mu(n-2^{k-1}) = 2^k$; this is the case since otherwise we would have, by Lemma 2, that $n - 2^{k-1} < 3 \cdot 2^{k-1}$ and hence that $n < 2^{k+1}$, contrary to the initial hypothesis of case M.1.

Next we consider the case $2^k < i \le n/2$. Now $\mu(i) = 2^{k-1}$ and $\mu(n-i)$ lies between 2^{k-1} and $\mu(l)$.

- *M.1.4* If $\mu(l) = \mu(n-i) = 2^{k-1}$, by applying the induction hypothesis on all four terms of $p(i)p(n-i) < p(2^k)p(l)$ we obtain

$$\left(1 + p\left(2^{k-1}\right)p\left(i - 2^{k-1}\right)\right)\left(1 + p\left(2^{k-1}\right)p\left(n - i - 2^{k-1}\right)\right)$$
$$< \left(1 + p\left(2^{k-1}\right)p\left(2^{k-1}\right)\right)\left(1 + p\left(2^{k-1}\right)p\left(l - 2^{k-1}\right)\right).$$

So it is enough to show the following two inequalities.

$$p\left(i-2^{k-1}\right)p\left(n-i-2^{k-1}\right)<p\left(2^{k-1}\right)p\left(l-2^{k-1}\right) \tag{5}$$
$$p\left(i-2^{k-1}\right)+p\left(n-i-2^{k-1}\right)\leq p\left(2^{k-1}\right)+p\left(l-2^{k-1}\right) \tag{6}$$

Equation (5) holds by the induction hypothesis since $\mu(l)=2^{k-1}$. Equation (6) holds by repeatedly applying Claim D.

M.1.5 Suppose $\mu(l)=\mu(n-i)=2^k$. Expanding the inequality $p(i)p(n-i)<p(2^k)p(l)$ as above, we see that we need to prove the following.

$$p\left(i-2^{k-1}\right)p\left(n-i-2^{k}\right)<p\left(2^{k-1}\right)p\left(l-2^{k}\right) \tag{7}$$
$$p\left(2^{k-1}\right)\left(p\left(i-2^{k-1}\right)-p\left(2^{k-1}\right)\right)\leq p\left(2^{k}\right)\left(p\left(l-2^{k}\right)-p\left(n-i-2^{k}\right)\right) \tag{8}$$

For Equation (7), it is enough to show that $\mu(l-2^{k-1})=2^{k-1}$, and this follows without difficulty from Lemma 2. For the second inequality we use again the induction hypothesis on D and that $p(2^{k-1})<p(2^k)$.

M.1.6 Finally, suppose $\mu(n-i)=2^{k-1}$ and $\mu(l)=2^k$. As in the previous two cases, we decompose the initial inequality using the induction hypothesis and we obtain two more inequalities. The first of them follows by the same argument as in case M.1.4. The second one is

$$p\left(2^{k-1}\right)p\left(i-2^{k-1}\right)+p\left(2^{k-1}\right)p\left(n-i-2^{k-1}\right)<p\left(2^{k-1}\right)p\left(2^{k-1}\right) +p\left(2^{k}\right)p\left(l-2^{k}\right). \tag{9}$$

By the induction hypothesis on D we know that

$$p(2^{k-1})(p(i-2^{k-1})+p(n-i-2^{k-1}))\leq p(2^{k-1})(p(2^{k-1})+p(l-2^{k-1})),$$

and by the induction hypothesis on M we have $p(2^{k-1})p(l-2^{k-1})<p(2^k)p(l-2^k)$, hence by combining these two inequalities we obtain (9).

This finishes the case $2^k\leq n/2$.

Case M.2: $2^k>n/2$

We show that $p(i)p(n-i)<p(2^k)p(l)$ for all i with $n/2\leq i<n, i\neq 2^k$. The bound on $n/2$ and Lemma 2 imply that $3\cdot 2^{k-1}\leq n<2^{k+1}$, hence $\mu(l)$ is either 2^{k-1} or 2^{k-2}.

We consider first that $i>2^k$. By the induction hypothesis it is enough to show that

$$(1+p(\mu(i))p(i-\mu(i)))p(n-i)<\left(1+p\left(2^{k-1}\right)p\left(2^{k-1}\right)\right)p(l), \tag{10}$$

and since $p(n-i) < p(l)$, it suffices to show

$$p\left(\mu\left(i\right)\right)p\left(i-\mu\left(i\right)\right)p\left(n-i\right) \le p\left(2^{k-1}\right)p\left(2^{k-1}\right)p\left(l\right). \tag{11}$$

The value of $\mu(i)$ is either 2^{k-1} or 2^k, since $2^k < i < n$. This leads us to consider two cases.

M.2.1 If $\mu(i) = 2^{k-1}$, to prove inequality (11) it is enough, by the induction hypothesis, to show that $\mu(n-2^{k-1}) = 2^{k-1}$. This holds by combining Lemma 2 and the inequalities $3\cdot 2^{k-1} \le n < 2^{k+1}$, which imply that $4\cdot 2^{k-2} \le n-2^{k-1} < 3\cdot 2^{k-1}$.

M.2.2 If $\mu(i) = 2^k$, using the induction hypothesis and the fact that $2^k > n-2^k$ we obtain the following chain of inequalities

$$\begin{aligned} p\left(l\right)p\left(2^{k-1}\right)^2 &\ge \left(1+p\left(i-2^k\right)p\left(n-i\right)\right)p\left(2^{k-1}\right)^2 \\ &\ge p\left(i-2^k\right)p\left(n-i\right)\left(1+p\left(2^{k-1}\right)^2\right). \end{aligned}$$

The last term equals $p(i-2^k)p(n-i)p(2^k)$, as needed to prove (11).

We consider now the case $n/2 \le i < 2^k$. The value of $\mu(i)$ is 2^{k-1}. We apply the induction hypothesis to all four terms of $p(i)p(n-i) < p(2^k)p(l)$ and as we did in Case M.1, to prove the resulting inequality it suffices to prove the following two.

$$p\left(2^{k-1}\right)p\left(i-2^{k-1}\right)p\left(\mu\left(n-i\right)\right)p\left(n-i-\mu\left(n-i\right)\right) < p\left(2^{k-1}\right)^2 p\left(\mu\left(l\right)\right)p\left(l-\mu\left(l\right)\right) \tag{12}$$

$$p\left(2^{k-1}\right)p\left(i-2^{k-1}\right)+p\left(\mu\left(n-i\right)\right)p\left(n-i-\mu\left(n-i\right)\right) \le p\left(2^{k-1}\right)^2+p\left(\mu\left(l\right)\right)p\left(l-\mu\left(l\right)\right) \tag{13}$$

Since $\mu(l) \le \mu(n-i) \le \mu(n/2) = 2^{k-1}$, there are three possibilities according to the values of $\mu(l)$ and $\mu(n-i)$

M.2.3 Suppose first that $\mu(l) = \mu(n-i) = 2^{k-2}$. Inequality (12) reduces to $p(i-2^{k-1})p(n-i-2^{k-2}) < p(2^{k-1})p(l-2^{k-2})$, which holds by the induction hypothesis since $\mu(l+2^{k-2}) = 2^{k-1}$, as is easily checked. Inequality (13) can be written as

$$\begin{aligned} &p\left(2^{k-2}\right)\left(p\left(n-i-2^{k-2}\right)-p\left(l-2^{k-2}\right)\right) \\ &\quad\le p\left(2^{k-1}\right)\left(p\left(2^{k-1}\right)-p\left(i-2^{k-1}\right)\right), \end{aligned}$$

which follows from the induction hypothesis on D.

M.2.4 Suppose that $\mu(l) = 2^{k-2}$ and $\mu(n-i) = 2^{k-1}$. Inequality (12) follows easily from the induction hypothesis. As for inequality (13), note first that

$$p\left(n-i-2^{k-1}\right)-p\left(l-2^{k-1}\right) < p\left(2^{k-1}\right)-p\left(i-2^{k-1}\right)$$

by the induction hypothesis on D. This implies

$$p\left(2^{k-1}\right)p\left(i-2^{k-1}\right)+p\left(2^{k-1}\right)p\left(n-i-2^{k-1}\right)<p\left(2^{k-1}\right)p\left(2^{k-1}\right)+p\left(2^{k-1}\right)p\left(l-2^{k-1}\right),$$

which gives the desired result when combined with $p(2^{k-2})p(l-2^{k-2}) > p(2^{k-1})p(l-2^{k-1})$, which holds by the induction hypothesis.

M.2.5 Finally, the case that $\mu(l) = \mu(n-i) = 2^{k-1}$ is proved in a way analogous to that of case M.2.3. □

Acknowledgments Supported by projects MTM2008-03020 and DGR2009-SGR1040.

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