



# Lower bounds on the number of crossing-free subgraphs of $K_N$

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## Abstract

We improve previous lower bounds on the number of simple polygonizations, and other kinds of crossing-free subgraphs, of a set of  $N$  points in the plane by analyzing a suitable configuration. We also prove that the number of crossing-free perfect matchings and spanning trees is minimum when the points are in convex position. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

A set of points in the plane always admits a simple polygonization. In fact, unless the points are in convex position, there is more than one such polygonization. Let  $\Phi(P)$  be the number of simple polygonizations of a point set  $P$ , and let  $\Phi(N)$  be the maximum of  $\Phi(P)$  among all sets  $P$  of  $N$  points. The problem of computing  $\Phi(N)$  was introduced by Newborn and Moser [11] who showed that

$$c10^{N/3} \leq \Phi(N) \leq c'6^N (\lfloor N/2 \rfloor)!,$$

where  $c$  and  $c'$  are suitable constants. Since then both bounds have been improved as we summarize below.

Take a set  $P$  of  $N$  points in the plane and consider the rectilinear straight-line drawing of the complete graph  $K_N$  they define. A subgraph of  $K_N$  is said to be *crossing-free* if its edges intersect only at the vertices. A fundamental result was proven by Ajtai et al. [1]: the number of crossing-free subgraphs of any plane drawing of  $K_N$  (even if one allows nonrectilinear edges) never exceeds a fixed exponential in  $N$ , namely  $10^{13N}$ . Further improvements on the upper bound have been obtained by considering the number of triangulations of a set of  $N$  points. Since every crossing-free subgraph can be extended to a triangulation and a triangulation has at most  $3N$  edges, a bound of  $\alpha^N$  on the number of triangulations implies a corresponding bound of  $2^{3N}\alpha^N = (8\alpha)^N$  on the number of crossing-free graphs of any kind. An

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upper bound of  $173\,000^N$  on the number of triangulations was proven by Smith [14]. A sharper bound of  $2^{13N}$  has been recently established by Seidel [13], thus  $2^{16N}$  is an upper bound for the number of crossing-free subgraphs of any plane drawing of  $K_N$ . Moreover, Fekete's lemma (see, for example, [15]) establishes that  $\lim_{N \rightarrow \infty} f(N)^{1/N}$  exists if  $f(m+n) \geq f(m)f(n) \forall m, n$ . Taking  $m$  points for which  $\Phi(m)$  is achieved, and other  $n$  points, separated from the previous  $m$  points by a straight line, for which  $\Phi(n)$  is achieved, we can form at least  $\Phi(m)\Phi(n)$  polygonizations on these  $m+n$  points. So,  $\Phi(m+n) \geq \Phi(m)\Phi(n)$ , and  $\lim_{N \rightarrow \infty} \Phi(N)^{1/N}$  exists and it is smaller than  $2^{16}$ .

On the other hand, lower bounds on  $\Phi(N)$  are obtained by analyzing specific configurations of points in the plane. First Akl [2], then Hayward [7], and more recently, García and Tejel [6] improved the lower bound to  $2.27^N$ ,  $3.26^N$  and  $3.605^N$ , respectively.

In this paper we introduce a new configuration of  $N$  points showing that

$$\lim_{N \rightarrow \infty} \Phi(N)^{1/N} \geq 4.642.$$

We also show that the same configuration provides a large number of crossing-free subgraphs of several kinds. In particular, we prove that the number of triangulations is  $\Omega(8^N/N^{O(1)})$ , the number of crossing-free perfect matchings is  $\Omega(3^N/N^{O(1)})$ , and the number of crossing-free spanning trees is  $\Omega(9.35^N)$ . These results are an improvement on those appearing in [14]. Finally, we prove that the number of crossing-free perfect matchings, and also the number of crossing-free spanning trees, is minimum when the points are in convex position.

## 2. Preliminaries

The basic configuration  $V_N$  that we analyze is shown in Fig. 1. It consists of  $N = 2n$  points,  $p_1, \dots, p_n$  on the upper line  $L_1$  and  $q_1, \dots, q_n$  on the lower line  $L_2$ . Both lines are convex with opposed concavity. Besides, for every  $i$  and  $j$  the line connecting  $p_i$  and  $p_j$  leaves all points of  $L_2$  below, and the line

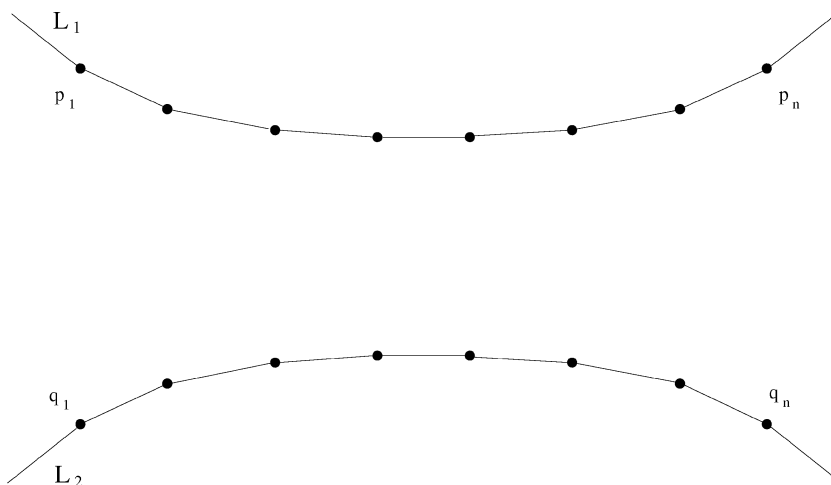


Fig. 1. The basic configuration  $V_N$ .

connecting  $q_i$  and  $q_j$  leaves all points of  $L_1$  above. The numbering of the points in both lines is from left to right.

For this configuration, the number of different crossing-free subgraphs only depends on  $n$ , and not on the exact position of the points on  $L_1$  or  $L_2$ . In addition, the number of crossings is not too large, it is  $2\binom{n}{4} + \binom{n}{2}^2 \sim (2n)^4/48$ , and it has many symmetries, which are needed in the combinatorial analysis.

We need, as prerequisites, several facts from enumerative combinatorics. First, the classical result that the number of triangulations of a convex polygon with  $n + 2$  vertices is the Catalan number  $C_n = \binom{2n}{n}/(n + 1) = \Theta(n^{-3/2}4^n)$ . Also, that the number of crossing-free perfect matchings of  $2n$  points in convex position (classically referred to as non-crossing configurations of chords on a circle) is again the Catalan number  $C_n$ . And finally, that the number of crossing-free spanning trees of  $n + 1$  points in convex position is equal to  $\binom{3n}{n}/(2n + 1) = \Theta(n^{-3/2}(27/4)^n)$ , a generalized Catalan number. The first result goes back to Euler; for the second one see [4,10]; the third one can be found in [4] (see also [12]).

Also, we need the special case of Lagrange’s inequality:  $(\sum_k a_k^2) \geq (\sum_k a_k)^2/n$  (see, for example, [9]) and the following theorem about the asymptotic behaviour of the coefficients of a power series given by Bender in [3].

**Theorem 2.1** (Bender). *Assume that the power series  $w(z) = \sum a_n z^n$  with nonnegative coefficients satisfies  $F(z, w) \equiv 0$ . Suppose there exist real numbers  $r > 0$  and  $s > a_0$  such that*

- (i) *for some  $\delta > 0$ ,  $F(z, w)$  is analytic whenever  $|z| < r + \delta$  and  $|w| < s + \delta$ ;*
- (ii)  *$F(r, s) = F_w(r, s) = 0$ ;*
- (iii)  *$F_z(r, s) \neq 0$ , and  $F_{ww}(r, s) \neq 0$ ; and*
- (iv) *if  $|z| \leq |r|$ ,  $|w| \leq |s|$ , and  $F(z, w) = F_w(z, w) = 0$ , then  $z = r$  and  $w = s$ .*

*Then*

$$a_n \sim \left( \frac{(r F_z)}{2\pi F_{ww}} \right)^{1/2} n^{-3/2} r^{-n},$$

*where the partial derivatives  $F_z$  and  $F_{ww}$  are evaluated at  $z = r$ ,  $w = s$ .*

From now on, a *matching* will be a crossing-free perfect matching, a *tree* will be a crossing-free spanning tree, and polygonizations will always be simple.

### 3. Polygonizations

Let  $C$  be a simple polygonal line visiting all the points of  $V_N$  starting at  $p_1$  and ending at  $q_n$ , and let  $k$  be the number of edges on  $C$  from  $L_1$  to  $L_2$  or from  $L_2$  to  $L_1$ . Let  $\{p_{i_1}, \dots, p_{i_k}\}$  and  $\{q_{j_1}, \dots, q_{j_k}\}$  be the extremes of these edges, with  $i_1 \leq i_2 \leq \dots \leq i_k$  and  $j_1 \leq j_2 \leq \dots \leq j_k$ ; the order in which  $\{p_{i_1}, \dots, p_{i_k}\}$  are visited in  $C$  can be quite arbitrary. Among all polygonal lines starting at  $p_1$  and ending at  $q_n$ , we consider only those in which the points  $\{p_{i_1}, \dots, p_{i_k}\}$  (and hence  $\{q_{j_1}, \dots, q_{j_k}\}$ ) are visited in exactly this order in the polygonal line (see Fig. 2). The family of such polygonal lines will be denoted by  $S$  and their number by  $g(n)$ .

Obviously, one can close these polygonal lines by adding a suitable extra point on  $L_2$  (see Fig. 2). Therefore, the number of polygonizations of the  $2n + 1$  points will be greater than  $g(n)$ .

If we take  $C \in S$  and consider how  $C$  visits the  $n$  points on  $L_2$  or  $L_1$ , we have the two curves shown in Fig. 3. Each of these curves visits  $n$  points in convex position such that any point  $i$  can be visited

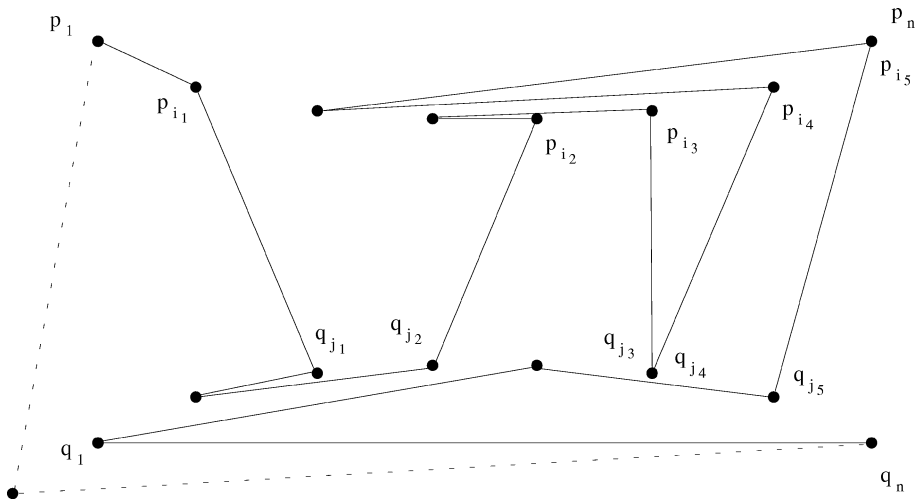


Fig. 2. Polygonizations in  $V_N$ .

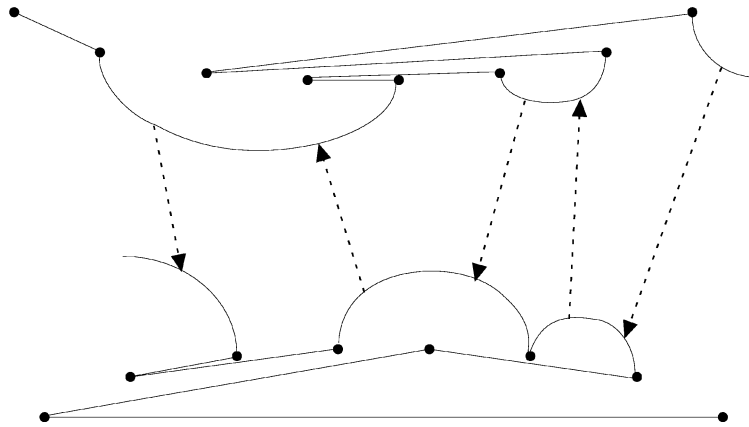


Fig. 3. Forming a polygonal line of  $S$  combining two curves of type 3.

from another point  $j$  via an edge, or via a jump outside the convex  $n$ -gon (this jump corresponds to the polygonal line that visits some points of the opposite line). None of the jumps can be enveloped by another jump because  $\{p_{i_1}, \dots, p_{i_k}\}$  are visited in exactly this order.

These two curves, and other types of curves visiting  $n$  points in convex position that we will use later, can be defined in the following way. Let us consider a convex polygon with  $n + 1$  vertices, numbered clockwise from 0 to  $n$ , and a simple closed curve  $C$  visiting them such that:

- (i) Any vertex  $j$  can be visited in  $C$  from another vertex  $i$  via a link inside the convex polygon (that we call an *edge*), or via a link outside the convex polygon (that we call a *jump*).
- (ii) When  $C$  is visited clockwise from vertex 0,  $C$  starts with a jump, and vertex 0 is reached in  $C$  via a jump, or via an edge from 1, or via an edge from  $n$ . Besides, if  $(i_1, j_1), \dots, (i_k, j_k)$  are the beginning and the end of the  $k$  jumps of  $C$ , with  $i_1 \leq i_2 \leq \dots \leq i_k$  in cyclic order, then  $j_1 \leq j_2 \leq \dots \leq j_k$  in cyclic order. (This condition means that a jump does not envelope another jump.)

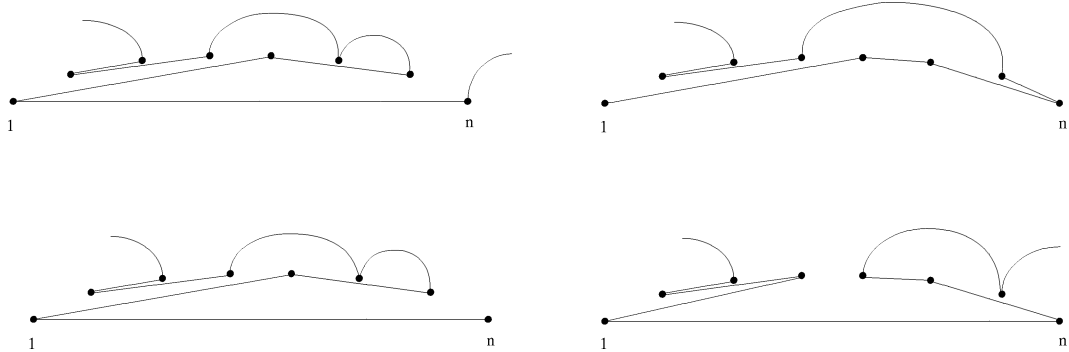


Fig. 4. Four types of curves.

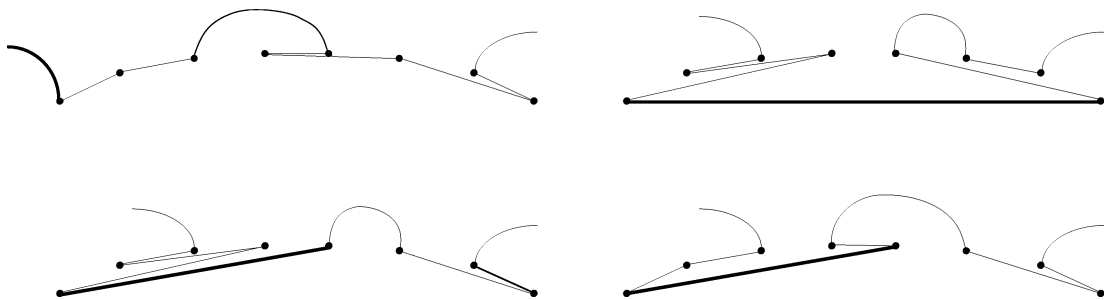


Fig. 5. Decomposition of a curve of type 1.

Two cycles  $C_1$  and  $C_2$  verifying these conditions are considered equal if the vertices  $\{0, 1, \dots, n\}$  are visited in the same order and with the same type of link (edge or jump).

For these simple cycles, if we omit the vertex 0 and the edge that arrives at it, if it exists, we have the following types of curves (see Fig. 4):

- Type 1: the curve starts and ends with jumps.
- Type 2: the curve starts with a jump and ends at 1.
- Type 3: the curve starts with a jump and ends at  $n$ .

In addition, we consider the following subtype of type 1:

- Type 4: as type 1, but 1 and  $n$  must be directly joined, and a vertex  $i$  is visited before 1, and a vertex  $j$  is visited after  $n$ .

The number of curves of type 1, 2, 3 and 4 will be denoted by  $g_1(n)$ ,  $g_2(n)$ ,  $g_3(n)$  and  $g_4(n)$ , respectively. From any polygonal line  $C \in S$  we can form two curves of type 3, the first one on the points on  $L_2$ , and the second one on the points on  $L_1$  in reverse order, both having the same number of jumps (see Figs. 2 and 3). Conversely, from a curve of type 3 on  $L_2$  with  $k$  jumps, and another curve of type 3 on  $L_1$  in reverse order with  $k$  jumps, we can form a single curve of  $S$  by sequentially merging the jumps of the two curves (see Fig. 3). If the number of curves of type 3 with exactly  $k$  jumps is denoted by  $g_3(n, k)$ , then  $g_3(n) = \sum_k g_3(n, k)$  and  $g(n) = \sum_k g_3(n, k)^2$ . We are ready for the following result.

**Theorem 3.1.** *The number of polygonizations of  $V_N$  is  $\Omega(4.64^N)$ .*

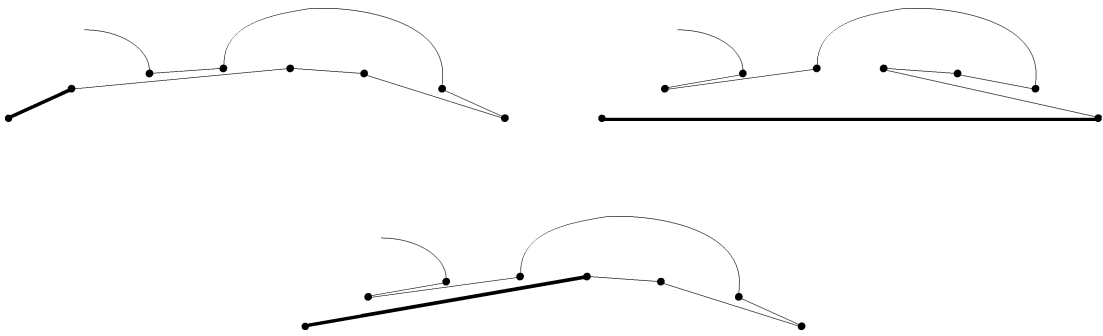


Fig. 6. Decomposition of a curve of type 2.

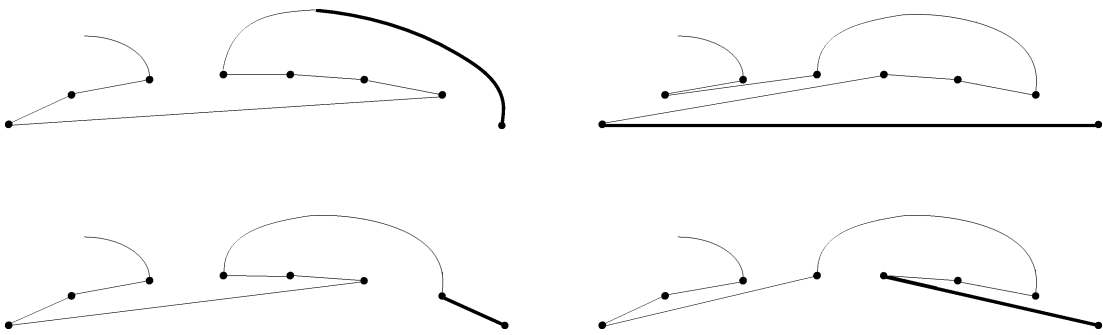


Fig. 7. Decomposition of a curve of type 3.

**Proof.** Recurrence formulas for  $g_1(n)$ ,  $g_2(n)$ ,  $g_3(n)$  and  $g_4(n)$  can be obtained since every curve of any of the four types is formed by curves of the same types using fewer points. For example, for a curve of type 2 (see Fig. 6), either 1 is joined to 2 (and then another curve of type 2 with  $n - 1$  points appears), or 1 is joined to  $n$  (and then a curve of type 3 with  $n - 1$  points appears), or 1 is joined to  $i + 2$  (and then a curve of type 1 with  $i$  points and a curve of type 2 with  $n - i + 1$  points appear). Figs. 5 and 7 show the possible decompositions of curves of types 1 and 3, respectively. The arguments for obtaining these decompositions are essentially the same as those detailed in [6] for a similar configuration of points. We then have the following recurrence formulas:

$$g_1(n) = g_3(n) + g_4(n) + \sum_{i=2}^{n-1} g_2(i)g_3(n - i) + \sum_{i=4}^{n-1} g_4(i)g_1(n - i), \quad n \geq 3,$$

$$g_2(n) = g_2(n - 1) + g_3(n - 1) + \sum_{i=1}^{n-3} g_1(i)g_2(n - i - 1), \quad n \geq 3,$$

$$g_3(n) = g_1(n - 1) + g_2(n - 1) + g_3(n - 1) + \sum_{i=1}^{n-3} g_1(i)g_2(n - i - 1), \quad n \geq 3,$$

$$g_4(n) = \sum_{i=2}^{n-2} g_2(i)g_2(n - i), \quad n \geq 4,$$

with the initial conditions:  $g_1(0) = g_2(0) = g_3(0) = g_4(0) = 0$ ,  $g_1(1) = g_2(1) = g_3(1) = 1$ ,  $g_4(1) = 0$ ,  $g_1(2) = g_3(2) = 2$ ,  $g_2(2) = 1$ ,  $g_4(2) = 0$  and  $g_4(3) = 0$ . These recurrence formulas are equivalent to

$$g_1(n) = g_3(n) + g_4(n) + \sum_{i=0}^n g_2(i)g_3(n-i) - g_3(n-1) + \sum_{i=0}^n g_4(i)g_1(n-i), \quad n \geq 3,$$

$$g_2(n) = g_2(n-1) + g_3(n-1) + \sum_{i=0}^{n-1} g_1(i)g_2(n-i-1) - g_1(n-2), \quad n \geq 3,$$

$$g_3(n) = g_2(n) + g_1(n-1), \quad n \geq 1,$$

$$g_4(n) = \sum_{i=0}^n g_2(i)g_2(n-i) - 2g_2(n-1), \quad n \geq 3,$$

and further manipulation gives

$$g_1(n) = 2g_2(n) - g_2(n-1) + g_1(n-1) - g_1(n-2) + 2g_4(n) + \sum_{i=0}^n g_4(i)g_1(n-i), \quad n \geq 3,$$

$$g_2(n) = 2g_2(n-1) + \sum_{i=0}^{n-1} g_1(i)g_2(n-i-1), \quad n \geq 3,$$

$$g_4(n) = \sum_{i=0}^n g_2(i)g_2(n-i) - 2g_2(n-1), \quad n \geq 3.$$

Now, let  $G_i(z) = \sum_{n \geq 0} g_i(n)z^n$  be the generating function of  $g_i(n)$ , for  $1 \leq i \leq 4$ . From the above equations we obtain

$$G_1(z)G_4(z) = (1 - z + z^2)G_1(z) + (z - 2)G_2(z) - 2G_4(z) + z,$$

$$G_1(z)G_2(z) = (1/z - 2)G_2(z) + z - 1,$$

$$G_4(z) = G_2(z)^2 - 2zG_2(z) + z^2 = (G_2(z) - z)^2.$$

By replacing the values of  $G_1(z)$  and  $G_4(z)$  in the first equation, obtained from the second and the third ones, the following equation holds:

$$F(z, G_2(z)) = G_2(z)^3 - zG_2(z)^2 + (-1 + 3z - z^2)G_2(z) + z(z - 1)^2 = 0.$$

Solving  $F(z, w) = F_w(z, w) = 0$ , we obtain  $r = 0.2154185247$  and  $s = 0.444014874$  as the roots closest to zero. Hence, by Theorem 2.1,

$$g_2(n) \sim c_2 n^{-3/2} (1/r)^n,$$

where  $c_2 = 0.1377744345$ . Obviously, from a curve of type 2 we obtain a curve of type 3 joining 1 to an extra point  $n + 1$ . Therefore,  $g_3(n) \sim c_3 n^{-3/2} (1/r)^n$ . Clearly,

$$\left( \sum_k g_3(n, k) \right)^2 \geq \sum_k g_3(n, k)^2 \geq \frac{(\sum_k g_3(n, k))^2}{n},$$

so we have

$$g_3(n)^2 > g(n) > \frac{g_3(n)^2}{n}.$$

It follows that

$$\lim_{n \rightarrow \infty} g(n)^{1/2n} = \frac{1}{r} = 4.642126305,$$

and this finishes the proof.  $\square$

#### 4. Triangulations, matchings and trees

In [14] a set of  $N$  points with  $\Omega(6.75^N)$  triangulations is given (this is a correction, using the results in [8], of the claimed value of  $\Omega(9.08^N)$ ). Also in [14] sets of  $N$  points are shown with  $\Omega(2.618^N)$  matchings and  $\Omega(7.10^N)$  trees. The configuration  $V_N$  allows us to increase these values.

**Theorem 4.1.**  $V_N$  admits (1)  $\Omega(8^N/N^{O(1)})$  triangulations; (2)  $\Omega(3^N/N^{O(1)})$  matchings; and (3)  $\Omega(9.35^N)$  trees.

**Proof.** In all three cases we compute either the exact value or an asymptotic lower bound for the respective number of triangulations, matchings and trees in  $V_N$ .

(1) Any triangulation of  $V_N$  has to use necessarily the edges  $p_1p_2, p_2p_3, \dots, p_n p_1, q_1q_2, q_2q_3, \dots, q_n q_1$  and  $p_1q_1, p_nq_n$ . Hence, we have a decomposition of the convex hull of  $V_N$  into two convex  $n$ -gons and one non-convex  $2n$ -gon  $Q_N = p_1p_2 \dots p_nq_nq_{n-1} \dots q_1$ . A triangulation of  $Q_N$  can be encoded as an ordered sequence of  $2n - 2$  triangles. The sign of a triangle within a triangulation is defined as follows: it is *positive* (pointing up) if it is  $q_iq_{i+1}p_j$ , and it is *negative* (pointing down) if it is  $p_i p_{i+1}q_j$ . It is clear that the number of positive triangles, as well as the number of negative triangles, is equal to  $n - 1$ . Hence, a triangulation of  $Q_N$  corresponds to a binary sequence of length  $2n - 2$  defined in the alphabet  $\{+, -\}$  with the same number of  $+$  and  $-$  signs. (See Fig. 8 for an example.) It follows that the number of triangulations in  $V_N$  is

$$C_{n-2}^2 \binom{2n-2}{n-1} = \Theta(8^{2n} n^{-7/2}) = \Theta(8^N N^{-7/2}).$$

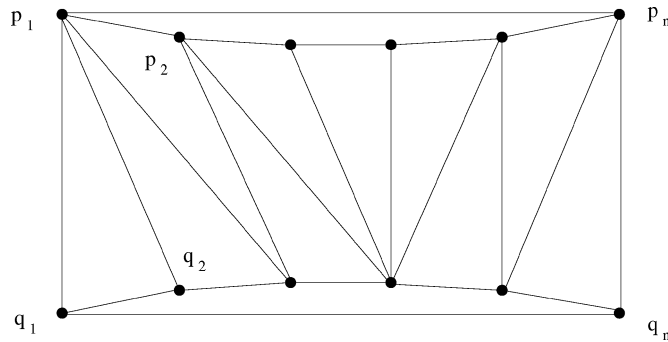


Fig. 8. A triangulation with code  $++-+----+--+$ .

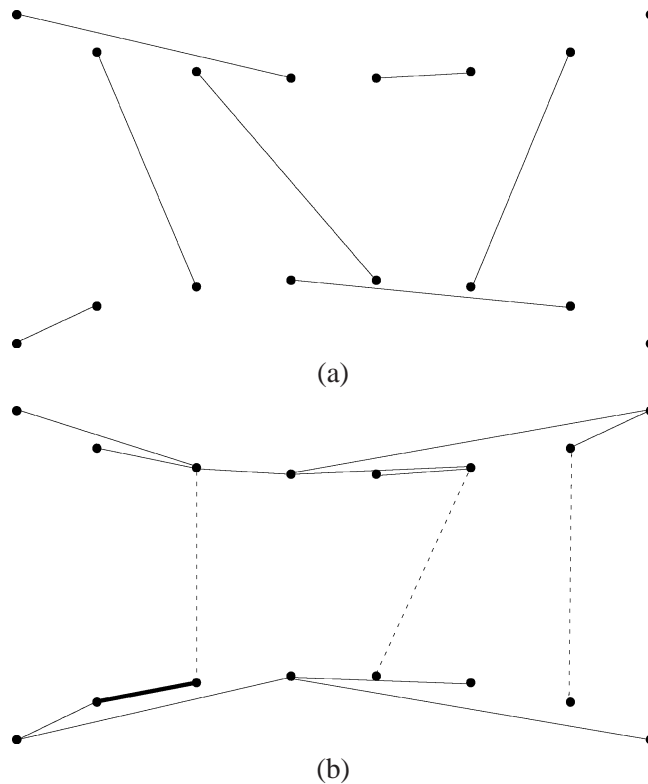


Fig. 9. Matchings and trees in  $V_N$ .

(2) In any matching of  $V_N$  there will be  $k$  points of  $L_1$  matched with  $k$  points of  $L_2$ , with  $n - k$  even and  $0 \leq k \leq n$ . The unmatched points will form two convex sets of size  $n - k$  (Fig. 9(a)). Thus the number of matchings is

$$\sum_{\substack{k=0 \\ n-k \text{ even}}}^n \binom{n}{k}^2 C_{(n-k)/2}^2.$$

This is a sum of positive unimodal terms which could be accurately estimated by standard methods [3]. However, it is enough for our purposes to locate the largest term in the sum. An elementary computation shows that it occurs when  $k \approx n/3$ . Stirling's estimate gives  $\binom{n}{n/3}^2 C_{n/3}^2 = \Theta(3^{2n} n^{-4}) = \Theta(3^N N^{-4})$ .

(3) Consider the configuration  $V_N = V_{2um}$ , where  $u$  is to be determined. Take any tree on  $L_1$  and any matching of size  $m$  between  $L_1$  and  $L_2$ . Now take any tree on the  $um - m$  unmatched points in  $L_2$ . Finally, add an extra edge in  $L_2$  to produce a tree in the whole configuration (see Fig. 9(b)). In this way we get

$$t_{um} t_{um-m} \binom{um}{m}^2 \sim \left[ \left( \frac{27}{4} \right)^{2u-1} \left( \frac{u^u}{(u-1)^{u-1}} \right)^2 \right]^{2um/2u}$$

different trees, where  $t_{n+1} = \binom{3n}{n}/(2n+1)$  is the number of trees for  $n$  points in convex position as mentioned above. Setting

$$g(u) = \left(\frac{27}{4}\right)^{(2u-1)/2u} u(u-1)^{(1-u)/u},$$

we aim at maximizing  $g(u)$ . Elementary calculus shows that the maximum is achieved at  $u_0 = 1 + 3\sqrt{3}/2$ . The desired value is then  $g(u_0) = 9.35$ , and the number of trees is at least  $9.35^{2um} = 9.35^N$ .  $\square$

We close this section by proving an absolute lower bound on the number of matchings and trees of any configuration (note that the result is obvious for the number of polygonizations).

**Theorem 4.2.** *The number of matchings of a set of  $N$  points in the plane is minimum when the points are in convex position. The same result holds for the number of trees.*

**Proof.** First observe that, in the case of matchings,  $N$  has to be even. As mentioned above, we use the fact that the number of matchings of  $2n$  points in convex position is equal to  $C_n$ , and also the fact that Catalan numbers satisfy the recurrence  $C_n = C_0C_{n-1} + C_1C_{n-2} + \dots + C_{n-1}C_0$  (see, for example, [9]). Now let  $P$  be a set of  $N = 2n$  points,  $p_1$  a point in the convex hull of  $P$  and  $p_2, \dots, p_{2n}$  an ordering of the remaining points in polar order with respect to  $p_1$ . If, in a matching of  $P$ , point  $p_1$  is joined to  $p_{2i}$  then, by induction on  $n$ , the remaining  $2n - 2$  points can be matched in at least  $C_{i-1}C_{n-i}$  ways. Hence, the number of matchings is at least  $C_0C_{n-1} + \dots + C_{n-1}C_0 = C_n$ .

The proof for the number of trees is quite similar, but uses the more complicated recurrence  $t_N = \sum_{i+j+k=N+1} t_i t_j t_k$  for the number  $t_N$  of trees of  $N$  points in convex position. To see why this recurrence holds, let  $T$  be any tree on the set  $P = \{p_1, p_2, \dots, p_N\}$ . Let  $u$  be the largest index such that  $p_1$  is connected to  $p_u$  in  $T$ . Then  $T$  induces a tree  $T_1$  on the vertex set  $\{p_u, \dots, p_N\}$ . If we remove the edge  $p_1 p_u$  from  $T$ , there exists  $v$  such that  $T$  induces a tree  $T_2$  on the set  $\{p_1, \dots, p_v\}$ , and another tree  $T_3$  on the set  $\{p_{v+1}, \dots, p_u\}$ . Now  $T$  can be uniquely reconstructed from  $T_1, T_2, T_3$ .

This argument can be used to prove the statement in the theorem if we take, as before,  $p_1$  in the convex hull of  $P$  and  $p_2, \dots, p_N$  in polar order with respect to  $p_1$ . Indeed, we can assume by induction that the result is true for all sets with less than  $N$  points. Then the number of trees in  $P$  is at least  $\sum_{i+j+k=N+1} t_i t_j t_k$ , which is precisely  $t_N$ .

## 5. Concluding remarks

We have analyzed a particular configuration of points in the plane with a number of crossing-free subgraphs of several kinds, which improves previous results. Whereas for triangulations and matchings in  $V_N$  our formulas are tight, a deeper analysis might show the existence of a larger number of trees and polygonizations in  $V_N$ . We can prove, however, an upper bound of  $O(5.61^N)$  on the number of polygonizations of  $V_N$ .

With the same techniques used in the paper, we can also prove that the number of crossing-free forests (i.e., acyclic graphs) in  $V_N$  is at least  $11.09^N$ , and that the total number of crossing-free subgraphs is at least  $(20 + 14\sqrt{2})^N = 39.8^N$ . We have omitted the proofs to avoid repetition.

On the other hand, we do not know how sharp these lower bounds are. Hayward [7] and Smith [14] conjectured  $4.5^N$  and  $6^N$  respectively as upper bounds for  $\Phi(N)$ . Smith also gives  $16^N$  as a possible upper bound for the number of triangulations.

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## Note added in proof

The upper bound on the number of triangulations has been reduced to  $279^n$  [16], and a sharper analysis of configuration  $V_N$  shows that it has  $\Omega(10.42^n)$  crossing-free spanning trees [17].

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