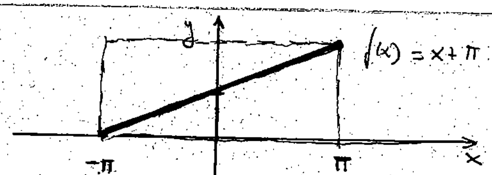
19. SÈRIES DE FOURIER

1. a) Trobeu la sèrie de Fourier de la funció $f(x) = x + \pi$ a l'interval $-\pi < x < \pi$.

b) Useu a) per demostrar que

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

S.



(a)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,dx = \frac{1}{\pi}\Big(\overbrace{\int_{-\pi}^{\pi} x\,dx}^{0 \text{ (funció senar)}} + \pi\int_{-\pi}^{\pi} dx\Big) = 2\pi$$

$$a_n = \frac{1}{\pi}\overbrace{\int_{-\pi}^{\pi} x\cos nx\,dx}^{0 \text{ (funció senar)}} + \frac{1}{\pi}\overbrace{\int_{-\pi}^{\pi}\pi\cos nx\,dx}^{0} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{x}{n}\,d(-\cos nx) = \frac{2}{\pi}\Big[-\frac{x}{n}\cos nx\Big]_0^{\pi} +$$

$$+ \frac{2}{n\pi}\int_0^{\pi}\cos nx\,dx = -\frac{2}{n}(-1)^n.$$

$$\mathcal{F}[f](x) = \pi + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

Nota: $\mathcal{F}[f](x) = x+\pi,\ \forall\ -\pi < x < \pi,$

$$\mathcal{F}[f](\pi) = \mathcal{F}[f](-\pi) = \frac{1}{2}\big(f(-\pi^+) + f(\pi^-)\big) = \frac{1}{2}(0+2\pi) = \pi$$

(b) $x = -\pi/2$

$$\mathcal{F}[f](-\tfrac{\pi}{2}) = \pi + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n}\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{\substack{\sin n\frac{\pi}{2}=0 \\ (\text{si } n \text{ parell})}} = \pi + 2\sum_{p\geq 1}\frac{(-1)^{2p+1}}{2p-1}\sin(2p-1)\frac{\pi}{2}$$

$$= \pi + 2\sum_{n=1}^{\infty}\frac{(-1)^p}{2p-1} = -\pi/2 + \pi = \pi/2$$

D'on:

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{p+1}}{2p-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

2. *

a) Trobeu la sèrie de Fourier de

$$f(x) = \begin{cases} 0 & \text{per} \quad -\pi < x < 0 \\ x^2 & \text{per} \quad 0 \leq x < \pi \end{cases}$$

b) Useu a) per demostrar que

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{i que} \quad \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

c) Useu b) per trobar una sèrie numèrica tal que la seva suma sigui $\pi^2/8$.

Solució.

a) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot dx + \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx \overset{\text{parts}}{=} \cdots =$$

$$= \frac{1}{\pi}\Big[\frac{x^2}{n}\overbrace{\sin(nx)}^{0}\Big]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi} x\sin nx\,dx \underset{\text{parts}}{=} \Big[\frac{2x}{n^2\pi}\cos nx\Big]_0^{\pi} - \frac{2}{n^2\pi}\overbrace{\int_0^{\pi}\cos nx\,dx}^{0}$$

$$= \frac{2}{n^2}(-1)^n.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx$$

$$\overset{\text{parts}}{=} \frac{1}{\pi}\Big[-\frac{x^2}{n}\cos nx\Big]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi} x\cos nx\,dx = -\frac{\pi}{n}(-1)^n + \frac{2}{n^2\pi}\int_0^{\pi} x\,d(\sin nx)$$

$$\overset{\text{parts}}{=} -\frac{\pi}{n}(-1)^n + \Big[\frac{2x}{n^2\pi}\overbrace{\sin nx}^{0}\Big]_0^{\pi} - \frac{2}{n^2\pi}\int_0^{\pi}\sin(nx)\,dx = -\frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}\big[\cos(nx)\big]_0^{\pi}$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}\big((-1)^n - 1\big) = \begin{cases} -\frac{\pi}{n}, & \text{si } n \text{ parell.} \\ \frac{\pi}{n} - \frac{4}{n^3\pi}, & \text{si } n \text{ senar.} \end{cases}$$

$$\mathcal{F}[f](x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n\cos(nx) - \sum_{m=1}^{\infty}\frac{\pi}{2m}\sin(2mx) +$$

$$+ \sum_{m=0}^{\infty}\left(\frac{\pi}{2m+1} - \frac{4}{(2m+1)^3\pi}\right)\sin(2m+1)x$$

$i:\ \mathcal{F}[f](x) = f(x),\ x\in(-\pi,\pi)$

$\mathcal{F}[f](\pm\pi) = \pi/2$

b) $x=0:\ \mathcal{F}[f](0) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n = \frac{\pi^2}{6} - \frac{2}{1^2} + \frac{2}{2^2} - \frac{2}{3^2} + \frac{2}{4^2} - \frac{2}{5^2} + \dots$

$= 0 \Longrightarrow \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{5^2} + \dots$

$x=\pi:\ \mathcal{F}[f](\pi) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2} \Longrightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2},$

i.e.:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

c) $\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{\pi^2}{4} = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \Longrightarrow \frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

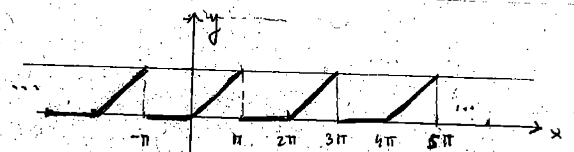
3. Trobeu la sèrie de Fourier de

$$f(x) = \begin{cases} 0 & \text{si} \quad -\pi < x < 0 \\ x & \text{si} \quad 0 \le x < \pi \end{cases}$$

a l'interval $-\pi < x < \pi$.

On convergeix la sèrie trobada en a), quan $x = \frac{7\pi}{2}$? I quan $x = 401\pi$?

Solució:



$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot dx + \frac{1}{\pi}\int_0^{\pi} x\,dx = \pi/2$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{1}{\pi}\underbrace{\left[\frac{x}{n}\sin(nx)\right]_0^{\pi}}_{=0} -$

$- \frac{1}{n\pi}\int_0^{\pi}\sin(nx)\,dx = \frac{1}{n^2\pi}\left[\cos(nx)\right]_0^{\pi} = \frac{1}{n^2\pi}\left((-1)^n - 1\right) = \begin{cases} 0, & n \text{ parell} \\ \frac{-2}{n^2\pi}, & n \text{ senar} \end{cases}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx =$

$= \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx)\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\cos(nx)\,dx = -\frac{1}{n}(-1)^n$

$\mathcal{F}[f](x) = \frac{\pi}{4} - \sum_{n=0}^{\infty}\frac{2}{(2n+1)^2\pi}\cos(2n+1)x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$

$\mathcal{F}[f](x) = f(x),\ x\in(-\pi,\pi)$

$\mathcal{F}[f](\pm\pi) = \frac{1}{2}\left(f(-\pi^+) + f(\pi^-)\right) = \frac{1}{2}(0+\pi) = \pi/2.$

$\mathcal{F}[f](401\pi) = \mathcal{F}[f](2\cdot(200\pi)+\pi) = \mathcal{F}[f](\pi) = \pi/2$

$\mathcal{F}[f](7\pi/2) = \mathcal{F}[f](4\pi - \pi/2) = \mathcal{F}[f](-\pi/2) = f(-\pi/2) = 0.$

4. * Trobeu, si és que existeix, una sèrie de Fourier que convergeix cap a $|\sin x|$ per a tot $x \in \mathbb{R}$.

Solució: $f(x) = |\sin x|$ quan $-\pi < x < \pi$, $|\sin x| = \begin{cases} -\sin x, & -\pi < x \le 0 \\ \sin x, & 0 \le x < \pi. \end{cases}$

$|\sin x|$ funció parella i per tant $b_n = 0\ \forall n = 1, 2, \dots$

$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{4}{\pi},$

$a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx = 0$

per $n>1$:

$a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\left(\sin(n+1)x - \sin(n-1)x\right)dx$

$= -\frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = -\frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} - \frac{1}{n+1} + \frac{1}{n-1}\right]$

$= \frac{1}{(n+1)\pi}\left(1 + (-1)^n\right) - \frac{1}{(n-1)\pi}\left(1 + (-1)^n\right) = \begin{cases} \frac{2}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \frac{-4/\pi}{n^2-1}, & \text{(si } n \text{ parell)} \\ 0, & \text{(si } n \text{ senar)} \end{cases}$

Per tant la sèrie de Fourier corresponent vé donada per:

$$\mathcal{F}[f](x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}$$

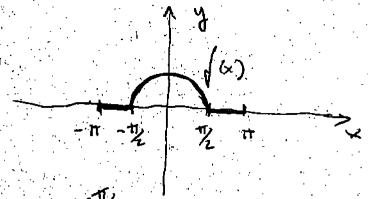
$f \in CT[-\pi,\pi]$ (de fet $C[-\pi,\pi]$) i $f' \in CT[-\pi,\pi]$. Llavors pel teorema de convergència (Dirichlet) $\mathcal{F}[f](x) = f(x)\ \forall x \in (-\pi,\pi)$. D'altra banda

$\mathcal{F}[f](\pm\pi) = \frac{1}{2}\left(f(-\pi^+)+f(\pi^-)\right) = 0$. Llavors $\mathcal{F}[f](x) = |\sin x|\ \forall x \in [-\pi,\pi]$. Si considerem l'extensió periòdica a tot $\mathbb{R}$ s'obté la funció $\mathbb{R}\ni x \mapsto |\sin x|$, que és contínua en $\mathbb{R}$ i així: $\mathcal{F}[f](x)=|\sin x|\ \forall x$.

5. Trobeu la sèrie de Fourier de la funció

$$f(x) = \begin{cases} 0 & \text{si } \frac{\pi}{2} \le |x| \le \pi \\ \cos x & \text{si } |x| < \frac{\pi}{2} \end{cases}$$

Solució:



funció parella $\Rightarrow b_n = 0\ \forall n \in \mathbb{N}$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} \cos x\, dx = \frac{2}{\pi}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \cos^2 x\, dx = \frac{2}{\pi}\int_0^{\pi/2} \frac{1+\cos 2x}{2}\, dx = \frac{2}{\pi}\left[\frac{x}{2}+\frac{\sin 2x}{4}\right]_0^{\pi/2} = \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2}\cos x \cos(nx)\, dx = \frac{1}{\pi}\int_0^{\pi/2}\left(\cos(n+1)x + \cos(n-1)x\right) dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} = \frac{1}{\pi}\left[\frac{\sin\left(\frac{n}{2}\pi+\frac{\pi}{2}\right)}{n+1} + \frac{\sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right)}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos\left(\frac{n\pi}{2}\right)}{n+1} - \frac{\cos\left(\frac{n\pi}{2}\right)}{n-1}\right] = \begin{cases} 0, & \text{si } n \text{ senar} \\ \dfrac{2(-1)^{\frac{n}{2}+1}}{\pi(n^2-1)}, & \text{si } n \text{ parell.} \end{cases}$$

Per tant, la sèrie és:

$$\mathcal{F}[f](x) = \frac{1}{\pi} + \frac{1}{2}\cos x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos(2nx).$$

6. * Sigui $f(x)$ continua a $(-L,L)$ i siguin a_n i b_n els seus coeficients de Fourier.

a) Proveu que si $S_M(x) = \frac{a_0}{2} + \sum_{n=1}^{M}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$, aleshores

$$\int_{-L}^{L} f(x)S_M(x)dx = L\left(\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2+b_n^2)\right).$$

b) Proveu que $\int_{-L}^{L} S_M^2(x)dx = L\left(\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2+b_n^2)\right)$.

c) Proveu que $2\int_{-L}^{L} f(x)S_M(x)dx - \int_{-L}^{L} S_M^2(x)dx \le \int_{-L}^{L}(f(x))^2dx$

d) Fent servir els apartats anteriors proveu la anomenada "desigualtat de Bessel":

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2) \le \frac{1}{L}\int_{-L}^{L}(f(x))^2dx.$$

Solució.

Agafarem: $\psi_0(x) = \frac{1}{2}$, $\psi_{2n-1}(x) = \sin\frac{n\pi x}{L}$, $\psi_{2n}(x) = \cos\frac{n\pi x}{L}$ $\Big\}$ $n = 0,1,2,\dots,2M$

$c_0 = a_0$, $c_{2n-1} = b_n$, $c_{2n} = a_n$

A més $\{\psi_n\}_{n=0,\dots,2M}$ són ortogonals respecte el producte $\langle f,g\rangle = \int_{-L}^{L} f(x)g(x)dx$

i amb $\langle\psi_n,\psi_m\rangle = \begin{cases} 0 & \text{si } n\ne m \\ L/2 & \text{si } n=m=0 \\ L & \text{si } n=m\ge 1 \end{cases}$

a) $$\int_{-L}^{L} f(x)S_M(x)\,dx = \langle f, S_M\rangle = \langle f, \sum_{n=0}^{2M} c_n\psi_n\rangle = \sum_{n=0}^{2M} c_n\langle f,\psi_n\rangle$$

$$= \sum_{n=0}^{2M} c_n^2\langle\psi_n,\psi_n\rangle = L\left(\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2+b_n^2)\right)$$

b) $$\int_{-L}^{L} S_M^2(x)\,dx = \langle S_M, S_M\rangle = \langle \sum_{n=0}^{2M} c_n\psi_n, \sum_{m=0}^{2M} c_m\psi_m\rangle =$$

$$= \sum_{n,m=0}^{2M} c_nc_m\langle\psi_n,\psi_m\rangle = \sum_{n=0}^{2M} c_n^2\langle\psi_n,\psi_n\rangle = L\left(\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2+b_n^2)\right)$$

c) $$0 \le \langle f-S_M, f-S_M\rangle = \langle f,f\rangle - 2\langle f,S_M\rangle + \langle S_M,S_M\rangle$$

d'on: $\langle f,f\rangle \ge 2\langle f,S_M\rangle - \langle S_M,S_M\rangle$,

que és la desigualtat buscada.

d) $\frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \frac{1}{L}\langle f,f\rangle \overset{c)}{\geq} \frac{2}{L}\langle f,s_M\rangle - \frac{1}{L}\langle s_M,s_M\rangle$

$$= \frac{a_0^2}{2} + \sum_{n=1}^{M}\left(a_n^2+b_n^2\right)$$

c),b)

I prenent límits quan $M\to\infty$, s'obté:

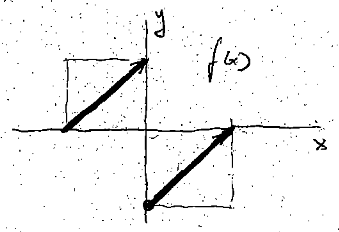
$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right) \leq \frac{1}{L}\int_{-L}^{L} f^2(x)\,dx.$$

7. Desenvolupeu la funció

$$f(x) = \begin{cases} x+1 & \text{per} \quad -1<x<0 \\ x-1 & \text{per} \quad 0\le x<1 \end{cases}$$

en sèrie de sinus o cosinus, segons convingui.

Solució:



f és una funció senar, llavors $\mathcal{F}[f](x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$.

amb:

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = 2\int_0^1 f(x)\sin(n\pi x)\,dx = 2\int_0^1 (x-1)\sin(n\pi x)\,dx$$

parts

$$= 2\int_0^1 (x-1)\, d\left(\frac{-1}{n\pi}\cos(n\pi x)\right) = \frac{2}{n\pi}\Big[(x-1)\cos(n\pi x)\Big]_0^1$$

$$= \frac{-2}{n\pi}(-1)^n$$

i finalment:

$$\mathcal{F}[f](x) = -\frac{2}{n\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\sin(n\pi x)$$

8. Desenvolupeu la funció

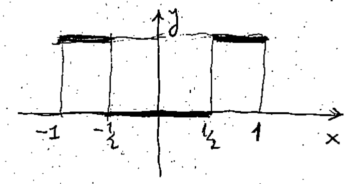
$$f(x) = \begin{cases} 0 & \text{per} \quad 0<x<1/2 \\ 1 & \text{per} \quad 1/2\le x<1 \end{cases}$$

en sèrie de cosinus en mig interval i en sèrie de sinus en mig interval.

Solució:

(i) Sèrie de cosinus $\Rightarrow$ Construïm l'extensió parella de $f(x)$ a $(-1,1)$, $f_p(x)$:

$$f_p(x) = \begin{cases} +f(-x), & -1<x<0 \\ f(x), & 0\le x<1 \end{cases}$$



$$\mathcal{F}[f_p](x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x)$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f_p(x)\,dx = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} 0\cdot dx + 2\int_{1/2}^{1} 1\cdot dx = 1.$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f_p(x)\cos(n\pi x)\,dx = 2\int_0^1 f(x)\cos(n\pi x)\,dx = 2\int_0^{1/2} 0\cdot\cos(n\pi x)\,dx$$

$$+ 2\int_{1/2}^{1} 1\cdot\cos(n\pi x)\,dx = \frac{2}{n\pi}\Big[\sin(n\pi x)\Big]_{1/2}^{1} = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0, & \text{si } n \text{ parell} \\ \dfrac{2(-1)^{\frac{n+1}{2}}}{n\pi}, & \text{si } n \text{ senar} \end{cases}$$

$$\mathcal{F}[f_p](x) = \frac{1}{2} + \sum_{m=0}^{\infty}\frac{2(-1)^{m+1}}{(2m+1)\pi}\cos(2m+1)\pi x$$

(ii) Sèrie en sinus. Extensió senar: $f_s(x) = \begin{cases} -f(-x), & -1<x<0 \\ f(x), & 0<x<1 \end{cases}$

amb

$$b_n = \frac{1}{1}\int_{-1}^{1} f_s(x)\sin(n\pi x)\,dx = 2\int_0^1 f(x)\sin(n\pi x)\,dx = 2\int_0^{1/2} 0\cdot\sin(n\pi x)\,dx$$

$$+ 2\int_{1/2}^{1} 1\cdot\sin(n\pi x)\,dx = \frac{-2}{n\pi}\Big[\cos(n\pi x)\Big]_{1/2}^{1} = -\frac{2}{n\pi}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \begin{cases} \dfrac{2}{n\pi}, & \text{si } n \text{ senar} \\ \dfrac{2\left((-1)^{n/2}-1\right)}{n\pi}, & \text{si } n \text{ parell} \end{cases}$$

d'on finalment:

$$\mathcal{F}[f_s] = \sum_{m=0}^{\infty}\frac{2}{(2m+1)\pi}\sin(2m+1)\pi x + \sum_{m=0}^{\infty}\frac{-2}{(2m+1)\pi}\sin 2(2m+1)\pi x.$$

9. Desenvolupeu la funció $f(x) = x^2$, $0 < x < \ell$

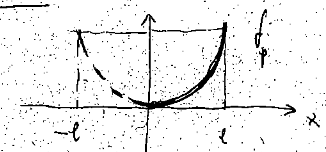
a) en sèrie de cosinus.

b) en sèrie de sinus.

c) en sèrie de Fourier.

Solució.

(a)



$$f_p(x) = \begin{cases} f(-x), & -\ell < x \leq 0 \\ f(x), & 0 < x < \ell \end{cases} \quad \text{(Extensió parella).}$$

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f_p(x)\,dx = \frac{2}{\ell}\int_0^\ell f(x)\,dx = \frac{2}{\ell}\int_0^\ell x^2\,dx = \frac{2}{\ell}\left[\frac{x^3}{3}\right]_0^\ell = \frac{2}{3}\ell^2$$

$$a_n = \frac{2}{\ell}\int_0^\ell x^2\cos\frac{n\pi x}{\ell}\,dx = \frac{2}{\ell}\cdot\frac{\ell}{n\pi}\int_0^\ell x^2\,d\left(\sin\frac{n\pi x}{\ell}\right) = \underbrace{\frac{2}{n\pi}\left[x^2\sin\frac{n\pi x}{\ell}\right]_0^\ell}_{=0} -$$

$$-\frac{4}{n\pi}\int_0^\ell x\,d\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right) = \frac{4\ell}{n^2\pi^2}\left[x\cos\frac{n\pi x}{\ell}\right]_0^\ell - \frac{4\ell}{n^2\pi^2}\underbrace{\int_0^\ell\cos\frac{n\pi x}{\ell}\,dx}_{=0}$$

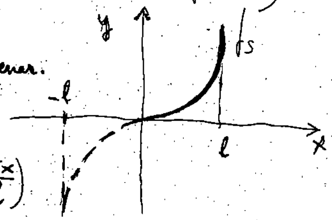
$$= \frac{4\ell^2}{n^2\pi^2}(-1)^n$$

$$\mathcal{F}[f_p](x) = \frac{1}{3}\ell^2 + \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{\ell},$$

i tenim que $\mathcal{F}[f_p](x) = x^2$, $0 < x < \ell$ (de fet també és cert per $-\ell < x < \ell$, donat que l'extensió periòdica en aquest cas coincideix amb la funció).

(b)

$$f_s(x) = \begin{cases} -f(-x) = -x^2, & -\ell < x \leq 0 \\ f(x) = x^2, & 0 < x < \ell \end{cases} \quad \text{Extensió senar.}$$



$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f_s(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{\ell}\int_0^\ell x^2\sin\left(\frac{n\pi x}{\ell}\right)dx = -\frac{2}{n\pi}\int_0^\ell x^2\,d\left(\cos\frac{n\pi x}{\ell}\right) = -\frac{2}{n\pi}\left[x^2\cos\frac{n\pi x}{\ell}\right]_0^\ell +$$

$$+\frac{4}{n\pi}\int_0^\ell x\,d\left(\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\right) = -\frac{2\ell^2}{n\pi}(-1)^n + \frac{4\ell}{n^2\pi^2}\overbrace{\left[x\sin\frac{n\pi x}{\ell}\right]_0^\ell}^{=0} +$$

$$+\frac{4\ell}{n^2\pi^2}\int_0^\ell d\left(\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right) = -\frac{2\ell^2}{n\pi}(-1)^n + \frac{4\ell^2}{n^3\pi^3}\left((-1)^n - 1\right) = \begin{cases} -\frac{2\ell^2}{n\pi}, & n \text{ parell.} \\ \frac{2\ell^2}{n\pi} - \frac{8\ell^2}{n^3\pi}, & n \text{ senar} \end{cases}$$

$$\mathcal{F}[f_s](x) = -\frac{2\ell^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n}\sin\frac{2n\pi x}{\ell} + \frac{\ell^2}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{2n-1} - \frac{8}{(2n-1)^3\pi}\right)\sin\frac{(2n-1)\pi x}{\ell}$$

Sèrie de sinus de x^2 en $0 < x < \ell$: $\mathcal{F}[f_s](x) = x^2 \ \forall\ 0 < x < \ell$.

c) $$\mathcal{F}[f](x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{\ell} + b_n\sin\frac{2n\pi x}{\ell}\right)$$

$$a_0 = \frac{2}{\ell}\int_0^\ell x^2\,dx = a_0(f_p) = \frac{2}{3}\ell^2$$

$$a_n = \frac{2}{\ell}\int_0^\ell x^2\cos\frac{2n\pi x}{\ell}\,dx = a_{2n}(f_p) = \frac{\ell^2}{n^2\pi^2}$$

$$b_n = \frac{2}{\ell}\int_0^\ell x^2\sin\frac{2n\pi x}{\ell}\,dx = b_{2n}(f_s) = -\frac{\ell^2}{n\pi}$$

$$\boxed{\mathcal{F}[f](x) = \frac{\ell^2}{3} + \frac{\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{\ell} - \frac{\ell^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{\ell} \quad \forall\ 0 < x < \ell}$$

10. * a) Trobeu la forma general de la sèrie de Fourier en cosinus i de la sèrie de Fourier en sinus a $[0, C]$ per a funcions que compleixen la relació $f(C - x) = f(x)$.

b) Mateixa pregunta per $f(C - x) = -f(x)$.

Solució.

a.1) $$\mathcal{F}[f_p](x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{C} \quad \text{on} \quad f_p(x) = \begin{cases} f(-x), & -C < x \leq 0 \\ f(x); & 0 < x < C. \end{cases}$$

$$a_0 = \frac{2}{C}\int_0^C f(x)\,dx = \frac{2}{C}\int_0^{C/2} f(x)\,dx + \frac{2}{C}\int_{C/2}^{C} f(x)\,dx = \left\{\begin{array}{l} \text{c.v.} \\ y = C - x \\ x = C/2 \Rightarrow y = C/2 \\ x = C \Rightarrow y = 0 \\ dx = -dy \end{array}\right\} =$$

$$= \frac{2}{C}\int_0^{C/2} f(x)\,dx + \frac{2}{C}\int_{C/2}^{0} f(C - y)(-dy)$$

$$= \frac{4}{C}\int_0^{C/2} f(x)\,dx.$$

$$a_n = \frac{2}{C}\int_0^C f(x)\cos\frac{n\pi x}{C}\,dx = \frac{2}{C}\int_0^{C/2} f(x)\cos\frac{n\pi x}{2}\,dx + \frac{2}{C}\int_0^C f(x)\cos\frac{n\pi x}{C}\,dx$$

$$= \left\{\begin{array}{c} \text{c.v.}: y = C - x,\ dy = -dx;\ x = C/2: y = C/2;\ x = C: y = 0 \\ \cos\frac{n\pi}{C}(C - y) = \cos\left(n\pi - \frac{n\pi y}{C}\right) = (-1)^n\cos\frac{n\pi y}{C} \end{array}\right\} =$$

$$= \frac{2}{C}\int_0^{C/2} f(x)\cos\frac{n\pi x}{C}\,dx + \frac{2}{C}(-1)^n\int_0^{C/2} f(C-y)\cos\frac{n\pi y}{C} =$$

$$= \frac{2}{C}\left(1+(-1)^n\right)\int_0^{C/2} f(y)\cos\frac{n\pi y}{C}\,dy = \begin{cases} 0, \text{ si } n \text{ és senar} \\ \frac{4}{C}\int_0^{C/2} f(y)\cos\frac{n\pi y}{C}\,dy, \; n \text{ parell}\end{cases}$$

Resumint

$$\left.\begin{aligned} a_0 &= \frac{4}{C}\int_0^{C/2} f(x)\,dx, \\ a_{2n+1} &= 0 \quad \forall n \geq 0, \\ a_{2n} &= \frac{4}{C}\int_0^{C/2} f(y)\cos\frac{2n\pi y}{C}\,dy \end{aligned}\right\}$$

d'on:

$$\mathcal{F}[f_p](x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n}\cos\frac{2n\pi x}{C}$$

a2) En sinus:

$$b_n = \frac{2}{C}\int_0^C f(x)\sin\frac{n\pi x}{C}\,dx = \frac{2}{C}\int_0^{C/2} f(x)\sin\frac{n\pi x}{C}\,dx + \frac{2}{C}\int_{C/2}^{C} f(x)\sin\frac{n\pi x}{C}\,dx$$

$$= \left\{\begin{aligned} &\text{c.v.}: y = C-x;\; x = C/2 : y = C/2;\; x = C : y = 0;\; dy = -dx; \\ &\sin\frac{n\pi x}{C} = \sin\frac{n\pi}{C}(C-y) = \sin\left(n\pi - \frac{n\pi}{C}y\right) = -\cos n\pi \sin\frac{n\pi y}{C} = \\ &= (-1)^n \sin\frac{n\pi y}{C} \end{aligned}\right\}$$

$$= \frac{2}{C}\int_0^{C/2} f(x)\sin\frac{n\pi x}{C}\,dx + \frac{2}{C}(-1)^{n+1}\int_0^{C/2} f(C-y)\sin\left(\frac{n\pi y}{C}\right)dy =$$

$$= \frac{2}{C}\left(1+(-1)^{n+1}\right)\int_0^{C/2} f(x)\sin\left(\frac{n\pi x}{C}\right)dx = \begin{cases} 0, \text{ si } n \text{ és parell} \\ \frac{4}{C}\int_0^{C/2} f(x)\sin\frac{n\pi x}{C}\,dx, \text{ si } n \text{ és senar}\end{cases}$$

Per tant, els coeficients queden:

$$\left\{\begin{aligned} b_{2n} &= 0 \quad \forall n \geq 1 \\ b_{2n+1} &= \frac{4}{C}\int_0^{C/2} f(x)\sin\frac{(2n+1)\pi x}{C}\,dx, \quad \forall n \geq 0 \end{aligned}\right.$$

d'on:

$$\mathcal{F}[f_s](x) = \sum_{n=0}^{\infty} b_{2n+1}\sin\frac{(2n+1)\pi x}{C}.$$

b.1) **En cosinus**

$$a_0 = \frac{2}{C}\int_0^C f(x)\,dx = \frac{2}{C}\int_0^{C/2} f(x)\,dx + \frac{2}{C}\int_{C/2}^{C} f(x)\,dx = \left\{\begin{aligned} &\text{c.v.} \\ &y = C-x \\ &\ldots \end{aligned}\right\}$$

$$= \frac{2}{C}\int_0^{C/2} f(x)\,dx + \frac{2}{C}\int_{C/2}^{0} f(C-y)\,(-dy) = \frac{2}{C}\int_0^{C/2} f(x)\,dx -$$

$$- \frac{2}{C}\int_0^{C/2} f(x)\,dx = 0$$

$$a_n = \frac{2}{C}\int_0^C f(x)\cos\frac{n\pi x}{C}\,dx = \frac{2}{C}\int_0^{C/2} f(x)\cos\frac{n\pi x}{C}\,dx +$$

$$+ \frac{2}{C}\int_{C/2}^{C} f(x)\cos\frac{n\pi x}{C}\,dx = \left\{\begin{aligned} &\text{c.v.}: y = C-x;\ldots \\ &\cos\frac{n\pi x}{C} = \cos\left(n\pi - \frac{n\pi y}{C}\right) = (-1)^n\cos\frac{n\pi y}{C} \end{aligned}\right.$$

$$= \frac{2}{C}\int_0^{C/2} f(x)\cos\frac{n\pi x}{C}\,dx + \frac{2}{C}(-1)^n\int_0^{C/2} \underbrace{f(C-y)}_{-f(y)}\cos\frac{n\pi y}{C}\,dy$$

$$= \frac{2}{C}\left(1+(-1)^{n+1}\right)\int_0^{C/2} f(x)\cos\frac{n\pi x}{C}\,dx = \begin{cases} = 0, \text{ si } n \text{ parell} \\ = \frac{4}{C}\int_0^{C/2} f(x)\cos\frac{n\pi x}{C}\,dx \text{ si } n \text{ senar}\end{cases}$$

Per tant: $a_{2n} = 0 \quad \forall n \geq 0,$

$$a_{2n+1} = \frac{4}{C}\int_0^{C/2} f(x)\cos\frac{(2n+1)\pi x}{C}\,dx \quad \forall n \geq 0$$

$$\mathcal{F}[f_p](x) = \sum_{n=0}^{\infty} a_{2n+1}\cos\frac{(2n+1)\pi x}{C}$$

b.2) En sinus:

$$b_n = \frac{2}{C}\int_0^{C/2} f(x)\sin\frac{n\pi x}{C}\,dx + \frac{2}{C}\int_{C/2}^{C} f(x)\sin\frac{n\pi x}{C}\,dx =$$

$$= \left\{\begin{aligned} &y = C-x,\; x = C/2 : y = C/2;\; x = C : y = 0 \\ &\sin\left(n\pi - \frac{n\pi y}{C}\right) = (-1)^{n+1}\sin\frac{n\pi y}{C}, \ldots \end{aligned}\right\}$$

$$= \frac{2}{C}\int_0^{C/2} f(x)\sin\frac{n\pi x}{C}\,dx + \frac{2}{C}(-1)^{n+1}\int_0^{C/2} \underbrace{f(C-y)}_{-f(y)}\sin\frac{n\pi y}{C}\,dy =$$

$$= \frac{2}{C}\left(1+(-1)^n\right)\int_0^{C/2} f(x)\sin\frac{n\pi x}{C}\,dx = \begin{cases} 0, & \text{si } n \text{ és senar} \\ \frac{4}{C}\int_0^{C/2} f(x)\sin\frac{n\pi x}{C}\,dx, & \text{si } n \text{ és parell} \end{cases}$$

$b_{2n+1} = 0 \quad \forall n \geq 0$

$$b_{2n} = \frac{4}{C}\int_0^{C/2} f(x)\sin\frac{2n\pi x}{C}\,dx \quad \forall n \geq 1$$

$$\mathcal{F}[f_s](x) = \sum_{n=1} b_{2n}\sin\frac{2n\pi x}{C}.$$

11. Desenvolupeu la funció $\cos xz$ en sèrie de Fourier en l'interval $[-\pi,\pi]$, on z és un paràmetre real.

Proveu les igualtats

$$\frac{1}{\sin \pi z} = \frac{2z}{\pi}\left(\frac{1}{2z^2} + \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k^2-z^2}\right)$$

$$\operatorname{cotg} \pi z = \frac{1}{\pi}\left(\frac{1}{z} - \sum_{k=1}^{\infty}\frac{2z}{k^2-z^2}\right)$$

i deduïu que

$$\pi = 2 + \sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{4k^2-1}$$

$$\pi = 4 - \sum_{k=1}^{\infty}\frac{8}{16k^2-1}.$$

Solució: Sigui $f(x) = \cos(xz)$, $x \in [-\pi,\pi]$, $z \in \mathbb{R}$. Notem que és una funció parella, per tant la seva sèrie de Fourier és una sèrie en cosinus ($b_n = 0 \ \forall n \geq 1$). Calculant els seus coeficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\cos(xz)\,dx = \frac{2}{\pi z}\int_0^{\pi} d(\sin(xz)) = \frac{2}{\pi z}\sin(\pi z)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\cos(xz)\cos(nx)\,dx = \frac{2}{2\pi}\int_0^{\pi}\left[\cos(z+n)x + \cos(z-n)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(z+n)\pi}{z+n} + \frac{\sin(z-n)\pi}{z-n}\right] = (-1)^n\left(\frac{1}{z+n} + \frac{1}{z-n}\right)\sin(\pi z)$$

Nota: suposem $z \notin \mathbb{N}$ (si $z = n \in \mathbb{N}$, llavors és trivial!)

La sèrie de Fourier queda doncs:

$$\mathcal{F}[f](x) = \frac{2z\sin\pi z}{\pi}\left(\frac{1}{2z^2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2-z^2}\cos(kx)\right)$$

$$\mathcal{F}[f](x) = \cos(zx); \quad \forall -\pi < x < \pi;$$

i $$\mathcal{F}[f](\pm\pi) = \frac{1}{2}\left(f(-\pi^+) + f(\pi^-)\right) = \frac{1}{2}\left(\cos(-\pi z) + \cos(\pi z)\right) = \cos(\pi z)$$

ara, agafant $x = 0 \in (-\pi,\pi)$

$$\mathcal{F}[f](0) = \frac{2z\sin\pi z}{\pi}\left(\frac{1}{2z^2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2-z^2}\right) = \cos(z\cdot 0) = 1 \Longrightarrow$$

$$\Longrightarrow \frac{1}{\sin\pi z} = \frac{2z}{\pi}\left(\frac{1}{2z^2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2-z^2}\right), \ (z \notin \mathbb{N}) \qquad (1)$$

i prenent $x = \pi$

$$\mathcal{F}[f](\pi) = \frac{2z\sin\pi z}{\pi}\left(\frac{1}{2z^2} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2-z^2}(-1)^k\right) =$$

$$= \frac{2z\sin\pi z}{\pi}\left(\frac{1}{2z^2} - \sum_{k=1}^{\infty}\frac{1}{k^2-z^2}\right) = \cos(\pi z) \Longrightarrow$$

$$\Longrightarrow \operatorname{cotg}\pi z = \frac{1}{\pi}\left(\frac{1}{z} - \sum_{k=1}^{\infty}\frac{2z}{k^2-z^2}\right); \ (z \notin \mathbb{N}) \qquad (2)$$

Prenent, en (1), $z = \frac{1}{2}$:

$$1 = \frac{1}{\pi}\left(\frac{1}{2\frac{1}{4}} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2-\frac{1}{4}}\right) \Longrightarrow \pi = 2 + 4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4k^2-1}$$

i $z = \frac{1}{4}$ en (2):

$$1 = \frac{1}{\pi}\left(4 - \sum_{k=1}^{\infty}\frac{2\cdot\frac{1}{4}}{k^2-\frac{1}{16}}\right) \Longrightarrow \pi = 4 - \sum_{k=1}^{\infty}\frac{8}{16k^2-1}$$