1. Calculeu el desenvolupament en sèrie de potències (termes de grau més baix, equació de recurrència i radi de convergència de la sèrie) per les equacions

(i) $y'' - 2xy' + y = 0$, en $x=0$.

(ii) $y'' - 2xy' + y = \sin(x)$, en $x=0$.

(iii) $x^2y' + y = 0$ en $x=1$

(iv) $y'' - \frac{\sin(x)}{x} y' + 4y = 0$ en $x=0$.

Solució:

(i) $y = \sum_{n\geq 0} a_n x^n \Longrightarrow y' = \sum_{n\geq 1} n a_n x^{n-1} \Longrightarrow y'' = \sum_{n\geq 2} n(n-1) a_n x^{n-2} = \sum_{n\geq 0} (n+2)(n+1) a_{n+2} x^n$

$$y'' - 2xy' + y = \sum_{n\geq 0} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n\geq 1} n a_n x^n + \sum_{n\geq 0} a_n x^n =$$

$$= 2a_2 + a_0 + \sum_{n\geq 1} \left((n+2)(n+1) a_{n+2} - (2n-1) a_n \right) x^n = 0$$

relació de recurrència:

$$a_2 = -\frac{1}{2} a_0, \quad a_{n+2} = \frac{2n-1}{(n+2)(n+1)} a_n \quad \text{per } n \geq 1$$

d'aquí:

$n=1$: $a_3 = \frac{1}{3\cdot 2} a_1$, $n=2$: $a_4 = \frac{3}{4\cdot 3} \cdot \frac{-1}{2} a_0 = \frac{-3}{4\cdot 3\cdot 2} a_0$,

$n=3$: $a_5 = \frac{5}{5\cdot 4} a_3 = \frac{5}{5\cdot 4\cdot 3\cdot 2} a_1$, $n=4$: $a_6 = \frac{7}{6\cdot 5} a_4 = -\frac{7\cdot 3}{6\cdot 5\cdot 4\cdot 3\cdot 2} a_0$,

$n=5$: $a_7 = \frac{9}{7\cdot 6} a_5 = \frac{9\cdot 5}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} a_1$, $n=6$: $a_8 = \frac{11}{8\cdot 7} a_6 = -\frac{11\cdot 7\cdot 3}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} a_0$,

$n=7$: $a_9 = \frac{13}{9\cdot 8} a_7 = \frac{13\cdot 9\cdot 5}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} a_1, \ldots$ i per inducció es comprova que:

$$a_{2n} = -\frac{(4n-5)(4n-9)\cdots 11\cdot 7\cdot 3}{(2n)!}, \quad n\geq 2, \quad a_{2n+1} = \frac{(4n-3)(4n-7)\cdots 13\cdot 9\cdot 5}{(2n+1)!}, \quad n\geq 1$$

amb $a_2 = -\frac{1}{2} a_0$, $a_1, a_0 \in \mathbb{R}$ indeterminats. La solució general és doncs:

$$\boxed{\begin{aligned} y(x) = {} & a_0 \left(1 - \frac{1}{2} x^2 - \sum_{n\geq 2} \frac{(4n-5)(4n-9)\cdots 11\cdot 7\cdot 3}{(2n)!} x^{2n} \right) + \\ & + a_1 \left(x + \sum_{n\geq 1} \frac{(4n-3)(4n-7)\cdots 13\cdot 9\cdot 5}{(2n+1)!} x^{2n+1} \right), \quad a_0, a_1 \in \mathbb{R} \end{aligned}}$$

i el radi de convergència de totes dues sèries es $R=+\infty$, per tant el camp de validesa de la solució és tot $\mathbb{R}$.

ii) $y=\sum_{n\geq 0} a_n x^n \Longrightarrow y'=\sum_{n\geq 1} n a_n x^{n-1} \Longrightarrow y''=\sum_{n\geq 2} n(n-1)a_n x^{n-2}$

ara, substituint a l'equació diferencial, $y''-2xy'+y=\sin x$:

$$\sum_{n\geq 2} n(n-1)a_n x^{n-2} - 2\sum_{n\geq 1} n a_n x^n + \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \Longleftrightarrow$$

$$\sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n - 2\sum_{n\geq 1} n a_n x^n + \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\Longleftrightarrow 2a_2 + a_0 + \sum_{n\geq 1}\left[(n+2)(n+1)a_{n+2} - (2n-1)a_n\right]x^n = \sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \Longleftrightarrow$$

$$2a_2+a_0+\sum_{n\geq 1}\left[(2n+2)(2n+1)a_{2n+2}-(4n-1)a_{2n}\right]x^{2n} + \sum_{n\geq 1}\left[(2n+1)2n\,a_{2n+1}-(4n-3)a_{2n-1}\right]x^{2n-1}$$
$$= \sum_{n\geq 1}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}.$$

d'on obtenim les relacions de recurrència següents:

$$a_2=-\frac{a_0}{2},\quad a_{2n+2}=\frac{4n-1}{(2n+2)(2n+1)}a_{2n},\quad a_{2n+1}=\frac{4n-3}{2n(2n+1)}a_{2n-1}+(-1)^{n+1}\frac{1}{(2n+1)!},$$

$n=1,2,3,4,\dots$; que produeixen:

$$a_4=-\frac{a_0}{8},\ a_6=-\frac{7}{240}a_0,\ a_8=-\frac{11}{1920}a_0,\ a_{10}=-\frac{11}{11520}a_0,\ a_{12}=-\frac{19}{138240}a_0,$$
$$a_3=\frac{1}{6}a_1+\frac{1}{6},\ a_5=\frac{1}{24}a_1+\frac{1}{30},\ a_7=\frac{1}{112}a_1+\frac{37}{5040},\ a_9=\frac{13}{8064}a_1+\frac{1}{756},$$
$$a_{11}=\frac{221}{887040}a_1+\frac{8161}{39916800},\dots;$$ $a_0, a_1\in\mathbb{R}$ indeterminats.

Solució:

$$\boxed{\begin{aligned} y(x) &= a_0\left(1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{7}{240}x^6-\frac{11}{1920}x^8-\frac{11}{11520}x^{10}-\frac{19}{138240}x^{12}+O(x^{14})\right)+\\ &\quad + a_1\left(x+\frac{1}{6}x^3+\frac{1}{24}x^5+\frac{1}{112}x^7+\frac{13}{8064}x^9+\frac{221}{887040}x^{11}+O(x^{13})\right)+\\ &\quad +\frac{1}{6}x^3+\frac{1}{30}x^5+\frac{37}{5040}x^7+\frac{1}{756}x^9+\frac{8161}{39916800}x^{11}+O(x^{13}),\\ &a_0, a_1\in\mathbb{R}.\end{aligned}}$$

amb radi de convergència $R=+\infty$

(iii) $x^2y' + y = 0$, en $x = 1$.

Busquem solucions en s.d.p de la forma $y = \sum_{n\geq 0} a_n (x-1)^n \left(\Longrightarrow y' = \sum_{n\geq 1} n a_n (x-1)^{n-1}\right)$, així:

$$(x-1+1)^2 y' + y = \left((x-1)^2 + 2(x-1)+1\right) \sum_{n\geq 1} n a_n (x-1)^{n-1} + \sum_{n\geq 0} a_n (x-1)^n =$$

$$= \sum_{n\geq 1} n a_n (x-1)^{n+1} + 2\sum_{n\geq 1} n a_n (x-1)^n + \sum_{n\geq 1} n a_n (x-1)^{n-1} + \sum_{n\geq 0} a_n (x-1)^n =$$

$$= \sum_{n\geq 2} (n-1) a_{n-1} (x-1)^n + 2\sum_{n\geq 1} n a_n (x-1)^n + \sum_{n\geq 0} (n+1) a_{n+1} (x-1)^n + \sum_{n\geq 0} a_n (x-1)^n$$

$$= a_1 + a_0 + (3a_1 + 2a_2)x + \sum_{n\geq 2} \left((n-1)a_{n-1} + (2n+1)a_n + (n+1)a_{n+1}\right) x^n = 0$$

d'aquí tenim la relació de recurrència:

$$a_1 = -a_0, \quad a_2 = \frac{3}{2} a_0, \quad a_{n+1} = -\frac{n-1}{n+1} a_{n-1} - \frac{2n+1}{n+1} a_n, \quad n \geq 2.$$

Calcularem els valors dels primers coeficients:

$$n=2: \quad a_3 = -\frac{1}{3} a_1 - \frac{5}{3} a_2 = \frac{1}{3} a_0 - \frac{5}{3}\cdot\frac{3}{2} a_0 = -\frac{13}{6} a_0,$$

$$n=3: \quad a_4 = -\frac{2}{4} a_2 - \frac{7}{4} a_3 = -\frac{2}{4}\cdot\frac{3}{2} a_0 + \frac{7}{4}\cdot\frac{13}{6} a_0 = \left(-\frac{3}{4} + \frac{91}{24}\right) a_0 = \frac{-3+91}{24} a_0 = \frac{73}{24} a_0,$$

$$n=4: \quad a_5 = -\frac{3}{5} a_3 - \frac{9}{5} a_4 = \left(\frac{3}{5}\cdot\frac{13}{6} - \frac{9}{5}\cdot\frac{73}{24}\right) a_0 = -\frac{167}{40} a_0$$

$$n=5: \quad a_6 = -\frac{4}{6} a_4 - \frac{11}{6} a_5 = \left(-\frac{4}{6}\cdot\frac{73}{24} + \frac{11}{6}\cdot\frac{167}{40}\right) a_0 = \left(-\frac{73}{36} + \frac{1837}{240}\right) a_0 =$$

$$= \frac{-1460+5511}{720} a_0 = \frac{4051}{720} a_0, \ldots; \quad a_0 \in \mathbb{R} \text{ indeterminat}$$

Solució:

$$\boxed{y(x) = a_0 \left(1 - (x-1) + 3/2\,(x-1)^2 - \frac{13}{6}(x-1)^3 + \frac{73}{24}(x-1)^4 - \frac{167}{40}(x-1)^5 + \frac{4051}{720}(x-1)^6 + O\left((x-1)^7\right)\right),\ a_0 \in \mathbb{R}} \quad (*)$$

amb radi de convergència $R \geq 1$ (de fet és $R = 1$).

Remarca. Per separació de variables es $y = C e^{\frac{1}{x}}$, $C \in \mathbb{R}$ i desenvolupant al voltant de $x = 1$:

$$y(x) = Ce^{\frac{1}{x}} = Ce^{\frac{1}{1+(x-1)}} = Ce^{1-(x-1)+(x-1)^2-(x-1)^3+\dots+(-1)^p(x-1)^p+\dots} =$$

$$= Ce\cdot \exp\left\{-(x-1)+(x-1)^2-(x-1)^3+\dots+(-1)^p(x-1)^p+\dots\right\} = C'\sum_{m\geq 1}\frac{1}{m!}\left(-(x-1)+\dots\right)^m$$

$(c'\equiv ce)$

$$= c'\left\{1+\left(-(x-1)+(x-1)^2-\dots\right)+\frac{1}{2!}\left(-(x-1)+(x-1)^2-\dots\right)^2+\frac{1}{3!}\left(-(x-1)+(x-1)^2-\dots\right)^3+\dots\right.$$

$$= C'\left\{1-(x-1)+(1+1/2!)(x-1)^2-(1+1+1/3!)(x-1)^3+\dots\right\} = C'\left(1-(x-1)+\frac{3}{2}(x-1)^2-\frac{13}{6}(x-1)^3+\right.$$

i.e., obtenim la sèrie (*)

(iv) $y''-\frac{\sin x}{x}y'+4y=0$ en $x=0$,

$$f(x)=\frac{\sin x}{x}=\sum_{k\geq 0}c_k x^k=\sum_{n\geq 0}(-1)^n\frac{x^{2n}}{(2n+1)!}$$

Busquem solucions en sèrie de potències en $x=0$ de la forma:

$$y(x)=\sum_{m\geq 0}a_m x^m \Longrightarrow y'(x)=\sum_{m\geq 1} m\, a_m x^{m-1} \Longrightarrow y''(x)=\sum_{m\geq 2}m(m-1)a_m x^{m-2}.$$

a continuació, substituint a l'EDO:

$$y''-\frac{\sin x}{x}y'+4y=\sum_{m\geq 0}(m+2)(m+1)a_{m+2}x^m-\left(\sum_{k\geq 0}c_k x^k\right)\left(\sum_{m\geq 0}(m+1)a_{m+1}x^m\right)+$$

$$+4\sum_{m\geq 0}a_m x^m \overset{(1)}{=} \sum_{m\geq 0}\left((m+2)(m+1)a_{m+2}-\sum_{k=0}^{m}(m-k+1)c_k a_{m-k+1}+4a_m\right)x^m$$

on $c_{2m}=\frac{(-1)^m}{(2m+1)!}$, $c_{2m+1}=0$, $m=0,1,2,3,\dots$ Arribem doncs a la relació de recurrència

$$a_{m+2}=\frac{\sum_{k=0}^{m}(m-k+1)c_k a_{m-k+1}-4a_m}{(m+2)(m+1)}$$

$$=\frac{1}{(m+2)(m+1)}\left(\sum_{n=0}^{[m/2]}(m-2n+1)\frac{(-1)^n}{(2n+1)!}a_{m-2n+1}-4a_m\right)$$

Aquí fem el producte de convolució de les dues sèries, i.e.:

$$\left(\sum_{k\geq 0}c_k x^k\right)\left(\sum_{m\geq 0}(m+1)a_{m+1}x^m\right)=\sum_{k\geq 0}\sum_{m\geq 0}c_k(m+1)a_{m+1}x^{m+k}=\left\{\begin{array}{l}m+k=n\\ \Leftrightarrow m=n-k\end{array}\right\}$$

$$=\sum_{n\geq 0}\sum_{k=0}^{n}c_k(n-k+1)a_{n-k+1}$$

essent $[m/2]$ la part entera de $m/2$, recordem: $[x] = E(x) = \sup\left\{ z \in \mathbb{Z} : z \leq x \right\}$.

Obtindrem els primers coeficients, donant a m els valors $m = 0, 1, 2, 3$ i 4; així:

$m = 0$: $a_2 = \frac{1}{2}\left(\frac{1}{1!}a_1 - 4a_0\right) = \underline{\frac{1}{2}a_1 - 2a_0}$, ($a_0, a_1 \in \mathbb{R}$ indeterminats)

$m = 1$: $a_3 = \frac{1}{6}\left(2a_2 - 4a_1\right) = \frac{1}{6}\left(2\left(\frac{1}{2}a_1 - 2a_0\right) - 4a_1\right) = \frac{1}{6}\left(-4a_0 - 3a_1\right) = \underline{-\frac{2}{3}a_0 - \frac{1}{2}a_1}$,

$m = 2$: $a_4 = \frac{1}{12}\left(3a_3 - \frac{a_1}{3!} - 4a_2\right) = \frac{1}{12}\left(-2a_0 - \frac{3}{2}a_1 - \frac{1}{6}a_1 - 2a_1 + 8a_0\right) = \underline{\frac{1}{2}a_0 - \frac{11}{36}a_1}$,

$m = 3$: $a_5 = \frac{1}{20}\left(4\left(\frac{1}{2}a_0 - \frac{11}{36}a_1\right) - \frac{1}{3}\left(\frac{1}{2}a_1 - 2a_0\right) - 4\left(-\frac{2}{3}a_0 - \frac{1}{2}a_1\right)\right)$

$= \frac{1}{20}\left(2a_0 - \frac{11}{9}a_1 - \frac{1}{6}a_1 + \frac{2}{3}a_0 + \frac{8}{3}a_0 + 2a_1\right) = \frac{1}{20}\left(\left(2 + \frac{2}{3} + \frac{8}{3}\right)a_0 + \right.$

$\left. + \left(-\frac{11}{9} - \frac{1}{6} + 2\right)a_1\right) = \frac{1}{20}\left(\frac{6+2+8}{3}a_0 + \frac{-22-3+36}{18}a_1\right) = \underline{\frac{4}{15}a_0 + \frac{11}{360}a_1}$,

$m = 4$: $a_6 = \frac{1}{30}\left(5a_5 - \frac{3}{3!}a_3 + \frac{1}{5!}a_1 - 4a_4\right) = \frac{1}{30}\left(5\left(\frac{4}{15}a_0 + \frac{11}{360}a_1\right) - 4\left(\frac{1}{2}a_0 - \frac{11}{36}a_1\right) - \right.$

$\left. - \frac{1}{2}\left(-\frac{2}{3}a_0 - \frac{1}{2}a_1\right) + \frac{1}{120}a_1\right) = \frac{1}{30}\left(\frac{4}{3}a_0 + \frac{11}{72}a_1 - 2a_0 + \frac{11}{9}a_1 + \frac{1}{3}a_0 + \right.$

$\left. + \frac{1}{4}a_1 + \frac{1}{120}a_1\right) = \frac{1}{30}\left(\left(\frac{4}{3} - 2 + \frac{1}{3}\right)a_0 + \left(\frac{11}{72} + \frac{11}{9} + \frac{1}{4} + \frac{1}{120}\right)a_1\right) =$

$= \frac{1}{30}\left(\frac{4-6+1}{3}a_0 + \frac{55+440+90+3}{360}a_1\right) = \frac{1}{30}\left(-\frac{1}{3}a_0 + \frac{588}{360}a_1\right) = \underline{-\frac{1}{90}a_0 + \frac{49}{900}a_1}$,

, ... i la solució s'escriu com:

$$\boxed{\begin{aligned} y(x) &= a_0\left(1 - 2x^2 - \frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{4}{15}x^5 - \frac{1}{90}x^6 + O(x^7)\right) + \\ &\quad + a_1\left(x + \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{11}{36}x^4 + \frac{11}{360}x^5 + \frac{49}{900}x^6 + O(x^7)\right), \\ &a_0, a_1 \in \mathbb{R} \end{aligned}}$$

amb radi de convergència de les dues sèries $R = +\infty$ □