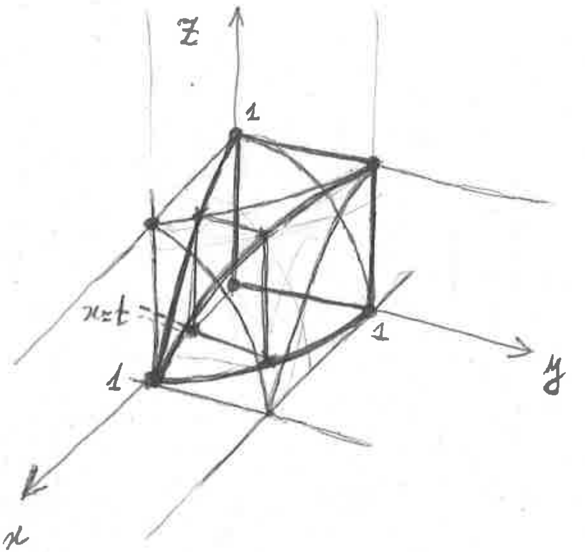
23) Useu coordenades cartesianes, cilíndriques o esfèriques (o bé el principi de Cavalieri per calcular el volum dels dominis de $\mathbb{R}^3$ limitats per les superfícies que s'indiquen.

a) $x^2+z^2=1,\ x^2+y^2=1$

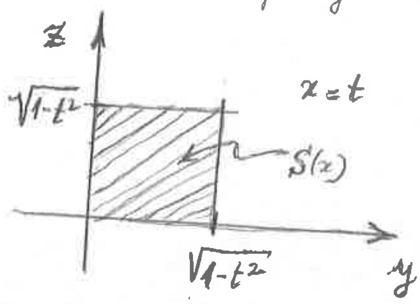


Parametrització de la intersecció per $y\geq 0, z\geq 0$.

$x=t$

$y=\sqrt{1-t^2}$

$z=\sqrt{1-t^2}$



Per tant: $S(x)=1-x^2$

$$V=8\cdot\int_0^1 S(x)\,dx=8\int_0^1(1-x^2)\,dx$$

$$=8\left[x-\frac{x^3}{3}\right]_0^1=8\left(1-\frac{1}{3}\right)=\boxed{\frac{16}{3}}$$

Tho podem calcular com una integral triple. En efecte, en el 1er octant (veure figura): $0\leq z\leq\sqrt{1-x^2}$, $0\leq y\leq\sqrt{1-x^2}$, $0\leq x\leq 1$. Aleshores

$$V=8\int_0^1 dx\int_0^{\sqrt{1-x^2}}dy\int_0^{\sqrt{1-x^2}}dz\ ..$$

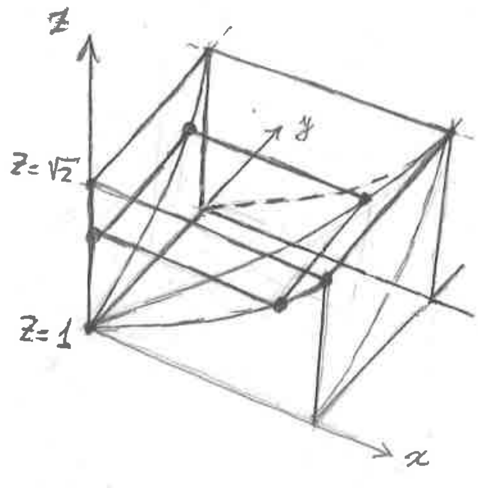
$$=8\int_0^1(1-x^2)\,dx$$

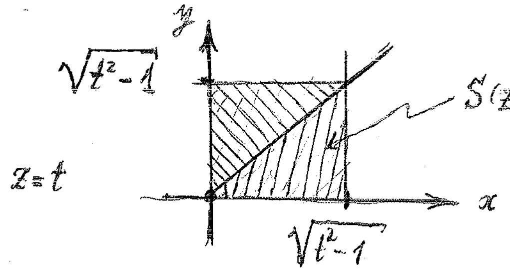
$$=8\left(x-\frac{x^3}{3}\right)\Big|_{x=0}^{x=1}=\boxed{\frac{16}{3}}$$

b) $z^2-x^2=1$ Parametrització de la intersecció

$z^2-y^2=1$ per $x\geq 0, y\geq 0, z\geq 0$.

$z=\sqrt{2}$ $\quad x=\sqrt{t^2-1},\ y=\sqrt{t^2-1},\ z=t,\ (t\geq 1)$.





$S(z) = z^2-1, \ (1 \leq z \leq \sqrt{2})$

aplicant el principi de Cavalieri:

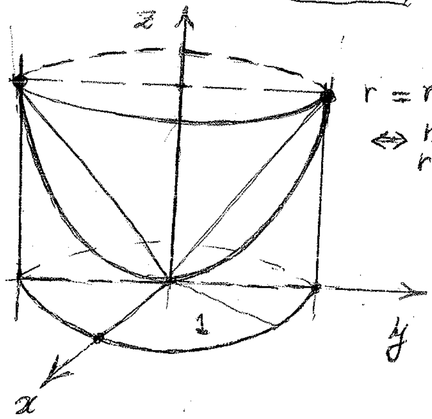
$$V = 4\cdot\int_1^{\sqrt{2}} S(z)\,dz = 4\cdot\int_1^{\sqrt{2}}(z^2-1)\,dz = 4\left(\frac{z^3}{3}-z\right)\Big|_1^{\sqrt{2}} = 4\left(\frac{2\sqrt{2}}{3}-\sqrt{2}+\frac{2}{3}\right) = \boxed{\frac{4}{3}\left(2-\sqrt{2}\right)} \quad \square$$

Alternativament:

$$\frac{V}{8} = \int_0^1 dx\int_0^x dy\int_{\sqrt{1+x^2}}^{\sqrt{2}} dz = \int_0^1 dx\int_0^x\left(\sqrt{2}-\sqrt{1+x^2}\right)dy = \int_0^1\left(\sqrt{2}\,x - x\sqrt{1+x^2}\right)dx$$

$$= \left[-\frac{1}{3}\left(1+x^2\right)^{3/2}+\frac{\sqrt{2}}{2}x^2\right]_0^1 = -\frac{1}{3}2\sqrt{2}+\frac{\sqrt{2}}{2}+\frac{1}{3} = \frac{1}{6}\left(2-\sqrt{2}\right) \Longrightarrow \boxed{V = \frac{4}{3}\left(2-\sqrt{2}\right)}.$$

c) $z^2 = x^2+y^2, \ z = x^2+y^2, \ (z \geq 0)$

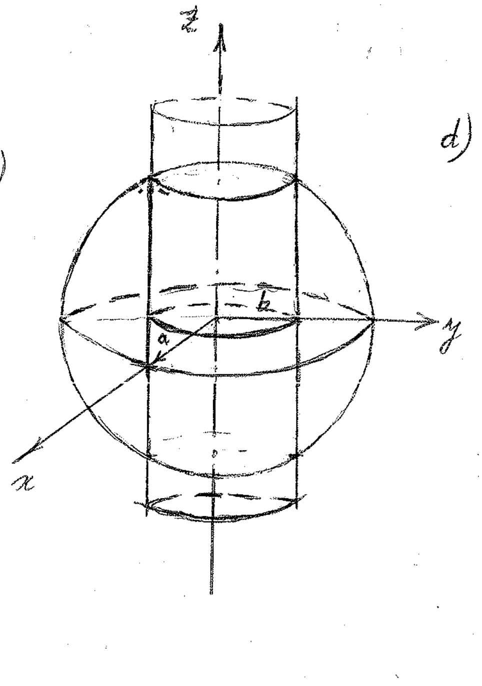


$r = r^2$
$\Leftrightarrow r=0$ ó $r=1$

$$V = \int_0^{2\pi} d\theta\int_0^1 r\,dr\int_{r^2}^{r} dz = 2\pi\int_0^1\left(r^2-r^3\right)dr = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$= \boxed{\frac{\pi}{6}} \quad \square$$

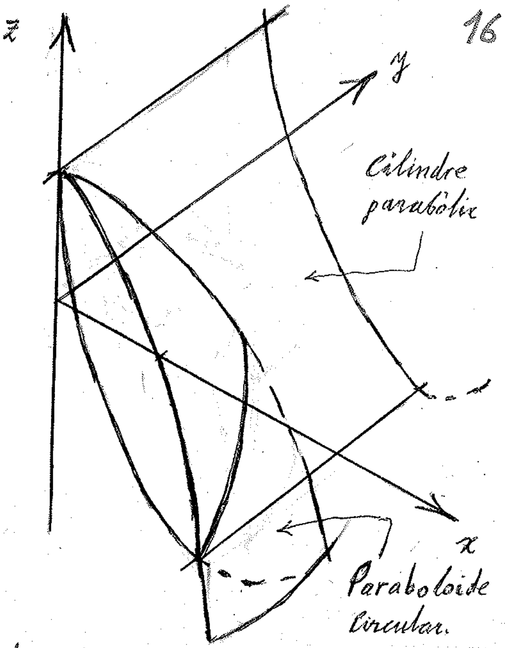
d)



d) Part de l'esfera $x^2+y^2+z^2=a^2$ que és exterior al cilindre $x^2+y^2=b^2$ $(a>b>0)$.

$$\frac{V}{2} = \int_0^{2\pi} d\theta\int_b^a r\,dr\int_0^{\sqrt{a^2-r^2}} dz = 2\pi\int_b^a r\left(a^2-r^2\right)^{1/2} dr$$

$$= -\frac{2}{3}\left(a^2-r^2\right)^{3/2}\pi\Big|_b^a = \frac{2\pi}{3}\left(a^2-b^2\right)^{3/2} \Longrightarrow \boxed{V = \frac{4\pi}{3}\left(a^2-b^2\right)^{3/2}}$$



e) $z = x^2-4x+1, \quad 1-z = x^2+y^2$

Intersecció de les superfícies per $y \geq 0$

$$1-x^2-y^2 = x^2-4x+1 \Rightarrow y^2 = 4x-2x^2$$

quan, $y \geq 0$, llavors $y = \sqrt{4x-2x^2}$, i tenim la següent parametrització de la intersecció de les superfícies (per $y \geq 0$)

$$\left.\begin{aligned} x &= t \\ y &= \sqrt{4t-2t^2} \\ z &= t^2-4t+1. \end{aligned}\right\} \text{; amb } 0 \leq t \leq 2.$$

La projecció d'aquesta parametrització sobre el pla $z=0$ ve donada per: $y = \sqrt{4x-2x^2}$. Amb la qual cosa, per calcular el volum, fem:

Aleshores:

$$\frac{V}{2} = \int_0^2 dx \int_0^{\sqrt{4x-x^2}} dy \int_{x^2-4x+1}^{1-x^2-y^2} dz = \int_0^2 dx \int_0^{\sqrt{4x-2x^2}} \left(1-x^2-y^2-x^2+4x-1\right) = \int_0^2 dx \int_0^{\sqrt{4x-2x^2}} dy \left(4x-2x^2-y^2\right)$$

$$= \int_0^2 dx \left(4xy - 2x^2y - \frac{y^3}{3}\right)\Bigg|_0^{\sqrt{4x-2x^2}} = \int_0^2 dx \left(4x\sqrt{4x-2x^2} - 2x^2\sqrt{4x-2x^2} - \frac{1}{3}(4x-2x^2)\sqrt{4x-2x^2}\right)$$

$$= \int_0^2 \sqrt{4x-2x^2}\left(4x-2x^2-\frac{4}{3}x+\frac{2}{3}x^2\right) dx = \frac{2}{3}\int_0^2 \left(4x-2x^2\right)^{3/2} = \frac{2^4}{3}\int_0^2 x^{3/2}\left(1-\frac{x}{2}\right)^{3/2} dx$$

$$= \left\{\begin{aligned} x &= 2\sin^2 t \\ dx &= 4\sin t \cos t \, dt \\ &\cdots \end{aligned}\right\} = \frac{64}{3}2^{3/2}\int_0^{\pi/2} \sin^4 t \cos^4 t \, dt = \frac{2^6 \cdot 2^{3/2}}{3 \cdot 2^4}\int_0^{\pi/2} \sin^4(2t)\, dt$$

$$\overset{(*)}{=} \frac{8\sqrt{3}}{3} \cdot \frac{3\pi}{16} = \frac{\pi\sqrt{3}}{2} \Longrightarrow \boxed{V = \pi\sqrt{2}}$$

(*) $$\int_0^{\pi/2} \sin^4(2t)\, dt = \int_0^{\pi/2} \sin^2(2t)\left(1-\cos^2(2t)\right) dt = \int_0^{\pi/2} \left[\sin^2(2t) - \frac{1}{4}\sin^2(4t)\right] dt$$

$$= \int_0^{\pi/2} \left[\frac{1-\cos(4t)}{2} - \frac{1-\cos(8t)}{8}\right] dt = \frac{\pi}{4} - \frac{\pi}{16} = \frac{3\pi}{16}$$

f) $x^2 = z$, $y^2 = x$, $z^2 = y$

$x^2 = az$, $y^2 = ax$, $z^2 = ay$ $(a>1)$

$$\mathcal{D}: \left.\begin{array}{l} y^2/a \leq x \leq y^2 \\ z^2/a \leq y \leq z^2 \\ x^2/a \leq z \leq x^2 \end{array}\right\} ;$$

Això suggereix el canvi següent:

$$u = x/y^2, \quad v = y/z^2, \quad w = z/x^2$$

Llavors $\mathcal{D}'$: $\frac{1}{a} \leq u \leq 1$, $\frac{1}{a} \leq v \leq 1$, $\frac{1}{a} \leq w \leq 1$ és el domini en

Càlcul del Jacobià de la transformació:

$$(x,y,z) \mapsto (u,v,w) = f(x,y,z) = \left(\frac{x}{y^2} \ \frac{y}{z^2} \ \frac{z}{x^2}\right),$$ d'on podem calcular $Df(x,y,z)$,

però de fet, necessitem: $(u,v,w) \mapsto (x,y,z) = f^{-1}(u,v,w)$ i el determinant de la corresponent matriu Jacobiana, i.e. $\det Df^{-1}(u,v,w)$.

$$f \circ f^{-1}(u,v,w) = (u,v,w) \Rightarrow Df \circ f^{-1}(u,v,w)\, Df^{-1}(u,v,w) = Df(x,y,z)\, Df^{-1}(u,v,w) = Id$$

$$\Rightarrow Df^{-1}(u,v,w) = Df(x,y,z)^{-1} \Rightarrow \det Df^{-1}(u,v,w) = \frac{1}{\det Df(x,y,z)}$$

(i.e., apliquem la regla de la cadena a la identitat: $f f^{-1} = Id$). Fent els càlculs:

$$\det Df(x,y,z) = \begin{vmatrix} 1/y^2 & -2x/y^3 & 0 \\ 0 & 1/z^2 & -2y/z^3 \\ -2z/x^3 & 0 & 1/x^2 \end{vmatrix} = \frac{1}{x^3y^3z^3} \begin{vmatrix} y & -2x & 0 \\ 0 & z & -2y \\ -2z & 0 & x \end{vmatrix} = \frac{1}{x^3y^3z^3}(xyz - 8xyz) =$$

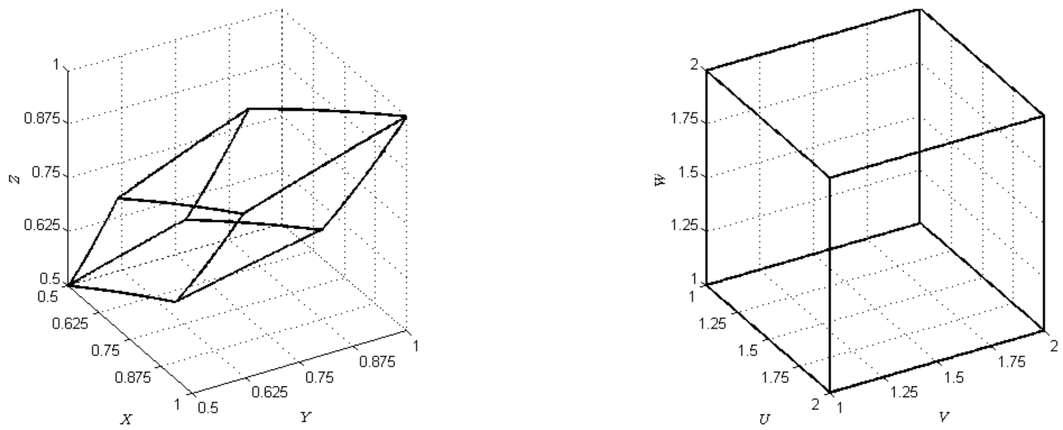
$$= -\frac{7}{x^2y^2z^2} \Longrightarrow \left|\det Df^{-1}(u,v,w)\right| = \frac{x^2y^2z^2}{7} = \frac{1/7}{u^2v^2w^2}$$

Aleshores:

$$V = \iiint_{\mathcal{D}} dx\,dy\,dz = \iiint_{\mathcal{D}' = f(\mathcal{D})} \left|\det Df^{-1}(u,v,w)\right| du\,dv\,dw = \frac{1}{7} \iiint_{\mathcal{D}' = [\frac{1}{a},1]\times[\frac{1}{a},1]\times[\frac{1}{a},1]} \frac{du\,dv\,dw}{u^2v^2w^2}$$

$$= \frac{1}{7} \int_{1/a}^{1} \frac{du}{u^2} \int_{1/a}^{1} \frac{dv}{v^2} \int_{1/a}^{1} \frac{dw}{w^2} = \frac{1}{7}\left(\int_{1/a}^{1} \frac{du}{u^2}\right)^3 = \frac{1}{7}\left(\left[-\frac{1}{u}\right]_{1/a}^{1}\right)^3 = \boxed{\frac{(a-1)^3}{7}}$$

□



(a) $\mathcal{D}$: Domini definit per les superfícies,
$x^2 = z, \quad y^2 = x, \quad z^2 = y,$
$x^2 = az, \quad y^2 = ax, \quad z^2 = ay.$

(b) $\mathcal{D}' : 1 \leq u \leq a,\ 1 \leq v \leq a,\ 1 \leq w \leq a.$

Figura 1. Transformació del domini $\mathcal{D}$ en $\mathcal{D}'$ pel canvi (1).

Nota

Alternativament, es comprova d'immediat que el canvi

$$u = \frac{y^2}{x}, \qquad v = \frac{z^2}{y}, \qquad w = \frac{x^2}{z}, \tag{1}$$

transforma el domini original $\mathcal{D}$, en $\mathcal{D}' = [1,a] \times [1,a] \times [1,a]$. És a dir, en un cub d'aresta $a-1$ (veure Figura 1). Aleshores el Jacobià corresponent surt més senzill. En efecte:

$$\det \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\dfrac{y^2}{x^2} & 2\dfrac{y}{x} & 0 \\ 0 & -\dfrac{z^2}{y^2} & 2\dfrac{z}{y} \\ 2\dfrac{x}{z} & 0 & -\dfrac{x^2}{z^2} \end{vmatrix} = -7,$$

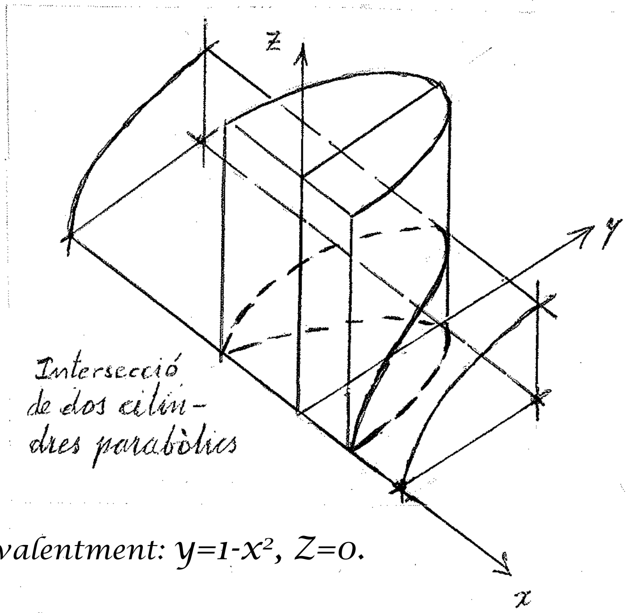
d'on:

$$\left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\det \dfrac{\partial(u,v,w)}{\partial(x,y,z)}\right|} = \frac{1}{7}.$$

Llavors el càlcul del volum es simplifica encara més:

$$\begin{aligned} V = \iiint_{\mathcal{D}} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z &= \iiint_{\mathcal{D}'} \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w \\ &= \frac{1}{7} \int_1^a \mathrm{d}u \int_1^a \mathrm{d}v \int_1^a \mathrm{d}w = \frac{1}{7} \left(\int_1^a \mathrm{d}u \right)^3 = \frac{(a-1)^3}{7}. \end{aligned}$$

g) $z^2 = y, \quad x^2 = 1-y$



Intersecció de dos cilindres parabòlics

Parametritzem la intersecció per a $z \geq 0$, $x \geq 0$

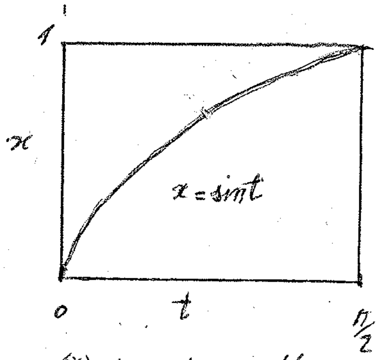
$$\left.\begin{aligned} x &= \sqrt{1-t} \\ y &= t \\ z &= \sqrt{t} \end{aligned}\right\} : \quad 0 \leq t \leq 1$$

Projecció sobre el pla $z=0$:
$x = \sqrt{1-t}$, $y = t$, $z=0$, o equivalentment: $y = 1-x^2$, $z=0$.

Aleshores:

$$\frac{V}{4} = \int_0^1 dx \int_0^{1-x^2} dy \int_0^{\sqrt{y}} dz = \int_0^1 dx \int_0^{1-x^2} dy \sqrt{y} = \frac{2}{3} \int_0^1 dx \left[y^{3/2} \right]_0^{1-x^2}$$

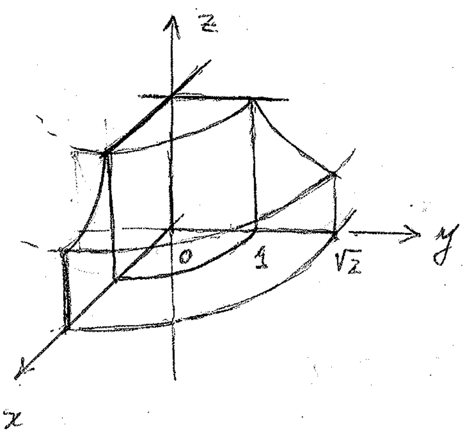
$$= \frac{2}{3} \int_0^1 \left(1-x^2\right)^{3/2} dx = \overset{\text{c.v. }(*)}{\left\{ \begin{aligned} x &= \sin t \\ dx &= \cos t \, dt \end{aligned} \right\}} = \frac{2}{3} \underbrace{\int_0^{\pi/2} \cos^4 t \, dt}_{= \, 3\pi/16} = \frac{\pi}{8} \Rightarrow \boxed{V = \frac{\pi}{2}}.$$



(*) Canvi de variables a la integral

h) $x^2+y^2 = 1, \quad x^2+y^2 = 2, \quad z(x^2+y^2) = 1, \quad z = 0.$

En coordenades cilíndriques: $\quad 1 \leq r \leq \sqrt{2}, \quad 0 \leq z \leq \frac{1}{r^2}$

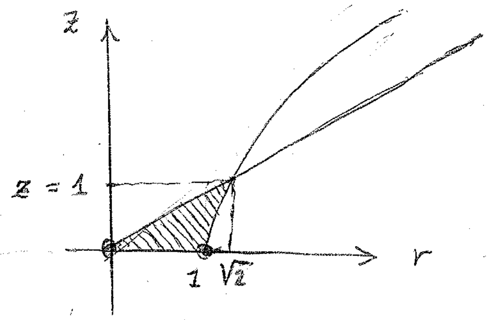


$$V = \int_0^{2\pi} d\theta \int_1^{\sqrt{2}} r\, dr \int_0^{1/r^2} dz = 2\pi \int_1^{\sqrt{2}} \frac{dr}{r}$$

$$= 2\pi \Big[\ln r \Big]_1^{\sqrt{2}} = \boxed{\pi \ln 2}$$

i) $x^2+y^2 = 2z^2, \quad x^2+y^2 = z^2+1 \quad (x \geq 0, y \geq 0, z \geq 0)$

Apliquem coordenades cilíndriques (pàg. següent)



$$V = \int_0^{\pi/2} d\theta \int_0^1 r\,dr \int_0^{r/\sqrt{2}} dz + \int_0^{\pi/2} d\theta \int_1^{\sqrt{2}} r\,dr \int_{\sqrt{r^2-1}}^{r/\sqrt{2}} dz$$

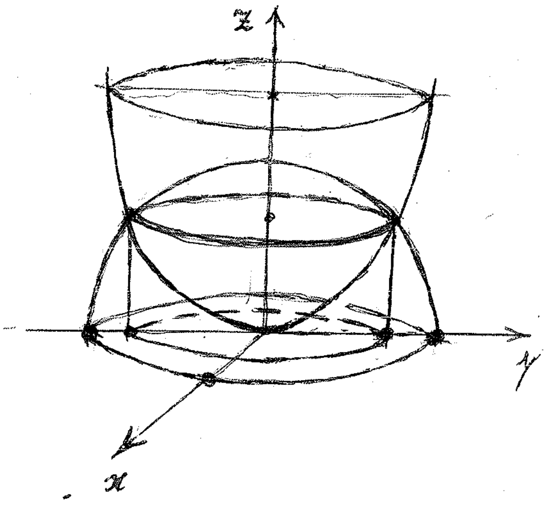
$$= \frac{\pi}{2}\int_0^1 \frac{r^2}{\sqrt{2}}\,dr + \frac{\pi}{2}\int_1^{\sqrt{2}} \left(\frac{r^2}{\sqrt{2}} - r\sqrt{r^2-1}\right) dr$$

$$= \frac{\pi}{4}\sqrt{2}\left[\frac{r^3}{3}\right]_0^1 + \frac{\pi}{2}\left[\frac{\sqrt{2}}{2}\cdot\frac{r^3}{3} - \frac{1}{3}(r^2-1)^{3/2}\right]_1^{\sqrt{2}} = \frac{\sqrt{2}\pi}{12} + \frac{\pi}{2}\left[\frac{2}{3} - \frac{1}{3} - \frac{\sqrt{2}}{6}\right] = \boxed{\frac{\pi}{6}}$$

Alternativament (més "fàcil"!):

$$V = \int_0^{\pi/2} d\theta \int_0^1 dz \int_{\sqrt{2}z}^{\sqrt{z^2+1}} r\,dr = \frac{\pi}{4}\int_0^1 (z^2+1-2z^2)\,dz = \frac{\pi}{4}\int_0^1 (1-z^2)\,dz$$

$$= \frac{\pi}{4}\left[z - \frac{z^3}{3}\right]_0^1 = \frac{\pi}{4}\left(1-\frac{1}{3}\right) = \boxed{\frac{\pi}{6}}$$

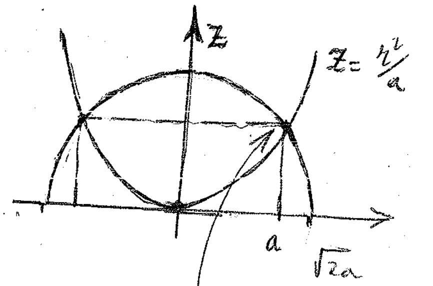


(j) $x^2+y^2+z^2 \le 2a^2$, $z \ge \frac{x^2+y^2}{a}$ $(a>0)$. En cilíndriques:

$$V = \int_0^{2\pi} d\theta \int_0^a r\,dr \int_{r^2/a}^{\sqrt{2a^2-r^2}} dz = 2\pi \int_0^a \left(r\sqrt{2a^2-r^2} - \frac{r^3}{a}\right) dr$$

$$= 2\pi\left[-\frac{1}{3}(2a^2-r^2)^{3/2} - \frac{r^4}{4a}\right]_0^a$$

$$= 2\pi\left[\frac{1}{3}(2\sqrt{2}-1) - \frac{1}{4}\right]a^3 = \boxed{2\pi a^3\left(\frac{2^{3/2}}{3} - \frac{7}{12}\right)}$$



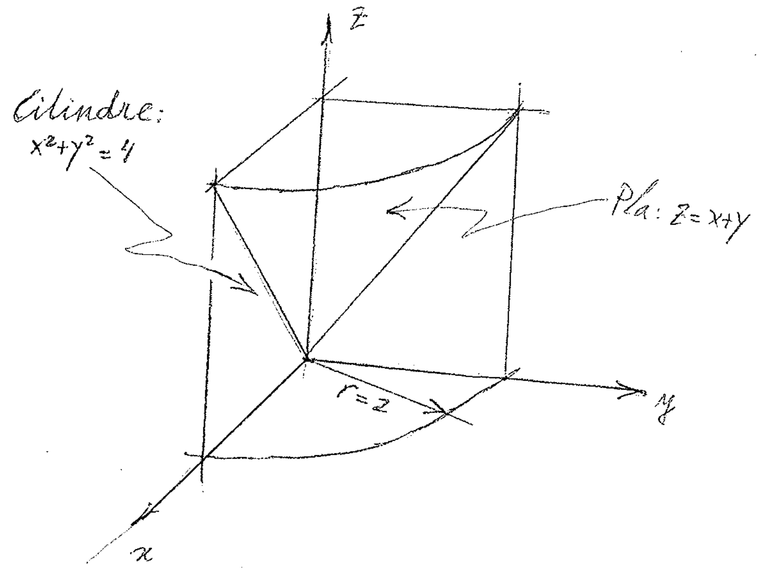
$az + z^2 = 2a^2$

$\Longleftrightarrow z^2 + az - 2a^2 = 0$

d'on:

$$z = \frac{-a \pm \sqrt{a^2+8a^2}}{2} = \frac{-a\pm 3a}{2} = \begin{cases} a \to r = a \\ -2a \text{ (No)} \end{cases}$$

(k) $x^2+y^2=4,\ z=x+y\ (x\geq 0,\ y\geq 0,\ z\geq 0)$



$$V=\int_0^{\pi/2} d\theta \int_0^2 r\,dr \int_0^{r\cos\theta + r\sin\theta} dz$$

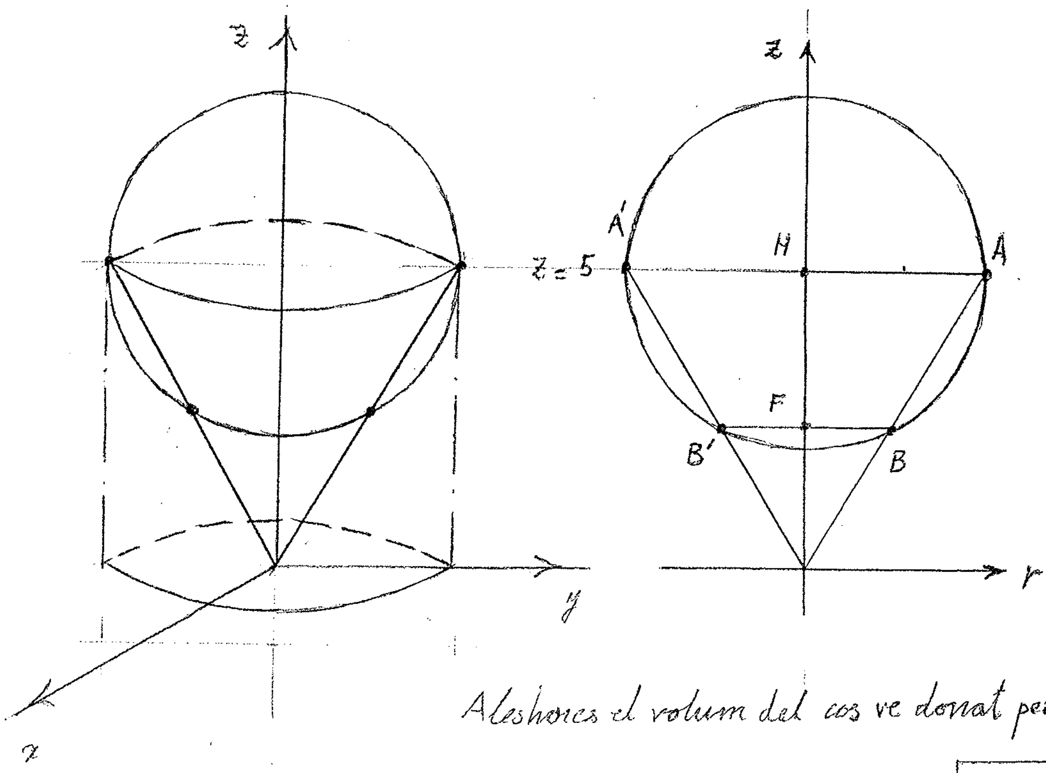
$$=\int_0^{\pi/2} d\theta \int_0^2 r^2(\cos\theta+\sin\theta)\,dr$$

$$=\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta \int_0^2 r^2\,dr$$

$$=\Big[\sin\theta-\cos\theta\Big]_0^{\pi/2}\cdot\left[\frac{r^3}{3}\right]_0^2=\boxed{\frac{16}{3}}\ \square$$

Exercici: substituir $z=x+y \to z=x+y-1$ i repetir el càlcul...

(l) Con de gelat definit per: $x^2+y^2\leq z^2/5,\ 0\leq z\leq 5+\sqrt{5-x^2-y^2}$.



Alçada dels punts d'intersecció A, B, A', B'

$$(z-5)^2=5-z^2/5$$
$$\iff 6z^2-50z+100=0.$$

d'on, $z=5$: alçada de A, A'

$z=\frac{10}{3}$: " " B, B'

i els 'radis' respectius són:

$r=\sqrt{5}$: distància $\overline{HA}=\overline{A'H}$

$r=\frac{2\sqrt{5}}{3}$: " $\overline{FB}=\overline{B'F}$

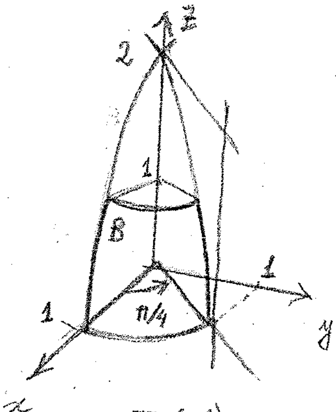
Aleshores el volum del cos ve donat per

$$V=\underbrace{\frac{2}{3}\pi\sqrt{5}^3}_{\text{Volum de la semiesfera}}+\underbrace{\frac{1}{3}\pi\sqrt{5}^2\,5}_{\text{Volum del con.}}=\boxed{\left(\frac{10\sqrt{5}}{3}+\frac{25}{3}\right)\pi}$$

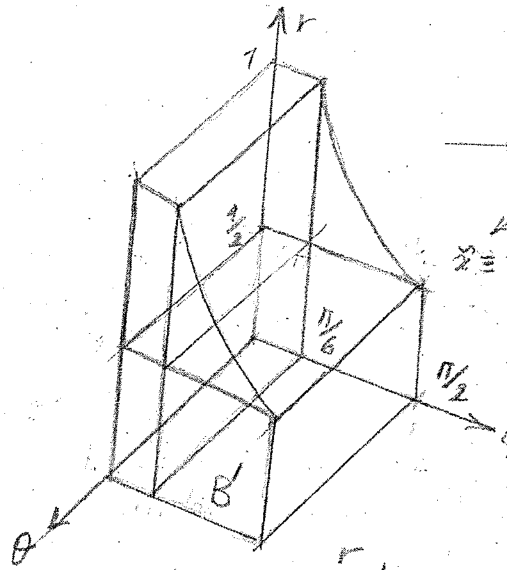
22) Adapteu les coordenades esfèriques per calcular les següents integrals triples.

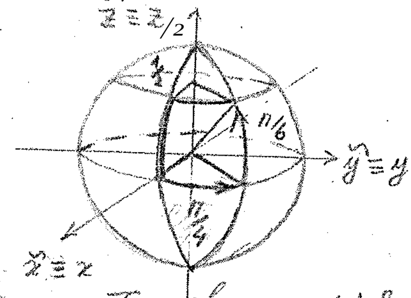
(a) $I = \iiint_B 16z\,dx\,dy\,dz$, $B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2/4 \leq 1,\ 0 \leq z \leq 1,\ 0 \leq y \leq x\}$

Coordenades esfèriques adaptades: $T(r,\theta,\varphi) = (r\cos\theta\cos\varphi,\ r\sin\theta\cos\varphi,\ r\sin\varphi)$.



$B = T(B')$





Transformació del domini B mitjançant les coordenades esfèriques adaptades.

$x^2+y^2+z^2/4 \leq 1 \iff 0 \leq r \leq 1$

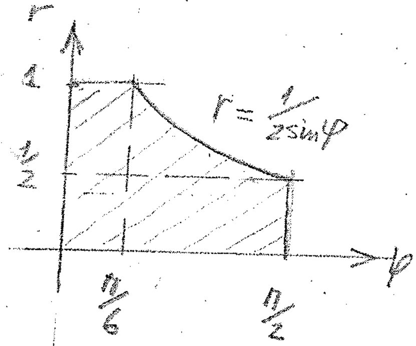
$0 \leq z \leq 1 \iff 0 \leq r \leq \frac{1}{2\sin\varphi}$, amb $0 \leq \varphi \leq \frac{\pi}{2}$

$0 \leq y \leq x \iff 0 \leq r\cos\theta\cos\varphi \leq r\sin\theta\cos\varphi$

amb $r > 0$, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$

$\iff 0 \leq \sin\theta \leq \cos\theta$ amb $0 \leq \theta \leq 2\pi$

$\iff 0 \leq \theta \leq \frac{\pi}{4}$.



Aleshores $I = I_1 + I_2$, amb:

$$I_1 \overset{(*)}{=} 64\int_0^{\pi/4} d\theta \int_0^{\pi/6} \sin\varphi\cos\varphi\,d\varphi \int_0^1 r^3\,dr = 64\,\frac{\pi}{4}\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/6}\left[\frac{r^4}{4}\right]_0^1 = 16\pi\,\frac{1}{8}\cdot\frac{1}{4} = \frac{\pi}{2}$$

$$I_2 = 64\int_0^{\pi/4} d\theta \int_{\pi/6}^{\pi/2} \sin\varphi\cos\varphi\,d\varphi \int_0^{\frac{1}{2\sin\varphi}} r^3\,dr = \frac{64}{16}\,\frac{\pi}{16}\int_{\pi/6}^{\pi/2} \frac{\cos\varphi}{\sin^3\varphi}\,d\varphi = \frac{\pi}{8}\left[-\frac{1}{\sin^2\varphi}\right]_{\pi/6}^{\pi/2}$$

$$= \frac{\pi}{8}(4-1) = \frac{3\pi}{8}$$

Llavors: $\boxed{I = I_1 + I_2 = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}.}$

(*) $\det DT(r,\theta,\varphi) = 2r^2\cos\varphi$.

$z = 2r\sin\varphi$. Aleshores $16z\,dx\,dy\,dz = 16\cdot 2r\sin\varphi\cdot 2r^2\cos\varphi\,d\theta\,d\varphi\,dr = 64r^3\sin\varphi\cos\varphi\,d\theta\,d\varphi\,dr$.

Alternativament:

$$\frac{1}{8}\int_{1/2}^{1} r\,dr = \frac{1}{16}\left(1-\frac{1}{4}\right)=\frac{3}{64}$$

$I = I_1' + I_2'$,

amb: $$I_1' = 64\int_0^{\pi/4} d\theta \int_{1/2}^{1} r^3 dr \underbrace{\int_0^{\arcsin\frac{1}{2r}} \cos\varphi \sin\varphi\, d\varphi}_{=\left.\frac{\sin^2\varphi}{2}\right|_0^{\arcsin\frac{1}{2r}}} = 64\int_0^{\pi/4} d\theta \left(\int_{1/2}^{1} r^3 \frac{1}{8r^2}\,dr\right) = \cancel{64}\cdot\frac{3}{\cancel{64}}\cdot\frac{\pi}{4} = \frac{3\pi}{4}$$

$$I_2' = 64\underbrace{\int_0^{\pi/4} d\theta}_{=\frac{\pi}{4}} \underbrace{\int_0^{1/2} r^3 dr}_{=\frac{1}{64}} \underbrace{\int_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi}_{=\left.\frac{\sin^2\varphi}{2}\right|_0^{\pi/2}} = \cancel{64}\,\frac{\pi}{4}\cdot\frac{1}{\cancel{64}}\cdot\frac{1}{2} = \frac{\pi}{8}$$

Aleshores: $\boxed{I = I_1' + I_2' = \frac{3\pi}{4} + \frac{\pi}{8} = \frac{7\pi}{8}}$

Exercici: intenteu aquest mateix apartat, fent servir coordenades cilíndriques ☺

(b) $I = \iiint_B \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dx\,dy\,dz$, $B = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$.

Solució: adaptem les coordenades esfèriques: $x = ar\cos\theta\cos\varphi$, $y = br\sin\theta\cos\varphi$, $z = cr\sin\varphi$, $T:(r,\theta,\varphi) \to (x,y,z) = T(r,\theta,\varphi)$

Aleshores: $B' = \{(r,\theta,\varphi) \in [0,+\infty) \times [0,2\pi] \times [-\frac{\pi}{2},\frac{\pi}{2}] : 0 \le r \le 1\} = B_1(0,0,0)$, és el domini transformat i llavors la integral I es pot calcular com:

$$I = \iiint_{B=T(B')} \underbrace{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)}_{=r^2} dx\,dy\,dz = \iiint_{B'} r^2 \underbrace{\left|\det \frac{\partial(x,y,z)}{\partial(\theta,\varphi,r)}\right|}_{=abc\, r^2\cos\varphi} d\theta\, d\varphi\, dr$$

Per què?

$$= abc\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi \int_0^1 r^4 dr = abc\, 2\pi \left[\sin\varphi\right]_{-\pi/2}^{\pi/2} \left[\frac{r^5}{5}\right]_0^1 = \boxed{\frac{4\pi}{5}abc}.$$

24) Sigui B la bola de centre $(0,0,0)$ i radi R. Denotem per $T(x,y,z)$ la temperatura en el punt (x,y,z) i suposem que és proporcional a la distància del punt a l'origen. En quins punts de B la temperatura coincideix amb la temperatura mitjana de la bola?

Solució

$$\langle T \rangle = \frac{\alpha}{\frac{4}{3}\pi R^3} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi \, d\varphi \int_0^R r^3 \, dr = \alpha \, \frac{4\pi \frac{R^4}{4}}{\frac{4}{3}\pi R^3} = \alpha \, \frac{3}{4} R$$

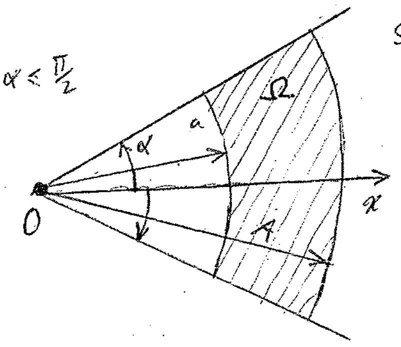
$$T(x,y,z) = \alpha \sqrt{x^2+y^2+z^2}$$
$$= \alpha r$$

$$T(x,y,z) = \alpha \, \frac{3}{4} R \iff \boxed{x^2+y^2+z^2 = \frac{9}{16} R^2}$$

25) Trobeu el centre de masses de les regions planes amb les densitats que s'indiquen

(a) Sector pla definit per una corona de radi interior a i radi exterior A, un angle d'obertura 2α i que és simètrica respecte l'eix x positiu, suposant densitat constant $\rho(x,y) = 1$.

$0 \leq \alpha \leq \frac{\pi}{2}$



Solució:

$$M = \iint_\Omega \overset{1}{\rho(x,y)} \, dx\,dy = \int_{-\alpha}^{\alpha} d\theta \int_a^A r \, dr = \frac{2\alpha}{2}\left(A^2-a^2\right) = \alpha\left(A^2-a^2\right)$$

$$X = \frac{1}{M} \iint_\Omega x\,\rho(x,y)\,dx\,dy = \frac{1}{M} \int_{-\alpha}^{\alpha} \cos\theta \, d\theta \int_a^A r^2 \, dr = \frac{2}{3M} (\sin\alpha) \cdot \left(A^3-a^3\right)$$

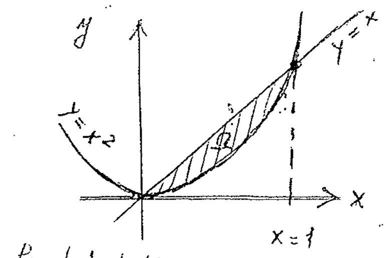
$$= \frac{2}{3} \cdot \frac{\sin\alpha}{\alpha} \cdot \frac{A^3-a^3}{A^2-a^2}.$$

$$Y = \frac{1}{M} \iint_\Omega y\,\rho(x,y)\,dx\,dy = \frac{1}{M} \int_{-\alpha}^{\alpha} \sin\theta \, d\theta \int_a^A r^2 \, dr = 0$$

Posició del Centre de Masses:

$$(X,Y) = \left(\frac{2}{3} \, \frac{\sin\alpha}{\alpha} \, \frac{A^3-a^3}{A^2-a^2}, \, 0 \right) \square$$

b) Regió entre $y=x^2$ i $y=x$ amb $\rho(x,y)=x+y$



Punt de tall: $x^2=x \iff x(x-1)=0$
$\iff x=0,\ x=1.$

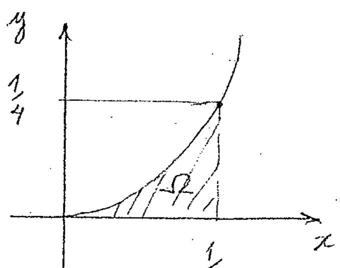
$$M=\iint_\Omega \rho(x,y)\,dx\,dy=\int_0^1 dx\int_{x^2}^{x}(x+y)\,dy$$
$$=\int_0^1 dx\left(xy+\frac{y^2}{2}\right)\Bigg|_{x^2}^{x}=\int_0^1 dx\left(x^2+\frac{x^2}{2}-x^3-\frac{x^4}{2}\right)$$
$$=\int_0^1\left(\frac{3}{2}x^2-x^3-\frac{x^4}{2}\right)dx=\frac{x^3}{2}-\frac{x^4}{4}-\frac{x^5}{10}\Bigg|_0^1=\frac{1}{2}-\frac{1}{4}-\frac{1}{10}=\frac{10-5-2}{20}=\frac{3}{20}$$

$$X=\frac{1}{M}\int_0^1 dx\int_{x^2}^{x}x(x+y)\,dy=\frac{1}{M}\int_0^1 dx\left[x^2y+x\frac{y^2}{2}\right]_{x^2}^{x}=\int_0^1\left(\frac{3x^3}{2}-x^4-\frac{x^5}{2}\right)dx$$
$$=\frac{1}{M}\left(\frac{3x^4}{8}-\frac{x^5}{5}-\frac{x^6}{12}\right)\Bigg|_0^1=\frac{1}{M}\left(\frac{3}{8}-\frac{1}{5}-\frac{1}{12}\right)=\frac{45-24-10}{120M}=\frac{11}{120M}=\frac{11}{120}\cdot\frac{20}{3}=\frac{11}{18}$$

$$Y=\frac{1}{M}\int_0^1 dx\int_{x^2}^{x}y(x+y)\,dy=\frac{1}{M}\int_0^1 dx\left[\frac{y^2}{2}x+\frac{y^3}{3}\right]_{x^2}^{x}=\frac{1}{M}\int_0^1\left(\frac{5x^3}{2}-\frac{x^5}{2}-\frac{x^6}{3}\right)dx$$
$$=\frac{1}{M}\left(\frac{5x^4}{24}-\frac{x^6}{12}-\frac{x^7}{21}\right)\Bigg|_0^1=\frac{1}{M}\left(\frac{1}{24}-\frac{1}{12}-\frac{1}{21}\right)=\frac{1}{M}\cdot\frac{35-14-8}{168}=\frac{13}{168}\cdot\frac{20}{3}=\frac{65}{126}.$$

∴ Posició del CDM:
$$(X,Y)=\left(\frac{11}{8},\frac{65}{126}\right).\ \square$$

c) Regió entre $y=0$ i $y=x^2$ $\left(0\le x\le \frac{1}{2}\right)$, amb $\rho(x,y)=1$.



$$M=\iint_\Omega \rho(x,y)\,dx\,dy=\int_0^{1/2}dx\int_0^{x^2}dy=\int_0^{1/2}x^2\,dx=\left[\frac{x^3}{3}\right]_0^{1/2}=\frac{1}{24}$$
$$X=\frac{1}{M}\iint_\Omega x\,\overset{1}{\overset{\shortparallel}{\rho(x,y)}}\,dx\,dy=\frac{1}{M}\int_0^{1/2}x\,dx\int_0^{x^2}dy=\frac{1}{M}\int_0^{1/2}x^3\,dx=\frac{1}{M}\left[\frac{x^4}{4}\right]_0^{1/2}=\frac{3}{8}$$
$$Y=\frac{1}{M}\iint_\Omega y\,\overset{1}{\overset{\shortparallel}{\rho(x,y)}}\,dx\,dy=\frac{1}{M}\int_0^{1/2}dx\int_0^{x^2}y\,dy=\frac{1/2}{M}\int_0^{1/2}x^4\,dx=\frac{1}{M}\left[\frac{x^5}{10}\right]_0^{1/2}=\frac{3}{40}$$

∴ Posició del CDM:
$$(X,Y)=\left(\frac{3}{8},\frac{3}{40}\right).\ \square$$

26) En cadascun dels casos següents, trobeu el centre de masses del sòlid A suposant distribució de masses homogènia.

(a) $A=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le R^2,\ x\ge 0,\ y\ge 0,\ z\ge 0\}$.

Solució. $\rho=\rho_0=\text{ctnt}$.

$$m(A)=\frac{4}{3}\pi R^3\rho_0\cdot\frac{1}{8}=\frac{\rho_0}{6}\pi R^3$$

$$X=\frac{1}{M}\iiint_A x\underbrace{\rho(x,y,z)}_{=\rho_0}\,dx\,dy\,dz=\frac{\rho_0}{M}\underbrace{\int_0^{\pi/2}\cos\theta\,d\theta}_{=1}\underbrace{\int_0^{\pi/2}\cos^2\varphi\,d\varphi}_{(*)}\underbrace{\int_0^R r^3\,dr}_{=R^4/4}$$

$$=\frac{\rho_0}{M}\cdot\frac{\pi}{4}\cdot\frac{R^4}{4}=\frac{\rho_0/16\,\pi R^4}{\rho_0/6\,\pi R^3}=\boxed{\frac{3}{8}R}$$

(*) Recordem que $\cos^2\varphi=\frac{1+\cos 2\varphi}{2}$, per tant:

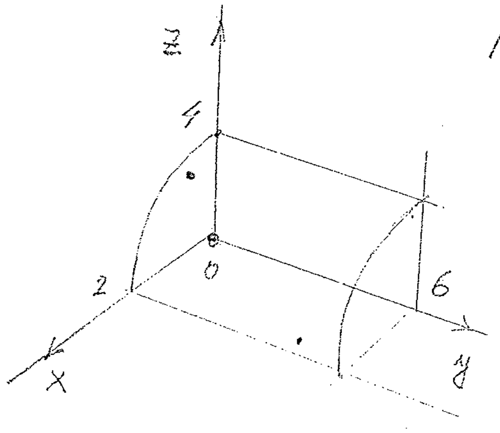
$$\int_0^{\pi/2}\cos^2\varphi\,d\varphi=\int_0^{\pi/2}\frac{1+\cos(2\varphi)}{2}\,d\varphi=\left[\frac{\varphi}{2}+\frac{\sin(2\varphi)}{4}\right]_0^{\pi/2}=\frac{\pi}{4}$$

$$Y=\frac{1}{M}\iiint_A y\,\rho(x,y,z)\,dx\,dy\,dz=\frac{\rho_0}{M}\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{\pi/2}\cos^2\varphi\,d\varphi\int_0^R r^3\,dr=\dots=\boxed{\frac{3}{8}R}$$

$$Z=\frac{1}{M}\iiint_A z\,\rho(x,y,z)\,dx\,dy\,dz=\frac{\rho_0}{M}\underbrace{\int_0^{\pi/2}d\theta}_{=\pi/2}\underbrace{\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi}_{=\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2}}\underbrace{\int_0^R r^3\,dr}_{=R^4/4}=\frac{\rho_0\frac{\pi}{16}R^4}{\rho_0\pi R^4}=\boxed{\frac{3}{8}R}$$

$\therefore$ Posició del CDM: $(X,Y,Z)=\left(\frac{3}{8}R,\frac{3}{8}R,\frac{3}{8}R\right)$

(b) $A=\{(x,y,z)\in\mathbb{R}^3 : 0\le x\le 2,\ 0\le y\le 6,\ 0\le z\le 4-x^2\}$, $\rho(x,y,z)=\rho_0=\text{ctnt}$.



$$M=\iiint_\Omega \underbrace{\rho(x,y,z)}_{=\rho_0}\,dx\,dy\,dz=\rho_0\int_0^2 dx\int_0^6 dy\int_0^{4-x^2}dz=6\rho_0\int_0^2(4-x^2)\,dx=$$

$$=6\rho_0\left(4x-\frac{x^3}{3}\right)\Big|_0^2=6\rho_0\left(8-\frac{8}{3}\right)=32\rho_0$$

$$X=\frac{1}{M}\iiint_\Omega x\underbrace{\rho(x,y,z)}_{=\rho_0}\,dx\,dy\,dz=\frac{1}{M}\int_0^2 x\,dx\int_0^6 dy\int_0^{4-x^2}dz$$

$$=\frac{6\rho_0}{M}\int_0^2 x(4-x^2)\,dx=\frac{6\rho_0}{M}\left(2x^2-\frac{x^4}{4}\right)\Big|_0^2=\frac{6\rho_0}{M}(8-4)=\frac{24\rho_0}{32\rho_0}=\boxed{\frac{3}{4}}$$

$$Y = \frac{1}{M}\iiint_{\Omega} y\,\rho(x,y,z)\,dx\,dy\,dz = \frac{\rho_0}{M}\int_0^2 dx \int_0^6 y\,dy \int_0^{4-x^2} dz = \frac{\rho_0}{M}\left[\frac{y^2}{2}\right]_0^6 \cdot \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{\rho_0}{M}\,18 \cdot \frac{16}{3} = \frac{6 \cdot 16 \cdot \rho_0}{32 \cdot \rho_0} = \boxed{3}$$

$$Z = \frac{1}{M}\iiint_{M} z\,\rho(x,y,z)\,dx\,dy\,dz = \frac{\rho_0}{M}\int_0^2 dx \int_0^6 dy \int_0^{4-x^2} z\,dz = \frac{6\rho_0}{M}\int_0^2 \frac{(4-x^2)^2}{2}\,dx$$

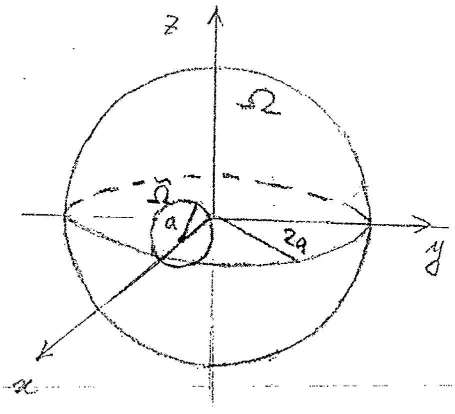
$$= \frac{3\rho_0}{M}\int_0^2 (16 - 8x^2 + x^4)\,dx = \frac{3\rho_0}{M}\left(16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right)\Bigg|_0^2 = \frac{3\rho_0}{M}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \frac{3\rho_0}{M}\cdot\frac{480-320+96}{15}$$

$$= \frac{3\rho_0}{32\rho_0}\cdot\frac{256}{15} = \boxed{\frac{8}{5}}$$

Posició del CDM: $(X,Y,Z) = \left(\frac{3}{4}, 3, \frac{8}{5}\right)$. □

(c) $A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 4a^2,\ (x-a)^2+y^2+z^2 \geq a^2\}$, $\rho(x,y,z) = \rho_0 = \text{ctnt.}$

Solució. Tenim: $\tilde{\Omega} = B_a(a,0,0)$, $\Omega = B_{2a}(0,0,0) \setminus \tilde{\Omega}$, amb:



$$m(\tilde{\Omega}) = \frac{4}{3}\pi a^3 \rho_0$$

$$m(\Omega) = m\left(B_{2a}(0,0,0)\right) - m(\tilde{\Omega}) = \frac{4}{3}\pi\, 8a^3\rho_0 - \frac{4}{3}\pi a^3 \rho_0$$

$$= \frac{4}{3}\pi\, 7a^3 \rho_0.$$

$$m(\Omega)X + m(\tilde{\Omega})\underbrace{\tilde{X}}_{=a} = \frac{4}{3}\pi\rho_0 7a^3 X + \frac{4}{3}\pi\rho_0 a^4 = 0 \Rightarrow X = -\frac{\frac{4}{3}\pi\rho_0 a^4}{\frac{4}{3}\pi\rho_0 7a^3} = -\frac{a}{7}$$

$$m(\Omega)Y + m(\tilde{\Omega})\underbrace{\tilde{Y}}_{=0} = 0 \Rightarrow Y = 0.$$

$$m(\Omega)Z + m(\tilde{\Omega})\tilde{Z} = 0 \Rightarrow Z = 0.$$

Posició del CDM: $(X,Y,Z) = (-a/7, 0, 0)$. □

28) Trobeu el centre de masses de la semiesfera definida per $x^2+y^2+z^2 \le R^2$ i $z \geq 0$, si la densitat en cada punt és proporcional a la distància d'aquest punt al centre.

Solució.

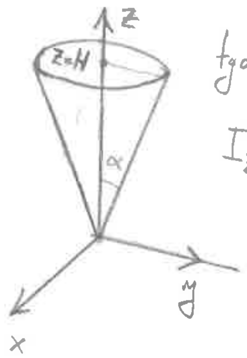
$$x^2+y^2+z^2 \le R^2,$$
$$z \geq 0$$
$$\rho(x,y,z) = \sqrt{x^2+y^2+z^2} =: r$$
$$M = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \int_0^R r^3 dr = 2\pi \frac{R^4}{4} = \boxed{\frac{\pi R^4}{2}}$$
$$X = \frac{1}{M} \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{0} \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \int_0^R r^4 dr = 0$$
$$Y = \frac{1}{M} \underbrace{\int_0^{2\pi} \sin\theta \, d\theta}_{0} \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \int_0^R r^4 dr = 0$$
$$Z = \frac{1}{M} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi \, d\varphi \int_0^R r^4 dr = \frac{\pi R^5/5}{\pi R^4/2} = \boxed{\frac{2}{5}R}$$

Posició del CDM: $(X,Y,Z) = (0,0,2R/5)$. □

34) Pels sòlids següents, calculeu els moments d'inèrcia que es demanen en cada cas tot suposant densitat homogènia igual a 1.

(a) Calculeu I_z pel sòlid limitat pel con de revolució d'alçada H i radi de base R donat per $x^2+y^2 \le R^2/H^2 \, z^2$ $(0 \le z \le H)$.

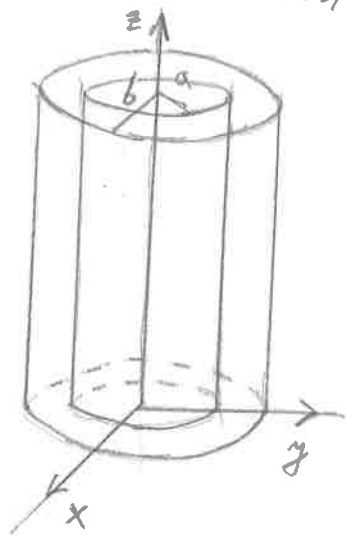
Solució.



$tg\alpha = \frac{R^2}{H^2}$, **ρ=1 dut**

$$I_z = \int_0^{2\pi} d\theta \int_0^R r^3 dr \underbrace{\int_{\frac{H}{R}r}^{H} \rho \, dz}_{H-\frac{H}{R}r} \overset{\rho=1}{=} 2\pi \int_0^R \left(Hr^3 - \frac{H}{R} r^4\right) dr = 2\pi H R^4 \left(\frac{1}{4}-\frac{1}{5}\right)$$
$$= \boxed{\frac{\pi H R^4}{10}} \quad □$$

(b) Calculeu I_z pel sòlid limitat per dos cilindres d'alçada h, $a^2 \le x^2+y^2 \le b^2$, $0 \le z \le h$

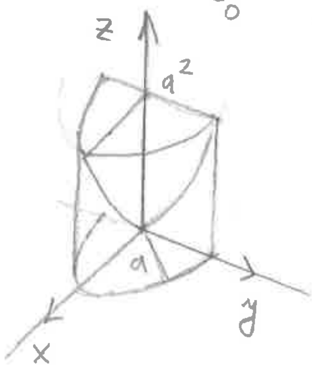


Solució.

$$I_z = \int_0^{2\pi} d\theta \int_a^b r^3 dr \int_0^h \underset{1}{\underset{\shortparallel}{\rho}}\, dz = 2\pi h \left[\frac{r^4}{4}\right]_a^b = \frac{\pi h}{2}(b^4 - a^4)$$

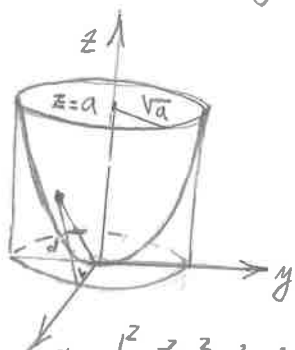
(c) Calculeu I_z pel sòlid limitat pel paraboloide $z = x^2+y^2$, el cilindre $x^2+y^2 = a^2$ $(z \ge 0)$

$$I_z = \int_0^{2\pi} d\theta \int_0^a r^3 dr \int_0^{r^2} \rho = 1\, dz = 2\pi \int_0^a r^5 dr = 2\pi \left(\frac{r^6}{6}\right)\Big|_0^a = \boxed{\frac{\pi a^6}{3}}$$



(d) Calculeu I_x, I_y, I_z pel sòlid tancat pel paraboloide $z = x^2+y^2$ i el pla $z = a$ $(a > 0)$

d_x: distància a l'eix x.



$$I_x = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r dr \int_{r^2}^{a} \rho d_x^2 dz$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r dr \int_{r^2}^{a} \overset{1}{\overset{\shortparallel}{\rho}} (z^2 + r^2 \sin^2\theta) dz$$

$$= \underbrace{\int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r dr \int_{r^2}^{a} z^2 dz}_{=: I_1} + \underbrace{\int_0^{2\pi} \sin^2\theta d\theta \int_0^{\sqrt{a}} r^3 dr \int_{r^2}^{a} dz}_{=: I_2}$$

$$= I_1 + I_2$$

$$d_x^2 = z^2 + r^2 - r^2\cos^2\theta = z^2 + r^2(1-\cos^2\theta) = z^2 + r^2\sin^2\theta.$$

$$I_1 := \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r dr \int_{r^2}^{a} z^2 dz = \frac{2\pi}{3} \int_0^{\sqrt{a}} r(a^3 - r^6) dr = \frac{2\pi}{3}\left(\frac{a^4}{2} - \frac{a^4}{8}\right) = \frac{\pi a^4}{4}.$$

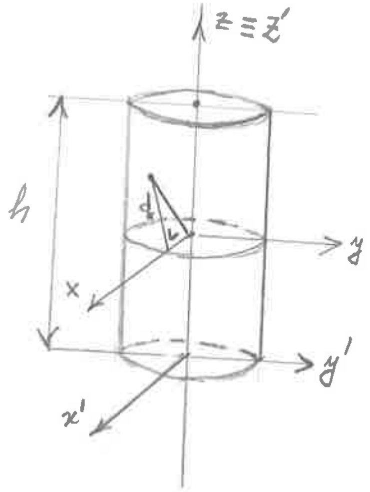
$$I_2 := \underbrace{\int_0^{2\pi} \sin^2\theta d\theta}_{\pi} \int_0^{\sqrt{a}} r^3 dr \int_{r^2}^{a} dz = \pi \int_0^{\sqrt{a}} r^3 (a - r^2) dr = \pi a^3 \left(\frac{1}{4} - \frac{1}{6}\right) = \frac{\pi a^3}{12}.$$

$$\therefore\ I_x = I_1 + I_2 = \frac{\pi a^3}{12}(1 + 3a). \ \square$$

$$I_y = I_x = \frac{\pi a^3}{12}(1+3a) \quad (\text{per simetria}). \ \square$$

$$I_z = \int_0^{2\pi} d\theta \int_0^{\sqrt{a}} r^3 dr \int_{r^2}^{a} \underbrace{\rho}_{=1} dz = 2\pi \int_0^{\sqrt{a}} r^3 (a-r^2) dr = 2\pi \left(a\frac{r^4}{4} - \frac{r^6}{6} \right)\Bigg|_0^{\sqrt{a}} = \boxed{\frac{\pi a^3}{6}}. \ \square$$

(e) Calculeu I_x, I_y, I_z pel cilindre $x^2+y^2 \leq R^2$, $-\frac{h}{2} \leq z \leq \frac{h}{2}$



$$I_x = \int_0^{2\pi} d\theta \int_0^R r\,dr \int_{-h/2}^{h/2} \rho d_x^2 \, dz$$

d_x: distància a l'eix x:
$d_x = z^2 + r^2 - r^2\cos^2\theta$
$= z^2 + r^2(1-\cos^2\theta)$
$= z^2 + r^2 \sin^2\theta.$

$$= \int_0^{2\pi} d\theta \int_0^R r\,dr \int_{-h/2}^{h/2} \underbrace{\rho}_{=1} (z^2 + r^2\sin^2\theta)\, dz$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{=2\pi} \underbrace{\int_0^R r\,dr}_{=R^2/2} \underbrace{\int_{-h/2}^{h/2} z^2 dz}_{=h^3/12} + \underbrace{\int_0^{2\pi} \sin^2\theta\, d\theta}_{=\pi} \underbrace{\int_0^R r^3 dr}_{=\frac{R^4}{4}} \underbrace{\int_{-h/2}^{h/2} dz}_{=h} = 2\pi \frac{R^2}{2}\frac{h^3}{12} + \pi\frac{R^4}{4}h = \frac{\pi R^2 h}{12}(h^2+3R^2). \ \square$$

$$I_y = I_x = \frac{\pi R^2 h}{12}(h^2+3R^2) \quad (\text{per simetria})$$

Teorema de Steiner $I_r = I_s + m(W)\, d^2(CM, r)$

Nota: $I_{x'} = I_{y'} = \underbrace{\frac{\pi R^2 h}{12}(h^2+3R^2)}_{\text{Moment d'inèrcia respecte de l'eix } x\text{, que passa pel CDM}} + \underbrace{\pi R^2 h}_{\text{Massa del cos } (\rho=1)} \cdot \underbrace{\frac{h^2}{4}}_{d^2(x,x')} = \pi R^2 h \left(\frac{h^2}{12} + \frac{h^2}{4} + 3R^2 \right) = \frac{\pi R^2 h}{12}(4h^2+3R^2). \ \square$

$$I_z = \int_0^{2\pi} d\theta \int_0^R r^3 dr \int_{-h/2}^{h/2} dz = 2\pi \frac{R^4}{4} h = \frac{\pi R^2 h}{2} h^2. \ \square$$

Nota: posant $M := \pi R^2 h$ (la "massa" del cos):

$$I_x = \frac{M}{12}(h^2+3R^2) = I_y, \quad I_{x'} = \frac{M}{12}(4h^2+3R^2) = I_{y'}, \quad I_z = \frac{M}{2}h^2. \ \square$$