(20) Calculeu per derivació directa les derivades parcials de les següents funcions

(20.1) $f(x,y) = x^{y^x}$

$f_x(x,y) = y^x x^{y^x-1} + y^x x^{y^x}(\ln x)(\ln y) = \underline{y^x x^{y^x}\left((\ln x)(\ln y) + \frac{1}{x}\right)}$.

$f_y(x,y) = \underline{x y^{x-1} x^{y^x} \ln x}$. $\square$

(20.2) $f(x,y) = x^{xy}$

$f_x(x,y) = xy\, x^{xy-1} + y\, x^{xy} \ln x = \underline{y\, x^{xy}(1+\ln x)}$.

$f_y(x,y) = x\, x^{xy} \ln x = \underline{x^{xy+1} \ln x}$. $\square$

(20.3) $f(x,y) = (x^2+y^2)^x$

$f_x(x,y) = 2x^2(x^2+y^2)^{x-1} + (x^2+y^2)^x \ln(x^2+y^2)$

$= \underline{(x^2+y^2)^x\left(\frac{2x^2}{x^2+y^2} + \ln(x^2+y^2)\right)}$.

$f_y(x,y) = x(x^2+y^2)^{x-1}2y = \underline{2xy(x^2+y^2)^{x-1}}$. $\square$

(20.4) $f(x,y) = xy\ln(x^2+y^2)$

$f_x(x,y) = \underline{y\ln(x^2+y^2) + \frac{2x^2y}{x^2+y^2}}$.

$f_y(x,y) = \underline{x\ln(x^2+y^2) + \frac{2xy^2}{x^2+y^2}}$. $\square$

(20.5) $f(x,y) = \ln\sqrt{x^2+y^2}\ \left(= \ln(x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2)\right)$

$f_x(x,y) = \underline{\frac{x}{x^2+y^2}}$, $f_y(x,y) = \underline{\frac{y}{x^2+y^2}}$. $\square$

20.6 $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}} \left(= (x^2+y^2)^{-\frac{1}{2}} \right)$

$$f_x(x,y) = -\tfrac{1}{2}(x^2+y^2)^{-3/2}\,2x = -\frac{x}{(x^2+y^2)^{3/2}}.$$

$$f_y(x,y) = -\tfrac{1}{2}(x^2+y^2)^{-3/2}\,2y = -\frac{y}{(x^2+y^2)^{3/2}}. \quad \square$$

20.7 $f(x,y) = \dfrac{xy}{x^2+y^2}$

$$f_x(x,y) = \frac{(x^2+y^2)y - 2x^2y}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}.$$

$$f_y(x,y) = \frac{(x^2+y^2)x - 2xy^2}{(x^2+y^2)^2} = \frac{x^3 - xy^2}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}. \quad \square$$

20.8 $f(x,y) = \dfrac{x^2+y^2}{x^2-y^2}$

$$f_x(x,y) = \frac{(x^2-y^2)2x - (x^2+y^2)2x}{(x^2-y^2)^2} = \frac{2x(\cancel{x^2}-y^2-\cancel{x^2}-y^2)}{(x^2-y^2)^2} = -\frac{4xy^2}{(x^2-y^2)^2}$$

$$f_y(x,y) = \frac{(x^2-y^2)2y + (x^2+y^2)2y}{(x^2-y^2)^2} = \frac{2y(x^2\cancel{-y^2}+x^2\cancel{+y^2})}{(x^2-y^2)^2} = \frac{4x^2y}{(x^2-y^2)^2}. \quad \square$$

20.9 $f(x,y) = \dfrac{\cos x + e^{xy}}{x^2+y^2}$

$$f_x(x,y) = \frac{(x^2+y^2)(-\sin x + ye^{xy}) - (\cos x + e^{xy})\,2x}{(x^2+y^2)^2} = \frac{e^{xy}(-2x+x^2y+y^3) - (x^2+y^2)\sin x - 2x\cos x}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{(x^2+y^2)(xe^{xy}) - (\cos x + e^{xy})2y}{(x^2+y^2)^2} = \frac{e^{xy}(-2y+x^3+xy^2) - 2y\cos x}{(x^2+y^2)^2}. \quad \square$$

20.10 $f(x,y,z) = e^x \tan(y^2 z)$

$$\underline{\frac{\partial f}{\partial x} = e^x \tan(y^2 z)}, \quad \underline{\frac{\partial f}{\partial y} \overset{(*)}{=} 2yz\, e^x \sec^2(y^2 z)}, \quad \underline{\frac{\partial f}{\partial z} = y^2 e^x \sec^2(y^2 z)}. \square$$

(*) Nota: recordem que: $\sec \alpha = \frac{1}{\cos \alpha}$ i que $D(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$.

20.11 $f(x,y,z) = x \sinh\left(\frac{y}{z}\right)$

$$\underline{f_x(x,y,z) = \sinh\left(\frac{y}{z}\right)}, \quad \underline{f_y(x,y,z) = \frac{x}{z} \cosh\left(\frac{y}{z}\right)}, \quad \underline{f_z(x,y,z) = -\frac{xy}{z^2} \cosh\left(\frac{y}{z}\right)}. \square$$

16 Aplicant la definició calculeu (si existeixen) les derivades parcials primeres de les següents funcions en el $(0,0)$.

16.1 $f(x,y) = \frac{xy}{x^2+y^2}$ si $(x,y) \neq (0,0)$ i $f(0,0) = 0$.

$$f_x(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

Per tant: $f_x(0,0) = 0 = f_y(0,0)$ $\square$

16.2 $f(x,y) = x \ln(x^2+y^2)$ si $(x,y) \neq (0,0)$ i $f(0,0) = 0$

$$f_x(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{t \ln t^2}{t} = \lim_{t\to 0} \ln t^2 = -\infty \Rightarrow \nexists\, f_x(0,0).$$

$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0 \cdot \ln t^2 - 0}{t} = \lim_{t\to 0} \frac{0}{t^2} = 0. \square$$

16.3 $f(x,y) = (x^2+y^2)^x$ si $(x,y) \neq (0,0)$ i $f(0,0) = 1$

$$f_x(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{e^{t \ln t^2} - 1}{t} \overset{(*)}{=} -\infty \Rightarrow \nexists\, f_x(0,0)$$

(*) $$\lim_{t\to 0} \frac{D(e^{t\ln t^2} - 1)}{D(t)} = \lim_{t\to 0} \frac{(2\ln|t| + 2)\, e^{2t \ln|t|}}{1} = +\infty \Rightarrow \lim_{t\to 0} \frac{e^{t\ln t^2} - 1}{t} = -\infty$$

$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{(t^2)^0 - 1}{t} = \lim_{t\to 0} \frac{1-1}{t} = 0. \square$$

(21) Calculeu les derivades parcials primeres de les següents funcions i doneu la seva matriu jacobiana.

(21.1) $f(x,y) = (e^{xy}+y,\ y^2x)$

$$Jf(x,y) = \begin{pmatrix} ye^{xy} & xe^{xy}+1 \\ y^2 & 2xy \end{pmatrix}. \square$$

(21.2) $f(x,y) = (\cos(x+2y),\ ye^{x+y},\ \cosh(xy^2))$

$$Jf(x,y) = \begin{pmatrix} -\sin(x+2y) & -2\sin(x+2y) \\ ye^{x+y} & (1+y)e^{x+y} \\ y^2\sinh(xy^2) & 2xy\sinh(xy^2) \end{pmatrix}. \square$$

(21.3) $f(x,y,z) = (z\tan(x^2+y^2),\ xy\ln z)$

$$Jf(x,y,z) = \begin{pmatrix} 2xz\sec^2(x^2+y^2) & 2yz\sec^2(x^2+y^2) & \tan(x^2+y^2) \\ y\ln z & x\ln z & \frac{xy}{z} \end{pmatrix}. \square$$

(23) D'acord amb la llei dels gasos perfectes tenim la relació $PV = kT$, on P és la pressió, V el volum, T la temperatura i k una constant. Demostreu que

$$\frac{\partial T}{\partial P}\frac{\partial P}{\partial V}\frac{\partial V}{\partial T} = -1.$$

($\frac{\partial T}{\partial V}$ vol dir que aïllem T com a funció de (P,V) i derivem respecte P. Idem per a les altres).

Solució: $T = T(P,V) = \frac{1}{k}PV$; $\frac{\partial T}{\partial P} = \frac{V}{k}$; $P = P(T,V) = k\frac{T}{V}$; $\frac{\partial P}{\partial V} = -k\frac{T}{V^2}$; $V = V(T,P) = k\frac{T}{P}$; $\frac{\partial V}{\partial T} = \frac{k}{P}$. Aleshores $\frac{\partial T}{\partial P}\frac{\partial P}{\partial V}\frac{\partial V}{\partial T} = \frac{V}{k}\left(-k\frac{T}{V^2}\right)\left(\frac{k}{P}\right) = -k\frac{T}{PV} = -k\frac{T}{kT} = -1. \square$