TEMA 2. Integració de funcions de vàries variables.

3) Apliqueu el principi de Cavalieri per calcular els següents volums a partir de l'àrea de seccions amb plans paral·lels als plans coordenats (triades de forma adequada)

(a) Volum envoltat per l'el·lipsoide: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solució. Fixant $0<x<a$: $\frac{y^2}{b^2\left(1-\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{x^2}{a^2}\right)}=1$, d'on $S(x)=\pi bc\left(1-\frac{x^2}{a^2}\right)$

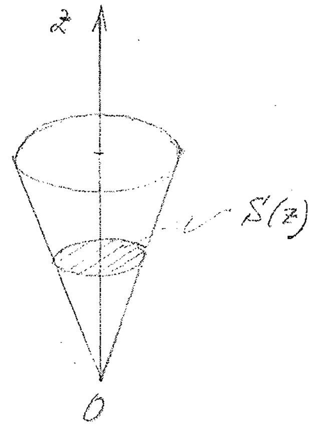
Nota: recordem que l'àrea d'una el·lipse ve donada per πAB on A i B són la longitud dels seus semieixos.

Llavors, aplicant el principi de Cavalieri: ...

$$V=2\int_0^a S(x)\,dx=2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx=2\pi bc\left(x-\frac{x^3}{3a^2}\right)\Bigg|_0^a=2\pi bc\left(a-\frac{a}{3}\right)$$

$$=\boxed{\frac{4}{3}\pi abc}$$

(b) Volum envoltat pel con de base el·líptica $x^2/a^2+\frac{y^2}{b^2}=z^2$, amb $0\leq z\leq h$



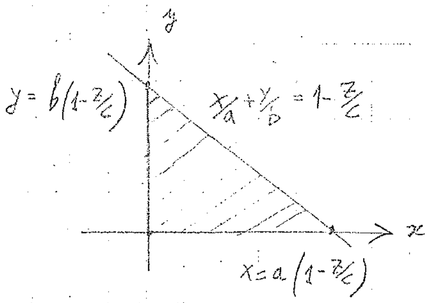
Solució. Fixant $0\leq z\leq h$: $\frac{x^2}{(az)^2}+\frac{y^2}{(bz)^2}=1$, d'on $S(z)=\pi abz^2$

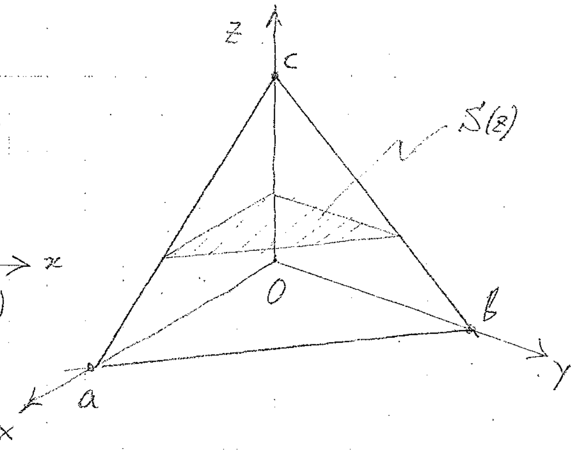
Alshores, aplicant el principi de Cavalieri:

$$V=\int_0^h S(z)\,dz=\int_0^h \pi abz^2\,dz=\pi ab\,\frac{z^3}{3}\Bigg|_0^h=\boxed{\frac{\pi}{3}abh^3}$$

Nota: $V=\frac{1}{3}\underbrace{\pi(ah)\cdot(bh)}_{\text{Àrea de la base}}\cdot h$

c) Volum de tetraedre limitat pels plans $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ $(a,b,c>0)$



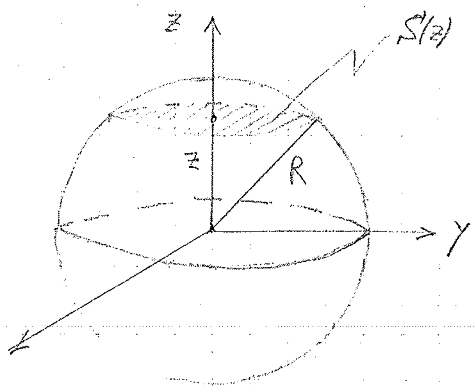


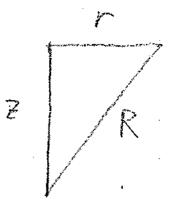
Solució. Fixant z: $S(z)=\frac{1}{2}ab\left(1-\frac{z}{c}\right)^2$,

d'on, aplicant el principi de Cavalieri:

$$V=\int_0^c S(z)\,dz=\frac{1}{2}ab\int_0^1\left(1-\frac{z}{c}\right)^2dz=-\frac{1}{6}abc\left(1-\frac{z}{c}\right)^3\Big|_0^c=\boxed{\frac{abc}{6}}$$

d) Volum envoltat pel casquet esfèric determinat per l'esfera $x^2+y^2+z^2=R^2$ i la condició $R-h\le z\le R$.





Solució.

$$S(z)=\pi r^2(z)=\pi\left(R^2-z^2\right)$$

D'on, aplicant el principi de Cavalieri:

$$V=\int_{R-h}^{R}S(z)\,dz=\pi\int_{R-h}^{R}\left(R^2-z^2\right)dz$$

$$=\pi\left(R^2z-\frac{z^3}{3}\right)\Big|_{R-h}^{R}=\pi\left(R^3-\cancel{\frac{R^3}{3}}-R^3+\cancel{R^2h}+\cancel{\frac{R^3}{3}}-\cancel{R^2h}+Rh^2-\frac{h^3}{3}\right)=\boxed{\frac{\pi h^2}{3}(3R-h)}$$

4) Generalitzeu el principi de Cavalieri al càlcul de volums en $\mathbb{R}^4$ i calculeu el volum de la bola 4-dimensional $B=\left\{(x,y,z,t)\in\mathbb{R}^4 : x^2+y^2+z^2+t^2\le R^2\right\}$

Solució: Fixem t: $0\le x^2+y^2+z^2\le R^2-t^2$, amb $-R\le t\le R$. Aleshores $V(t)=\frac{4}{3}\pi\left(R^2-t^2\right)^{3/2}$. D'on, aplicant el principi de Cavalieri: $M=\int_{-R}^{R}V(t)\,dt=\frac{4}{3}\pi\int_{-R}^{R}\left(R^2-t^2\right)^{3/2}dt=\frac{8}{3}\pi R^3\int_0^R\left(1-t^2/R^2\right)^{3/2}dt=\left\{\text{c.v. } t/R=\sin\theta\right\}=$

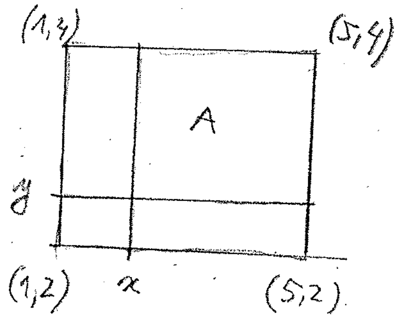
$$=\frac{8}{3}\pi R^4\int_0^{\pi/2}\cos^4\theta\,d\theta\overset{(*)}{=}\frac{8}{3}\pi R^4\int_0^{\pi/2}\left(\frac{1+\cos(2\theta)}{2}-\frac{1-\cos(4\theta)}{8}\right)d\theta=\frac{8}{3}\pi R^4\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\boxed{\frac{\pi^2R^4}{12}}$$

(*) Nota: $\cos^4\theta=\cos^2\theta\left(1-\sin^2\theta\right)=\cos^2\theta-\frac{1}{4}\left(2\sin\theta\cos\theta\right)^2=\cos^2\theta-\frac{1}{4}\sin^2(2\theta)=\frac{1+\cos(2\theta)}{2}-\frac{1-\cos(4\theta)}{8}$ fent servir les fórmules de l'angle doble.

8) Per a les regions $A \subset \mathbb{R}^2$ indicades, escriviu la integral doble $\iint_A f(x,y)\,dx\,dy$ en termes d'integrals iterades preses en diferents ordres $\int\left(\int f\,dx\right)dy$ i $\int\left(\int f\,dy\right)dx$, donats quins són els extrems d'integració per a x i y en cada cas.

a) A rectangle de vèrtexs $(1,2)$, $(5,2)$, $(5,4)$ i $(1,4)$.

Solució.

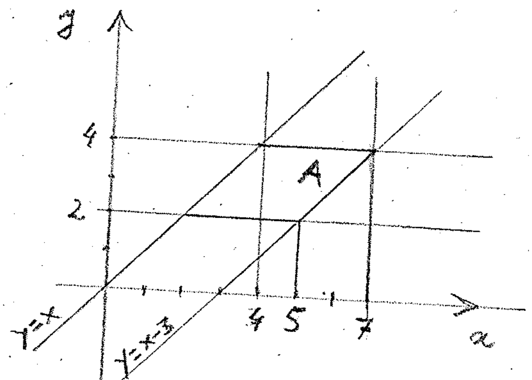


$$\iint_A f(x,y)\,dx\,dy = \int_1^5 dx \int_1^4 f(x,y)\,dy = \int_2^4 dy \int_1^5 f(x,y)\,dx$$

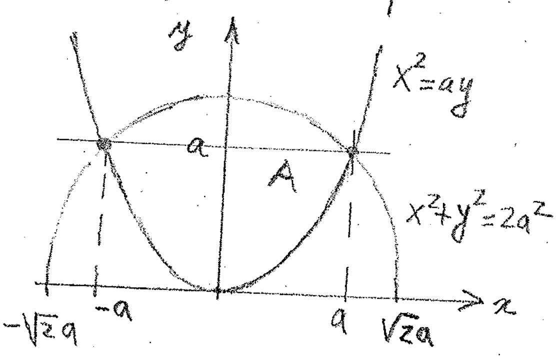
b) A paral·lelogram limitat per les rectes $y=x$, $y=x-3$, $y=2$, $y=4$.

$$I = \iint_A f(x,y)\,dx\,dy = \int_2^4 dx \int_2^x f(x,y)\,dy + \int_4^5 dx \int_2^4 f(x,y)\,dy$$

$$+ \int_5^7 dx \int_{x-3}^4 f(x,y)\,dy = \int_2^4 dy \int_y^{y+3} f(x,y)\,dx$$



c) A regió limitada per les corbes $x^2+y^2=2a^2$, $x^2=ay$ $(y\geq 0,\ a>0)$.



$$I = \iint_A f(x,y)\,dx\,dy = \int_{-a}^{a} dx \int_{\frac{x^2}{a}}^{\sqrt{2a^2-x^2}} f(x,y)\,dy$$

$$= \int_0^a dy \int_{-\sqrt{ay}}^{\sqrt{ay}} f(x,y)\,dx + \int_a^{\sqrt{2}a} dy \int_{-\sqrt{2a^2-y^2}}^{\sqrt{2a^2-y^2}} f(x,y)\,dx$$

$ay+y^2=2a^2 \Leftrightarrow y^2+ay-2a^2=0$

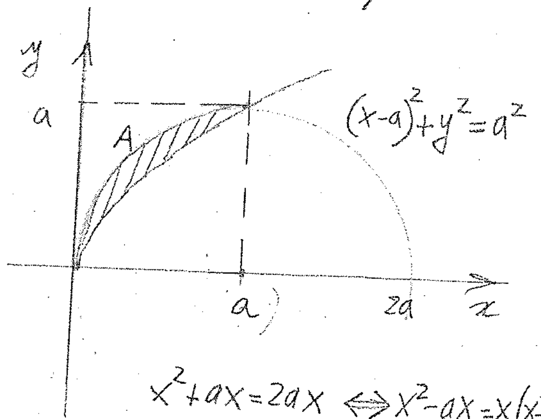
d'on: $y = \dfrac{-a \pm \sqrt{a^2+8a^2}}{2}$

$= \dfrac{-a\pm 3a}{2} = \begin{cases} a \\ -2a \text{ (No)} \end{cases}$

d) A regió limitada per les corbes $y^2 = ax$, $x^2+y^2 = 2ax$, $y=0$ $(y \geq 0, a>0)$

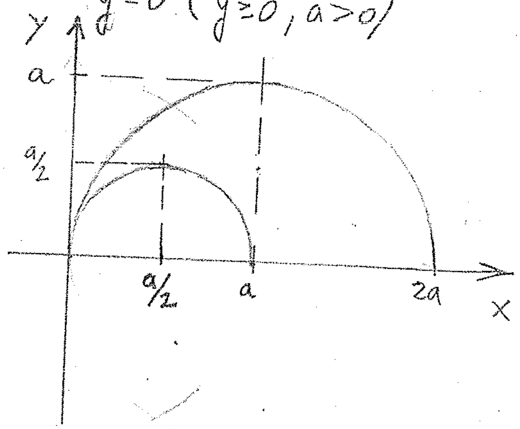
$$I = \iint_A f(x,y)\,dx\,dy = \int_0^a dx \int_{\sqrt{ax}}^{\sqrt{2ax-x^2}} f(x,y)\,dy$$

$$= \int_0^a dy \int_{a-\sqrt{a^2-y^2}}^{y^2/a} f(x,y)\,dx. \quad \square$$



$x^2+ax=2ax \Leftrightarrow x^2-ax=x(x-a)=0$
d'on $x=0$, $x=a$.

e) A regió limitada per les corbes $x^2+y^2=ax$, $x^2+y^2=2ax$, $y=0$ $(y \geq 0, a>0)$



$$I = \iint_A f(x,y)\,dx\,dy$$

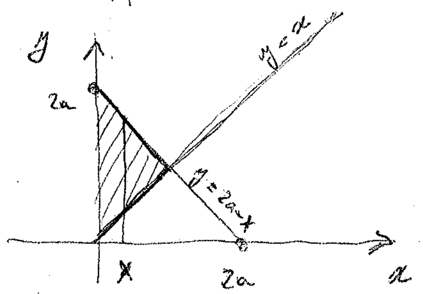
$$= \int_0^a dx \int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y)\,dy + \int_a^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy$$

$$= \int_0^{a/2} dy \int_{a-\sqrt{a^2-y^2}}^{a/2-\sqrt{a^2/4-y^2}} f(x,y)\,dx + \int_0^{a/2} dy \int_{a/2+\sqrt{a^2/4-y^2}}^{a} f(x,y)\,dx$$

$$+ \int_0^{a/2} dy \int_a^{a+\sqrt{a^2-y^2}} f(x,y)\,dx + \int_{a/2}^{a} dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx. \quad \square$$

12. Calculeu les següents integrals dobles en els dominis de $\mathbb{R}^2$ que s'indiquen.

(a) $\iint_A (x^2+y^2)\,dx\,dy$ A limitat per les rectes $y=x$, $x+y=2a$, $x=0$ $(a>0)$



$$\iint_A (x^2+y^2)\,dx\,dy = \int_0^a dx\int_x^{2a-x}(x^2+y^2)\,dy = \int_0^a dx\left(x^2y+\frac{y^3}{3}\right)\Big|_{y=x}^{y=2a-x}$$

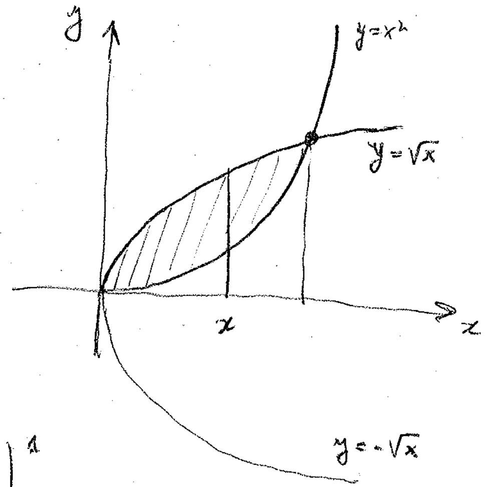
$$= \int_0^a dx\left(x^2(2a-x)+\frac13(2a-x)^3 - x^3 - \frac{x^3}{3}\right) \overset{(*)}{=} \int_0^a dx\left(\frac{8a^3}{3}-4a^2x+4ax^2-\frac83x^3\right)$$

$$=\left[\frac83 a^3x - 2a^2x^2+\frac43 ax^3 - \frac23 x^4\right]_0^a = \boxed{\frac{4a^4}{3}}\ \square$$

(*) $= 2ax^2 - x^3 + \frac83 a^3 - 4a^2x + 2ax^2 - \frac{x^3}{3} - x^3 - \frac{x^3}{3}$

$= \frac83 a^3 - 4a^2x+4ax^2-\frac83x^3$

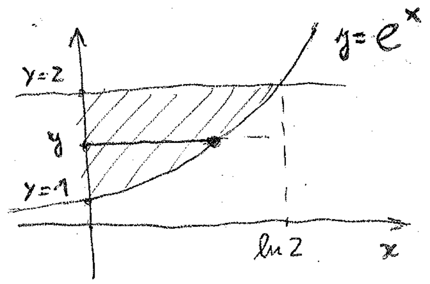
(b) $I=\iint_A (x+2y)\,dx\,dy$, A limitat per les corbes $y=x^2$, $y^2=x$



$$I=\iint_A(x+2y)\,dx\,dy = \int_0^1 dx\int_{x^2}^{\sqrt x}(x+2y)\,dy = \int_0^1 dx\,(xy+y^2)\Big|_{y=x^2}^{y=\sqrt x}$$

$$=\int_0^1 dx\,(x^{3/2}+x-x^3-x^4)\,dx = \frac25 x^{5/2}+\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^5}{5}\Big|_0^1$$

$$=\frac{8+10-5-4}{20}=\boxed{\frac{9}{20}}\ \square$$



c) $I=\iint_A e^{x+y}\,dx\,dy$ A limitat per les corbes: $y=e^x$, $x=0$, $y=2$

$$I=\int_1^2 e^y\,dy\int_0^{\ln y} e^x\,dx = \int_1^2 e^y\,dy\left[e^x\right]_0^{\ln y} = \int_1^2 e^y(y-1)\,dy$$

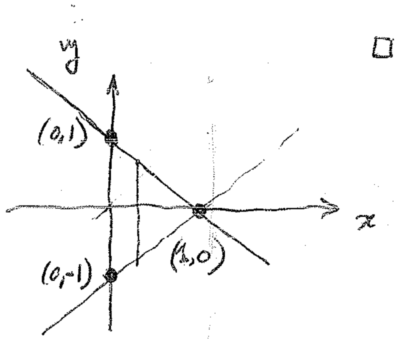
$$= ye^y - 2e^y\Big|_1^2 = \cancel{2e^2}-\cancel{2e^2} - e + 2e = \boxed{e}$$

$\parallel$

$e^y(y-2)$.

Alternativament: $\int_0^{\ln 2} e^x\,dx\int_{e^x}^2 e^y\,dy = \int_0^{\ln 2} dx\, e^x\left[e^y\right]_{e^x}^2 = \int_0^{\ln 2} e^x\left(e^2-e^{e^x}\right)dx =$

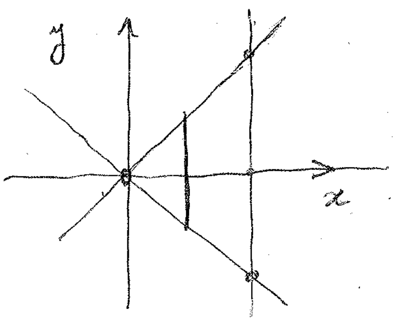
$$= e^{2+x} - e^{e^x}\Big|_0^{\ln 2} = -\cancel{e^2} + 2\cancel{e^2} + e - \cancel{e^2} = \boxed{e}$$ □



(d) $\iint_A e^y \, dx\,dy$, A triangle de vèrtexs: (1,0), (0,1), (0,-1).

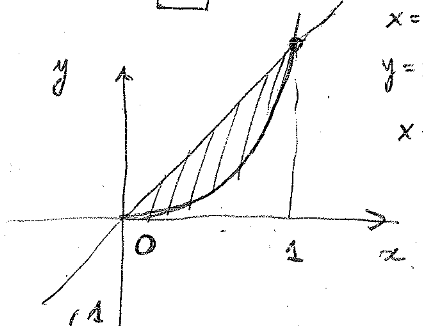
$$I = \iint_A e^y \, dx\,dy = \int_0^1 dx \int_{x-1}^{-x+1} e^y dx = \int_0^1 dx \left(e^{-x+1} - e^{x-1}\right) = -e^{-x+1} - e^{x-1}\Big|_0^1 = \boxed{e + \frac{1}{e} - 2}$$ □

e) $\iint xy^2 \, dx\,dy$, A limitat per les rectes $x=1$, $x=y$, $x+y=0$



$$I = \iint_A xy^2 \, dx = \int_0^1 x\,dx \int_{-x}^{x} y^2 dy = \int_0^1 dx \; x \left(\frac{x^3}{3}\right]_{y=-x}^{y=x}$$

$$= \frac{2}{3}\int_0^1 x^4 dx = \frac{2}{15}\left(x^5\Big|_0^1 = \boxed{\frac{2}{15}}\right.$$. □



$x = y$

$y = x^2$

$x = x^2 \Leftrightarrow x(x-1) = 0$

$x = 0$

$x = 1$

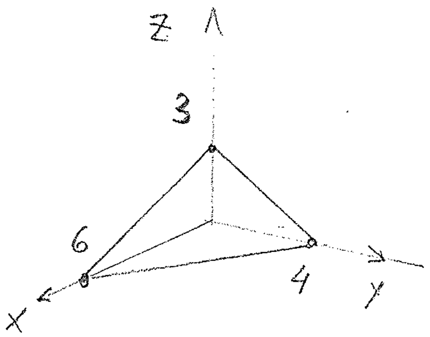
f) $\iint_A xy \, dx\,dy$, A limitat per les corbes

$$I = \iint_A xy \, dx\,dy = \int_0^1 x\,dx \int_{x^2}^{x} y\,dx = \int_0^1 dx \; x \left(\frac{y^2}{2}\Big|_{x^2}^{x}\right. = \frac{1}{2}\int_0^1 dx \left(x^3 - x^5\right) =$$

$$= \frac{1}{2}\left(\frac{}{4} - \frac{x^6}{6}\right)\Big|_0^1 = \frac{1}{2} \cdot \frac{3-2}{12} = \boxed{\frac{1}{24}}$$ □

14) Per les regions de $\mathbb{R}^3$ indicades escriviu la integral triple $\iiint_A f(x,y,z)\,dx\,dy\,dz =: I$ en termes d'integrals iterades preses en diferents ordres.

(a) A tetraedre limitat pels plans $x=0,\ y=0,\ z=0,\ 2x+3y+4z=12$.



$$I = \int_0^6 dx \int_0^{4-\frac{2}{3}x} dy \int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z)\,dz = \int_0^4 dy \int_0^{6-\frac{3}{2}y} dx \int_0^{3-\frac{1}{2}x-\frac{3}{4}y} f(x,y,z)\,dz$$

$$= \int_0^6 dx \int_0^{3-\frac{1}{2}x} dz \int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z)\,dy = \int_0^3 dz \int_0^{6-2z} dx \int_0^{4-\frac{2}{3}x-\frac{4}{3}z} f(x,y,z)\,dy$$

$$= \int_0^4 dy \int_0^{3-\frac{3}{4}y} dz \int_0^{6-\frac{3}{2}y-2z} f(x,y,z)\,dx = \int_0^3 dz \int_0^{4-\frac{4}{3}z} dy \int_0^{6-\frac{3}{2}y-2z} f(x,y,z)\,dx. \quad \square$$

- $4z = 12-2x-3y \to z = 3-\frac{1}{4}x-\frac{3}{4}y$

  $z=0,\ 2x+3y=12 \to y = 4-\frac{2}{3}x$

  $x = 6-\frac{3}{2}y$

- $3y = 12-2x-4z \to y = 4-\frac{2}{3}x-\frac{4}{3}z$

  $y=0,\ 2x+4z=12 \to z = 3-\frac{1}{2}x$

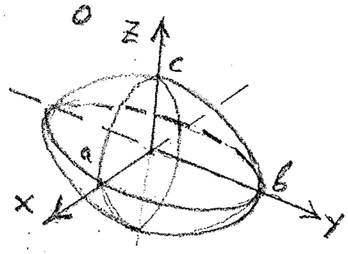
  $x = 6-2z$

- $2x = 12-3y-4z \to x = 6-\frac{3}{2}y-2z$

  $x=0:\ 3y+4z=12 \to z = 3-\frac{3}{4}y$

  $y = 4-\frac{4}{3}z$

(b) A interior del el·lipsoide



$$I = \int_{-a}^{a} dx \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} dy \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z)\,dz$$

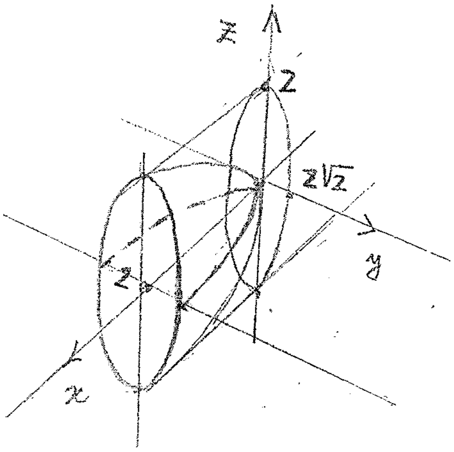
$$= \int_{-b}^{b} dy \int_{-a\sqrt{1-y^2/b^2}}^{a\sqrt{1-y^2/b^2}} dx \int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} f(x,y,z)\,dz$$

$$= \int_{-a}^{a} dx \int_{-c\sqrt{1-x^2/a^2}}^{c\sqrt{1-x^2/a^2}} dz \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z)\,dy = \int_{-c}^{c} dz \int_{-a\sqrt{1-z^2/c^2}}^{a\sqrt{1-z^2/c^2}} dx \int_{-b\sqrt{1-x^2/a^2-z^2/c^2}}^{b\sqrt{1-x^2/a^2-z^2/c^2}} f(x,y,z)\,dy$$

$$= \int_{-b}^{b} dy \int_{-c\sqrt{1-y^2/b^2}}^{c\sqrt{1-y^2/b^2}} dz \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z)\,dx = \int_{-c}^{c} dz \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} dy \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} f(x,y,z)\,dx. \quad \square$$

$$\frac{y^2}{4}+\frac{z^2}{2}=x$$

(c) A cos limitat per les superficies $y^2+2z^2=4x$, $x=2$.



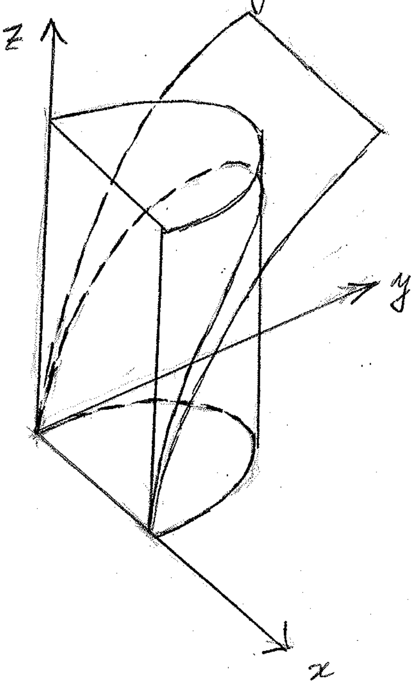
$$I=\int_{-2}^{2}dz\int_{-\sqrt{8-2z^2}}^{\sqrt{8+2z^2}}dy\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2}f(x,y,z)\,dx=\int_{-2\sqrt{2}}^{2\sqrt{2}}dy\int_{-\sqrt{4-y^2/2}}^{\sqrt{4-y^2/2}}dz\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2}f(x,y,z)\,dx$$

$$=\int_{0}^{2}dx\int_{-\sqrt{2x}}^{\sqrt{2x}}dz\int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}}f(x,y,z)\,dy=\int_{-2}^{2}dz\int_{z^2/2}^{2}dx\int_{-\sqrt{4x-2z^2}}^{\sqrt{4x-2z^2}}f(x,y,z)\,dy$$

$$=\int_{-2\sqrt{2}}^{2\sqrt{2}}dy\int_{\frac{1}{4}y^2}^{2}dx\int_{-\sqrt{2x-\frac{1}{2}y^2}}^{\sqrt{2x-\frac{1}{2}y^2}}f(x,y,z)\,dz=\int_{0}^{2}dx\int_{-2\sqrt{x}}^{2\sqrt{x}}dy\int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}}f(x,y,z)\,dz.\ \square$$

15) Calculeu les integrals triples següents en les regions de $\mathbb{R}^3$ que s'indiquen

a) $I=\iiint_A xz\,dx\,dy\,dz$, A limitat pel cilindre de base circular $x^2+y^2-2x=0$ i la superfície: $z^2=2y$ $(y,z\geq 0)$



$$I=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}}dy\int_0^{\sqrt{2y}}z\,dz$$

$$=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}}y\,dy$$

$$=\int_0^2 x\left(\frac{y^2}{2}\right]_0^{\sqrt{2x-x^2}}dx=\int_0^2\left(x^2-\frac{x^3}{2}\right)dx$$

$$=\left[\frac{x^3}{3}-\frac{x^4}{8}\right]_0^2=\frac{8}{3}-\frac{16}{8}=\frac{8}{3}-2=\boxed{\frac{2}{3}}\ \square$$

b) $I=\iiint_A zy\sqrt{x^2+y^2}\,dx\,dy\,dz$, $A=\left\{(x,y,z)\in\mathbb{R}^3 : 0\leq z\leq x^2+y^2,\ 0\leq y\leq\sqrt{2x-x^2}\right\}$

$$I=\int_0^2 dx\int_0^{\sqrt{2x-x^2}}y\sqrt{x^2+y^2}\,dy\int_0^{x^2+y^2}z\,dz=\int_0^2 dx\int_0^{\sqrt{2x-x^2}}y\sqrt{x^2+y^2}\left[\frac{z^2}{2}\right]_0^{x^2+y^2}dy$$

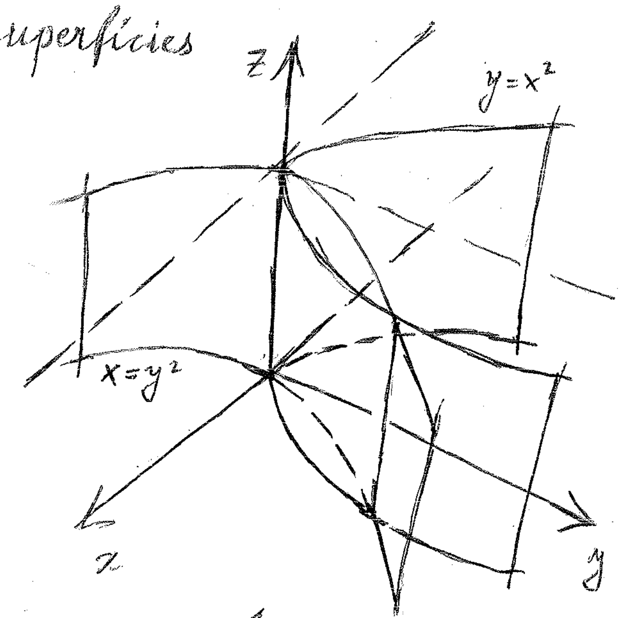
$$=\frac{1}{2}\int_0^2 dx\int_0^{\sqrt{2x-x^2}}y\left(x^2+y^2\right)^{5/2}dy=\frac{1}{14}\int_0^2 dx\left[\left(x^2+y^2\right)^{7/2}\right]_0^{\sqrt{2x-x^2}}=\frac{1}{14}\int_0^2 dx\left[\left(x^2+2x-x^2\right)^{7/2}-x^7\right]$$

$$=\frac{1}{14}\left[2\cdot 2^{7/2}\frac{x^{9/2}}{9}-\frac{x^8}{8}\right]_0^2=\frac{1}{14}\cdot\frac{1}{72}\cdot\left(16\cdot 2^8-9\cdot 2^8\right)=\frac{1}{14}\cdot\frac{1}{2^3\cdot 9}\cdot 7\cdot 2^8=\boxed{\frac{16}{9}}\ \square$$

c) $I=\iiint_A dx\,dy\,dz$, $A=\left\{(x,y,z)\in\mathbb{R}^3 : 1\leq x\leq 3,\ 1\leq y\leq 3,\ 0\leq z\leq xy\right\}$.

$$I=\int_1^3 dx\int_1^3 dy\int_0^{xy}dz=\int_1^3 x\,dx\int_1^3 y\,dy=\left(\int_1^3 x\,dx\right)^2=\left(\left[\frac{x^2}{2}\right]_1^3\right)^2=\left(\frac{9}{2}-\frac{1}{2}\right)^2=\boxed{16}\ \square$$

(d) $I = \iiint_A xyz\,dx\,dy\,dz$, A limitat per les superfícies



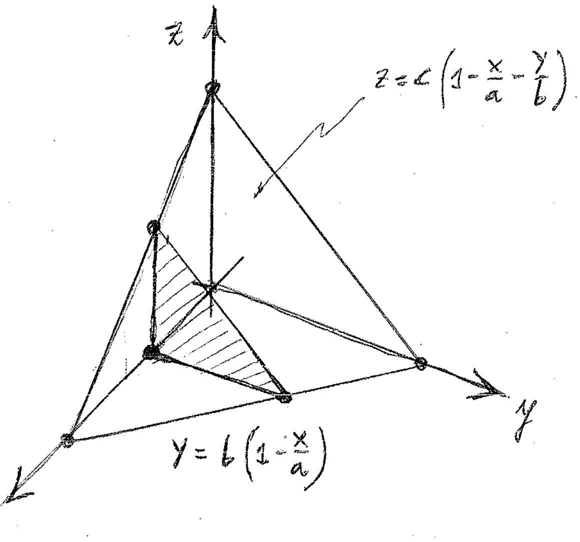
$y = x^2,\ x = y^2,\ z = xy,\ z = 0$

$$I = \int_0^1 x\,dx \int_{x^2}^{\sqrt{x}} y\,dy \int_0^{xy} z\,dz$$

$$= \frac{1}{2}\int_0^1 x^3\,dx \int_{x^2}^{\sqrt{x}} y^3\,dy$$

$$= \frac{1}{8}\int_0^1 dx\, x^3\left(x^2 - x^8\right) = \frac{1}{8}\int_0^1 \left(x^5 - x^{11}\right)dx = \frac{1}{8}\left(\frac{x^6}{6} - \frac{x^{12}}{12}\right)\Bigg|_0^1 = \frac{1}{8}\left(\frac{1}{6} - \frac{1}{12}\right) = \boxed{\frac{1}{96}} \quad \square$$

(e) $I = \iiint_A x\,dx\,dy\,dz$, A tetraedre format pels plans $x=0,\ y=0,\ z=0,\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, amb $a, b, c > 0$



$$I = \int_a^b x\,dx \int_0^{b(1-\frac{x}{a})} dy \int_0^{c(1-x/a-y/b)} dz$$

$$= c\int_0^a x\,dx \int_0^{b(1-x/a)} dy\left(1 - x/a - \frac{y}{b}\right)$$

$$= c\int_0^a dx\, x\left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)\Bigg|_0^{b(1-\frac{x}{a})}$$

$$= c\int_0^a x\left[b\left(1-\frac{x}{a}\right) - \frac{bx}{a}\left(1-\frac{x}{a}\right) - \frac{b}{2}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$\overset{(*)}{=} \frac{cb}{2}\int_0^a x\left(1-\frac{x}{a}\right)^2 dx = \frac{cb}{2}\int_0^a x\left(1 - \frac{2x}{a} - \frac{x^2}{a^2}\right)dx = \frac{cb}{2}\left(\frac{x^2}{2} - \frac{2x^3}{3a} + \frac{x^4}{4a^2}\right)\Bigg|_0^a$$

$$= \frac{cb}{2}\left(\frac{a^2}{2} - \frac{2a^2}{3} + \frac{a^2}{4}\right) = \boxed{\frac{cba^2}{24}}$$

(*) $\left(1-\frac{x}{a}\right)\left(1 - \frac{x}{a} - \frac{1}{2} + \frac{x}{2a}\right) = \frac{b}{2}\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}\right) = \frac{b}{2}\left(1-\frac{x}{a}\right)^2$,

16) Useu coordenades polars per calcular les següents integrals dobles.

$$x = r\cos\theta \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$
$$y = r\sin\theta$$

$$\iint_{T(D)} f(x,y)\,dx\,dy = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

a) $I = \iint_A (x^2+y^2)\,dx\,dy$; $A = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 4\}$

$$I = \int_0^{2\pi} d\theta \int_0^2 r^3\,dr = 2\pi\left[\frac{r^4}{4}\right]_0^2 = \boxed{8\pi}.$$

b) $I = \iint_A \cos(x^2+y^2)\,dx\,dy$, $A = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le \frac{\pi}{2}\}$

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{\pi/2}} r\cos(r^2)\,dr = 2\pi\,\frac{1}{2}\left[\sin(r^2)\right]_0^{\sqrt{\pi/2}} = \boxed{\pi}.$$

c) $I = \iint_A \frac{(x+y)^2}{x^2+y^2+2}\,dx\,dy$, $A = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 1\}$

$$I = \int_0^{2\pi} d\theta \int_0^1 \frac{r^3}{r^2+2}\,dr = 2\pi\int_0^1\left(1 - \frac{r}{r^2+2}\right)dr = 2\pi\left[\frac{r^2}{2} - \ln(r^2+2)\right]_0^1$$

$$= 2\pi\left[\frac{1}{2} - \ln 3 + \ln 2\right] = \boxed{2\pi\left[\frac{1}{2} + \ln\left(\frac{2}{3}\right)\right]} \ \square$$

d) $I = \iint_A \frac{dx\,dy}{(1+x^2+y^2)^2\sqrt{x^2+y^2}}$, $A = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le R^2\}$. (Indicació: useu propietats elementals del sin i cos per veure que $\sin(\arctan(R)) = \frac{R}{\sqrt{1+R^2}}$ i $\cos(\arctan(R)) = \frac{1}{\sqrt{1+R^2}}$).

$$I = \int_0^{2\pi} d\theta \int_0^R \frac{r\,dr}{(1+r^2)^2\sqrt{r^2}} = 2\pi \int_0^R \frac{dr}{(1+r^2)^2} = \begin{cases} r = tg\,t \Rightarrow dr = \dfrac{dt}{\cos^2 t} \\ r=0 \Rightarrow t=0 \\ r=R \Rightarrow t = \arctan(R) \end{cases}$$

$$= 2\pi \int_0^{\arctan(R)} \cos^4 t \, \frac{dt}{\cos^2 t} = 2\pi \int_0^{\arctan(R)} \frac{1+\cos(2t)}{2} dt = 2\pi \left(\frac{t}{2} + \frac{\sin(2t)}{4} \right) \Bigg|_0^{\arctan(R)}$$

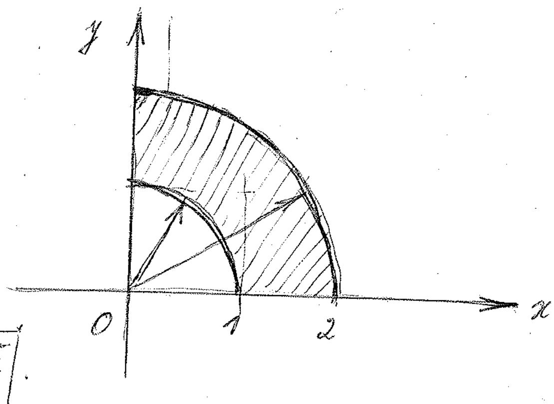
$$\overset{(*)}{=} \boxed{\pi \left(\arctan(R) + \frac{R}{1+R^2} \right)} \quad \square$$

(*) Notem que: $\sin(2t) = 2\sin t \cos t = \dfrac{2\sin t \cos^2 t}{\cos t} = 2\,tg\,t \cos^2 t = \dfrac{2\,tg\,t}{1+tg^2 t}$

e) $I = \iint_A \sqrt{x^2+y^2-9}\, dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 : 9 \leq x^2+y^2 \leq 25\}$,

$$I = \int_0^{2\pi} d\theta \int_3^5 \sqrt{r^2-9}\, r\,dr = \frac{2\pi}{3}(r^2-9)^{3/2} \Big|_3^5 = \frac{2\pi}{3} 4^3 = \boxed{\frac{128\pi}{3}} \quad \square$$

f) $I = \iint_A xy\,dx\,dy$; A intersecció amb el 1er quadrant de la corona circular de centre $(0,0)$, radi interior 1 i radi exterior 2

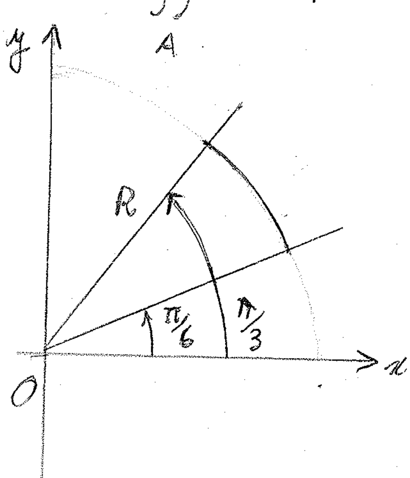


$$I = \int_0^{\frac{\pi}{2}} d\theta \int_1^2 r^3 \cos\theta \sin\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin(2\theta)\, d\theta \int_1^2 r^3 dr$$

$$= \left[-\frac{1}{2} \cos(2\theta) \right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^4}{4} \right]_1^2 = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \boxed{\frac{15}{8}} \quad \square$$

g) $I = \iint_A x(x^2+y^2)\,dx\,dy$, A sector circular de centre $(0,0)$ i radi R format per angles entre $\frac{\pi}{3}$ i $\frac{\pi}{6}$ amb l'eix x positiu.



$$I = \int_{\pi/6}^{\pi/3} d\theta \int_0^R r^4 \cos\theta \, dr = \int_{\pi/6}^{\frac{\pi}{3}} \cos\theta\, d\theta \int_0^R r^4 dr$$

$$= \left[\sin\theta \right]_{\pi/6}^{\pi/3} \cdot \left[\frac{r^5}{5} \right]_0^R = \boxed{\frac{R^5}{10}(\sqrt{3}-1)}$$

20.) Useu coordenades esfèriques per calcular les següents integrals triples

(a) $I=\iiint_B x^4y^2z^3\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le a^2\}=B_a(0,0,0)$ $(a>0)$

Solució: $$I=\int_0^{2\pi}\cos^4\theta\sin^2\theta\,d\theta\,\overbrace{\int_{-\pi/2}^{\pi/2}\cos^2\varphi\sin^3\varphi\,d\varphi}^{=0}\int_0^a r^{11}\,dr=0.\ \square$$

(b) $I=\iiint_B z(x^2+y^2)\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le a^2,\ z\ge 0\}$

Solució: $$I=\int_0^{2\pi}d\theta\int_0^{\pi/2}\cos^3\varphi\sin\varphi\int_0^a r^5\,dr=2\pi\left[-\frac{\cos^4\varphi}{4}\right]_0^{\pi/2}\cdot\left[\frac{r^6}{6}\right]_0^a=\boxed{\frac{\pi a^6}{12}}\ \square$$

(c) $I=\iiint_B \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$, $B=\{(x,y,z)\in\mathbb{R}^3 : a^2\le x^2+y^2+z^2\le b^2\}$

Solució: $$I=\int_0^{2\pi}d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi\int_a^b r^2\frac{dr}{r^3}=4\pi\ln\left(\frac{b}{a}\right)$$

(d) $I=\iiint_B \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}$, B el domini de l'apartat anterior.

Solució: $$I=\int_0^{2\pi}d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi\int_a^b r^3e^{-r^2}dr=4\pi\int_a^b r^3e^{-r^2}dr\overset{(*)}{=}2\pi\left(e^{-a^2}(a^2+1)-e^{-b^2}(b^2+1)\right)\ \square$$

(*) Parts: $$\int_a^b r^3e^{-r^2}dr=-\frac{1}{2}\int_a^b r^2\,d(e^{-r^2})=\left[-\frac{r^2e^{-r^2}}{2}\right]_a^b+\underbrace{\int_a^b re^{-r^2}dr}_{\left[-\frac{1}{2}e^{-r^2}\right]_a^b}=$$

$$=\frac{a^2e^{-a^2}}{2}-\frac{b^2e^{-b^2}}{2}+\frac{e^{-a^2}}{2}-\frac{e^{-b^2}}{2}=\frac{1}{2}(a^2+1)e^{-a^2}-(b^2+1)\frac{e^{-b^2}}{2}$$