(48) Mitjançant l'ús de desenvolupaments de Taylor de funcions d'1 variable (coneguts a priori), calculeu els desenvolupaments de Taylor en l'origen fins a termes de grau 2 inclosos de les següents funcions.

(a) $f(x,y) = e^{xy} \ln(1+x+y) \overset{(1)}{=} \left(1 + xy + \frac{1}{2!}(xy)^2 + \dots\right)\left(x+y - \frac{1}{2}(x+y)^2 + \dots\right)$

$$= x+y - \frac{1}{2}(x+y)^2 + \dots = x+y - \frac{x^2}{2} - xy - \frac{y^2}{2} + R_2(x,y) \quad \square$$

(1) on fem ús dels desenvolupaments, en $t=0$:

$$e^t = 1 + t/1! + t^2/2! + \dots + t^p/p! + \dots$$

$$\ln(1+t) = t - t^2/2 + t^3/3 - t^4/4 + \dots + (-1)^p \frac{t^{p+1}}{p+1} + \dots$$

(b) $f(x,y) = e^x \cos y \overset{(2)}{=} \left(1 + x + x^2/2! + \dots\right)\left(1 - y^2/2! + \dots\right)$

$$= 1 - y^2/2! + x + x^2/2! + \dots = 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + R_2(x,y) \quad \square$$

(2) on fem ús dels desenvolupaments, en $t=0$ de e^t (veure nota 1) i de:

$$\cos t = 1 - t^2/2! + t^4/4! - t^6/6! + \dots + (-1)^p t^{2p}/(2p)! + \dots$$

(c) $f(x,y) = \dfrac{1}{1+x+y} \overset{(3)}{=} 1 - x - y + (x+y)^2 + \dots = 1 - x - y + x^2 + 2xy + y^2 + R_2(x,y) \quad \square$

(3) On fem ús del desenvolupament, en $t=0$, de:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots + (-1)^p t^p + \dots$$

(d) $f(x,y,z) \overset{(4)}{=} e^{x+y}\sqrt{1+x}\,\cos(x+y+z) = \left(1 + x+y + \frac{1}{2}(x+y)^2 + \dots\right) \times$

$$\times \left(1 + x/2 - \frac{x^2}{8} + \dots\right) \times \left(1 - \frac{1}{2}(x+y+z)^2 + \dots\right) =$$

$$= \left(1+x+y+\frac{1}{2}(x+y)^2+\dots\right)\left(1+\frac{x}{2}-\frac{x^2}{8}+\dots\right)\left(1-\frac{(x+y+z)^2}{2}+\dots\right)$$

$$\begin{aligned} = \Big(1 &+ \frac{x}{2} - \frac{x^2}{8} \\ &+ x + \frac{x^2}{2} \\ &+ y + \frac{1}{2}xy \\ &+ \frac{x^2}{2} + xy + \frac{y^2}{2} + \dots\Big)\cdot\left(1-\frac{1}{2}(x+y+z)^2+\dots\right) \end{aligned}$$

$$= \left(1+\frac{3}{2}x+y+\frac{7}{8}x^2+\frac{3}{2}xy+\frac{y^2}{2}+\dots\right)\left(1-\frac{1}{2}(x+y+z)^2+\dots\right)$$

$$= 1+\frac{3}{2}x+y+\left(\frac{7}{8}-\frac{1}{2}\right)x^2+\left(\frac{3}{2}-1\right)xy+\left(\frac{1}{2}-\frac{1}{2}\right)y^2-xz-yz-\frac{z^2}{2}+\dots$$

$$= 1+\frac{3}{2}x+y+\frac{3}{8}x^2+\frac{1}{2}xy-xz-yz-\frac{z^2}{2}+R_2(x,y,z) \quad \square$$

(4) On hem fet ús dels desenvolupaments, en $t=0$, de e^t, $\cos t$ (veure notes 1 i 2) i:

$$(1+t)^{\frac{1}{2}} = 1+\binom{\frac{1}{2}}{1}t+\binom{\frac{1}{2}}{2}t^2+\dots+\binom{\frac{1}{2}}{p}t^p+\dots = 1+\frac{1}{2}t-\frac{1}{8}t^2+\dots+(-1)^{p+1}\frac{(2p-3)!!}{2^p\,p!}+\dots$$

on: $\binom{\frac{1}{2}}{1} = 1$, $\binom{\frac{1}{2}}{2} = \dfrac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = \dfrac{-1}{2^2\,2!}$, $\binom{\frac{1}{2}}{3} = \dfrac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \dfrac{3\cdot 1}{2^3\,3!}$, ...

..., $\binom{\frac{1}{2}}{p} = (-1)^{p-1}\dfrac{(2p-3)!!}{2^p\,p!}$, per $p=2,3,\dots$

(49) Calculeu el desenvolupament de Taylor en l'origen de les següents funcions fins l'ordre que s'indica en cada cas, doneu el valor de totes les derivades parcials de la funció en el (0,0) corresponents a l'ordre màxim fins al qual s'ha desenvolupat (per ex., si desenvolupem fins a ordre 5 volem $\frac{\partial^5 f}{\partial x^n \partial y^m}(0,0)$ amb $n+m=5$).

Solució.

Primer, recordem que, per desenvolupaments de funcions de dues variables en (x_0, y_0):

$$f(x,y) = \sum_{k=0}^{p} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(x_0,y_0)(x-x_0)^j (y-y_0)^{k-j} + R_p(x-x_0, y-y_0)$$

$$= \sum_{k=0}^{p} \sum_{j=0}^{k} coef_{j,k-j} (x-x_0)^j (y-y_0)^{k-j} + R_p(x-x_0, y-y_0),$$

d'on tenim que les derivades d'ordre k es poden calcular amb la fórmula:

$$\frac{\partial^k f}{\partial x^j \partial y^{k-j}}(0,0) = j!(k-j)!\, coef_{j,k-j}, \text{ amb } j=0,1,2,\dots,k.$$

(a) $f(x,y) = \ln(1+x^2-y)$, fins a ordre 3.

$$f(x,y) = (x^2-y) - \frac{1}{2}(x^2-y)^2 + \frac{1}{3}(x^2-y)^3 + \dots$$

$$= -y + x^2 - \frac{y^2}{2} + x^2 y - \frac{1}{3}y^3 + R_3(x,y)$$

Llavors les derivades d'ordre 3 vénen donades per:

$$\frac{\partial^3 f}{\partial x^3}(0,0) = 0,\ \frac{\partial^3 f}{\partial x^2 \partial y}(0,0) = 2,\ \frac{\partial^3 f}{\partial x \partial y^2}(0,0) = 0,\ \frac{\partial^3 f}{\partial y^3}(0,0) = -\frac{1}{3}3! = -2. \square$$

(òbviament, totes les derivades parcials "creuades" coincideixen al (0,0); i.e.:

$$f_{xxy}(0,0) = f_{xyx}(0,0) = f_{yxx}(0,0) = 2,$$

$$f_{xyy}(0,0) = f_{yxy}(0,0) = f_{yyx}(0,0) = -2.$$

(b) $f(x,y) = \cos(xy)$, fins ordre 8

$$f(x,y) = \cos(xy) = 1 - \frac{x^2y^2}{2!} + \frac{x^4y^4}{4!} + R_4(x,y),$$

d'on tenim que les derivades d'ordre 8 al $(0,0)$ són:

$$\frac{\partial^8 f}{\partial x^8}(0,0) = 0, \quad \frac{\partial^8 f}{\partial x^7 \partial y}(0,0) = 0, \quad \frac{\partial^8 f}{\partial x^6 \partial y^2}(0,0) = 0, \quad \frac{\partial^8 f}{\partial x^5 \partial y^3}(0,0) = 0,$$

$$\frac{\partial^8 f}{\partial x^4 \partial y^4}(0,0) = \frac{1}{\cancel{4!}}\cancel{4!}\,4! = 24, \quad \frac{\partial^8 f}{\partial x^3 \partial y^5}(0,0) = 0, \quad \frac{\partial^8 f}{\partial x^2 \partial y^6}(0,0) = 0,$$

$$\frac{\partial^8 f}{\partial x \partial y^7}(0,0) = 0, \quad \frac{\partial^8 f}{\partial y^8}(0,0) = 0. \square$$

(c) $f(x,y) = e^{x^2-y^2}$ fins ordre 8.

$$f(x,y) = e^{x^2-y^2} = 1 + \frac{1}{1}(x^2-y^2) + \frac{1}{2!}(x^2-y^2)^2 + \frac{1}{3!}(x^2-y^2)^3 + \frac{1}{4!}(x^2-y^2)^4 + \dots$$

$$= 1 + x^2 - y^2 + \frac{1}{2}x^4 - x^2y^2 + \frac{1}{2}y^4 + \frac{1}{6}x^6 - \frac{1}{2}x^4y^2 + \frac{1}{2}x^2y^4$$

$$- \frac{1}{6}y^6 + \frac{1}{24}x^8 - \frac{1}{6}x^6y^2 + \frac{1}{4}x^4y^4 - \frac{1}{6}x^2y^6 + \frac{1}{24}y^8$$

$$+ R_8(x,y).$$

Aleshores, les derivades parcials d'ordre 8 en $(0,0)$, resulten:

$$\frac{\partial^8 f}{\partial x^8}(0,0) = \frac{1}{24}8! = 8\cdot7\cdot6\cdot5 = 1680, \quad \frac{\partial^8 f}{\partial x^7 \partial y}(0,0) = 0, \quad \frac{\partial^8 f}{\partial x^6 \partial y^2}(0,0) = -\frac{1}{6}6!2! = -240.$$

$$\frac{\partial^8 f}{\partial x^5 \partial y^3}(0,0) = 0, \quad \frac{\partial^8 f}{\partial x^4 \partial y^4}(0,0) = \frac{1}{4}4!4! = 144, \quad \frac{\partial^8 f}{\partial x^3 \partial y^5}(0,0) = 0,$$

$$\frac{\partial^8 f}{\partial x^2 \partial y^6}(0,0) = -\frac{1}{6}2!6! = -240, \quad \frac{\partial^8 f}{\partial x \partial y^7}(0,0) = 0, \quad \frac{\partial^8 f}{\partial y^8}(0,0) = \frac{1}{24}8! = 1680.$$

$\square$

(51) Determineu el valor de $\lambda \in \mathbb{R}$ per tal que:

$$\lim_{(x,y)\to(0,0)} \frac{\arctan(x^2+y)-x^2-y-\lambda y^3}{(x^2+y^2)^{3/2}} = 0$$

Solució.

$$\lim_{(x,y)\to(0,0)} \frac{\arctan(x^2+y)-x^2-y-\lambda y^3}{(x^2+y^2)^{3/2}} \overset{(5)}{=} \lim_{(x,y)\to(0,0)} \frac{(x^2+y)-\frac{1}{3}(x^2+y)^3+\dots -x^2-y-\lambda y^3}{(x^2+y^2)^{3/2}}$$

(5) fem ús del desenvolupament, en $t=0$, de $\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} \dots + (-1)^p \frac{t^{2p+1}}{2p+1} + \dots$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{y}+\cancel{x^2}-(\lambda+\frac{1}{3})y^3-\cancel{x^2}-\cancel{y}+R_3(x,y)}{\|(x,y)\|^3} = \lim_{(x,y)\to(0,0)} \left(\frac{-(\lambda+\frac{1}{3})y^3}{\|(x,y)\|^3} + \underbrace{\frac{R_3(x,y)}{\|(x,y)\|^3}}_{\downarrow\ 0\ (\text{Taylor})} \right)$$

$$= \lim_{(x,y)\to(0,0)} \frac{R_3(x,y)}{\|(x,y)\|^3} = 0, \text{ si } \lambda = -\tfrac{1}{3}.$$

Si $\lambda \neq -\frac{1}{3}$ el límit no existeix. Per comprovar-ho és suficient considerar el límit a l'origen segons la recta $\{x=0\}$. En efecte:

$$\lim_{y\to 0} \frac{\arctan y - y - \lambda y^3}{|y|^3} = \lim_{y\to 0} \frac{\cancel{y} - \frac{1}{3}y^3 - \cancel{y} - \lambda y^3 + R_3(y)}{|y|^3} =$$

$$= \lim_{y\to 0} \left[\frac{R_3(y)}{|y|^3} - \left(\frac{1}{3}+\lambda\right)\frac{y^3}{|y|^3} \right] = \begin{cases} -(\frac{1}{3}+\lambda), & \text{si } y\to 0^+ \\ \frac{1}{3}+\lambda, & \text{si } y \to 0^- \end{cases}$$

Llavors el límit segons la recta $\{x=0\}$ no existeix i per tant no pot existir el límit a l'origen. □