(42) La resistència equivalent R corresponent a dues resistències R_1 i R_2 connectades en paral·lel verifica:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Usant l'aproximació lineal, estimeu la variació del valor de R si incrementem el valor de R_1 de $10\,\Omega$ a $10.5\,\Omega$ i decreixem el valor de R_2 de $15\,\Omega$ a $13\,\Omega$.

Solució:
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1+R_2}{R_1R_2} \iff R(R_1,R_2) = \frac{R_1R_2}{R_1+R_2}$$

$R_1^o = 10\,\Omega$, $\Delta R_1 = 0'5\,\Omega$; $R_2^o = 15\,\Omega$, $\Delta R_2 = -2\,\Omega$.

$$\frac{\partial R}{\partial R_1}(R_1,R_2) = \frac{R_2(R_1+R_2)-R_1R_2}{(R_1+R_2)^2} = \frac{R_2^2}{(R_1+R_2)^2}, \quad \frac{\partial R}{\partial R_2}(R_1,R_2) = \frac{R_1^2}{(R_1+R_2)^2}.$$

$$\frac{\partial R}{\partial R_1}(10,15) = \frac{225}{625} = \frac{9}{25}, \quad \frac{\partial R}{\partial R_2}(10,15) = \frac{100}{625} = \frac{4}{25}.$$

- Aproximació lineal

$$R(R_1^o+\Delta R_1, R_2^o+\Delta R_2) - R(R_1^o, R_2^o) \simeq \frac{\partial R}{\partial R_1}(R_1^o,R_2^o)\,\Delta R_1 + \frac{\partial R}{\partial R_2}(R_1^o,R_2^o)\cdot\Delta R_2$$

$$|R(10.5,13)-R(10,15)| \simeq \left|\frac{\partial R}{\partial R_1}(10,15)\cdot 0'5 + \frac{\partial R}{\partial R_2}(10,15)\cdot(-2)\right|$$
$$= \left|\frac{9}{50} - \frac{16}{50}\right| = \frac{7}{50} = 0.14$$

- Acotació de l'error:

$$|R(R_1^o+\Delta R_1, R_2^o+\Delta R_2) - R(R_1^o,R_2^o)| \leq \max_r \left|\frac{\partial R}{\partial R_1}(R_1,R_2)\right|\cdot|\Delta R_1| + \max_r \left|\frac{\partial R}{\partial R_2}(R_1,R_2)\right|\cdot|\Delta R_2|$$

on r és el segment que uneix (R_1^o, R_2^o) i $(R_1^o+\Delta R_1, R_2^o+\Delta R_2)$.

$$\max_r \left|\frac{\partial R}{\partial R_1}(R_1,R_2)\right| \leq \max_{\substack{10\leq R_1\leq 10.5\\ 13\leq R_2\leq 15}} \frac{R_2^2}{(R_1+R_2)^2} \leq \frac{15^2}{23^2} = \frac{225}{529} =: M_1$$

$$\max_{r} \left|\frac{\partial R}{\partial R_2}(R_1,R_2)\right| \leq \max_{\substack{10\leq R_1\leq 10.5\\ 13\leq R_2\leq 15}} \frac{R_1^2}{(R_1+R_2)^2} \leq \frac{10.5^2}{23^2} = \frac{441}{2116} =: M_2 .$$

Aleshores:

$$|R(10.5,13)-R(10,15)| \leq M_1\cdot|\Delta R_1| + M_2\cdot|\Delta R_2| = \frac{225}{529}\times 0.5 + \frac{441}{2116}\times 2$$

$$= \frac{450+882}{2116} = \frac{1332}{2116} = \frac{333}{529} = 0.629\,489\,603\,024\,575\ldots$$

D'altra banda, l'error "real" és:

$$|R(10.5,13)-R(10,15)| = \left|\frac{273}{47}-6\right| = \frac{|273-282|}{47} = \frac{9}{47} = 0.191\,489\,361\,702\,128\ldots \quad \square$$

(47) Calculeu totes les derivades parcials fins a ordre 2 de les següents funcions i doneu el seu desenvolupament de Taylor fins a termes de grau 2 inclosos entorn al punt que s'indica en cada cas.

Solució. Reordem (fórmula de Taylor a grau 2):

$$f(a+h) = f(a) + \langle \nabla f(a), h\rangle + \frac{1}{2}\langle h, D^2 f(a) h\rangle + R_2(h)$$

(a) Taylor de $f(x,y) = \sin(xy)$ entorn al punt $(1,\frac{\pi}{2})$

$$\frac{\partial f}{\partial x}(x,y) = y\cos(xy) \ ; \ \frac{\partial f}{\partial y}(x,y) = x\cos(xy) ; \ \frac{\partial^2 f}{\partial x^2}(x,y) = -y^2\sin(xy);$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y) = \cos(xy) - xy\sin(xy) \ ; \ \frac{\partial^2 f}{\partial y^2}(x,y) = -x^2\sin(xy),$$

$$\frac{\partial f}{\partial x}(1,\tfrac{\pi}{2}) = 0, \ \frac{\partial f}{\partial y}(1,\tfrac{\pi}{2}) = 0, \ \frac{\partial^2 f}{\partial x^2}(1,\tfrac{\pi}{2}) = -\frac{\pi^2}{4}, \ \frac{\partial^2 f}{\partial x\partial y}(1,\tfrac{\pi}{2}) = -\frac{\pi}{2}, \ \frac{\partial^2 f}{\partial y^2}(1,\tfrac{\pi}{2}) = -1.$$

$$\boxed{f(x,y) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right) - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2 + R_2\left(x-1, y-\frac{\pi}{2}\right).}$$

(b) $f(x,y) = x^y$ en un entorn del punt $(x,y)=(1,1)$.

$$\frac{\partial f}{\partial x}(x,y) = y x^{y-1}, \quad \frac{\partial f}{\partial y}(x,y) = x^y \ln x, \quad \frac{\partial^2 f}{\partial x^2}(x,y) = y(y-1)x^{y-2},$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = y x^{y-1}\ln x + x^{y-1} = x^{y-1}(1+y\ln x),$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = x^y (\ln x)^2.$$

$$f(1,1)=1, \quad \frac{\partial f}{\partial x}(1,1) = 1, \quad \frac{\partial f}{\partial y}(1,1) = 0, \quad \frac{\partial^2 f}{\partial x^2}(1,1) = 0, \quad \frac{\partial^2 f}{\partial x \partial y}(1,1) = 1, \quad \frac{\partial^2 f}{\partial y^2}(1,1) = 0.$$

$$\boxed{f(x,y) = 1 + (x-1) + (x-1)(y-1) + R_2(x-1,y-1) = x + (x-1)(y-1) + R_2(x-1,y-1).}$$

(c) $f(x,y) = e^{x/y}$ en un entorn del punt $(0,1)$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{y}e^{x/y}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2}e^{x/y}, \quad \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{1}{y^2}e^{x/y},$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -\frac{1}{y^2}e^{x/y} - \frac{x}{y^3}e^{x/y} = -\frac{1}{y^2}e^{x/y}\left(1+\frac{x}{y}\right).$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2x}{y^3}e^{x/y} + \frac{x^2}{y^4}e^{x/y} = \frac{x}{y^3}e^{x/y}\left(2+\frac{x}{y}\right)$$

$$f(0,1) = 1, \quad \frac{\partial f}{\partial x}(0,1) = 1, \quad \frac{\partial f}{\partial y}(0,1) = 0, \quad \frac{\partial^2 f}{\partial x^2}(0,1) = 1, \quad \frac{\partial^2 f}{\partial x \partial y}(0,1) = -1,$$

$$\frac{\partial^2 f}{\partial y^2}(0,1) = 0.$$

$$\boxed{f(x,y) = e^{x/y} = 1 + x + \tfrac{1}{2}x^2 - x(y-1) + R_2(x,y-1).}$$

(d) Taylor de $f(x,y,z) = e^{-x}\sin(yz)$ entorn del punt $(0,1,\pi)$.

$$\frac{\partial f}{\partial x}(x,y,z) = -e^{-x}\sin(yz), \quad \frac{\partial f}{\partial y}(x,y,z) = z e^{-x}\cos(yz), \quad \frac{\partial f}{\partial z}(x,y,z) = y e^{-x}\cos(yz),$$

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = e^{-x}\sin(yz), \quad \frac{\partial^2 f}{\partial x \partial y}(x,y,z) = -z e^{-x}\cos(yz), \quad \frac{\partial^2 f}{\partial x \partial z}(x,y,z) = -y e^{-x}\cos(yz)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y,z) = -z^2 e^{-x}\sin(yz), \quad \frac{\partial^2 f}{\partial y \partial z}(x,y,z) = e^{-x}\cos(yz) - zy e^{-x}\sin(yz),$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z) = -y^2 e^{-x}\sin(yz).$$

$\frac{\partial f}{\partial x}(0,1,\pi)=0,\ \frac{\partial f}{\partial y}(0,1,\pi) = -\pi,\ \frac{\partial f}{\partial z}(0,1,\pi) = -1,\ \frac{\partial^2 f}{\partial x^2}(0,1,\pi) = 0,\ \frac{\partial^2 f}{\partial x \partial y}(0,1,\pi) = \pi,$

$,\ \frac{\partial^2 f}{\partial x \partial z}(0,1,\pi) = 1,\ \frac{\partial^2 f}{\partial y \partial z}(0,1,\pi) = -1,\ \frac{\partial^2 f}{\partial y^2}(0,1,\pi) = 0,\ \frac{\partial^2 f}{\partial z^2}(0,1,\pi) = 0$

$$\boxed{f(x,y,z) = e^{-x}\sin(yz) = -\pi(y-1) - (z-\pi) + \pi x(y-1) + x(z-\pi) - (y-1)(z-\pi) + R_2(x, y-1, z-\pi)}$$