Calcul-2 Problemes — Classe 11-02-2014

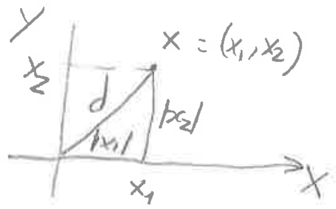
Pas de $\mathbb{R}$ a $\mathbb{R}^n$ 1^h Problemes.

1] $x_i \in \mathbb{R}\ \forall i=1,2,\dots,n \Rightarrow x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n = \overbrace{\mathbb{R}\times\cdots\times\mathbb{R}}^{n}$, x_i: coordenada i-èsima del punt x

$x=(x_1,x_2,\dots,x_n),\ y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$: $x+y=(x_1,x_2,\dots,x_n)+(y_1,y_2,\dots,y_n)=(x_1+y_1,x_2+y_2,\dots,x_n+y_n)$

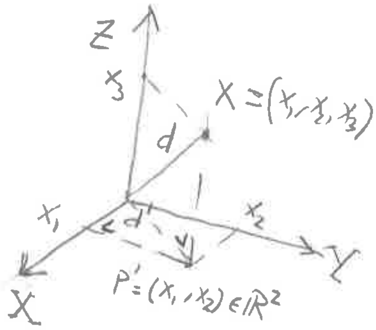
$x=(x_1,x_2,\dots,x_n),\ \lambda\in\mathbb{R}$: $\lambda x=\lambda\cdot(x_1,x_2,\dots,x_n)=(\lambda x_1,\lambda x_2,\dots,\lambda x_n)$: producte per un escalar.

$(\mathbb{R},+,\cdot)$ és un $\mathbb{R}$-e.v. (Àlgebra).



c.p. 1) $x_1,x_2\in\mathbb{R}\Rightarrow x=(x_1,x_2)\in\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$ és un punt de $\mathbb{R}^2$

2) $x_1,x_2,x_3\in\mathbb{R}\Rightarrow x=(x_1,x_2,x_3)\in\mathbb{R}^3=\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ és un punt de $\mathbb{R}^3$



2] $x=(x_1,x_2,\dots,x_n),\ y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$. Definim el seu producte escalar com:

$$\langle x,y\rangle = x\cdot y = x_1y_1+x_2y_2+\dots+x_ny_n=\sum_{i=1}^n x_iy_i\in\mathbb{R}$$

Propietats:

1] $\langle x,x\rangle\geq 0\ \forall x\in\mathbb{R}^n$, $\langle x,x\rangle=0 \Leftrightarrow x=0$.

2] $\forall x,y\in\mathbb{R}^n$ $\langle x,y\rangle=\langle y,x\rangle$ (Simetria).

3] $\forall x\in\mathbb{R}^n, \forall\lambda\in\mathbb{R}$ $\langle\lambda x,y\rangle=\lambda\langle x,y\rangle$.

4] $\forall x,y,z\in\mathbb{R}^n$ $\langle x+z,y\rangle=\langle x,y\rangle+\langle z,y\rangle$ (Bilinealitat).

Comprovació immediata. Per exemple, per comprovar aquesta última fem:

$$\langle x+z,y\rangle=(x_1+z_1)y_1+(x_2+z_2)y_2+\dots+(x_n+z_n)y_n = x_1y_1+x_2y_2+\dots+x_ny_n+z_1y_1+z_2y_2+\dots+z_ny_n$$
$$=\langle x,y\rangle+\langle z,y\rangle.$$

3] Donat $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, definim la norma de x com:

$$\|x\|=+\sqrt{\langle x,x\rangle}=\sqrt{x_1^2+x_2^2+\dots+x_n^2}=\sqrt{\sum_{i=1}^n x_i^2}$$

c.p. $n=1$, $x\in\mathbb{R}$: $\|x\|=\sqrt{x^2}=|x|=\begin{cases}x, & x>0\\ -x, & x\leq 0\end{cases}$

$n=2$, $x=(x_1,x_2)\in\mathbb{R}^2$: $\|x\|=\sqrt{x_1^2+x_2^2}$: distància de l'origen al punt $x\in\mathbb{R}^2$ Pitàgoras $d=\sqrt{x_1^2+x_2^2}=\|x\|$

$n=3$, $x=(x_1,x_2,x_3)\in\mathbb{R}^3$: $\|x\|=\sqrt{x_1^2+x_2^2+x_3^2}$: " " " " " $x\in\mathbb{R}^3$: Pitàgoras $d=\sqrt{d'^2+x_3^2}=\sqrt{x_1^2+x_2^2+x_3^2}=\|x\|$

Propietats:

1) $\|x\| \geq 0 \ \forall x \in \mathbb{R}^n$,

2) $\forall \lambda \in \mathbb{R}, \forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$: $\|\lambda x\| = |\lambda| \, \|x\|$.

3) $\forall x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$: $|\langle x, y\rangle| \leq \|x\| \, \|y\|$ (Desigualtat de Cauchy-Schwartz).

4) $\forall x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$: $\|x+y\| \leq \|x\| + \|y\|$ (" triangular).

Demostració

1, 2) Immediates, a partir de les definicions

3) $\|x+ty\|^2 = \langle x+ty, x+ty\rangle = \langle x, x+ty\rangle + \langle ty, x+ty\rangle = \langle x,x\rangle + \langle x,ty\rangle + \langle ty,x\rangle + \langle ty,ty\rangle$

$= \|x\|^2 + 2t\langle x,y\rangle + t^2\|y\|^2 \geq 0 \quad \forall x,y \in \mathbb{R}^n, \forall t \in \mathbb{R}$

Si $y \neq 0$, agafant $t = -\dfrac{\langle x,y\rangle}{\|y\|^2}$, tenim:

$$\|x\|^2 + 2t\langle x,y\rangle + t^2\|y\|^2 = \|x\|^2 - 2\frac{\langle x,y\rangle^2}{\|y\|^2} + \frac{\langle x,y\rangle^2}{\|y\|^2} = \|x\|^2 - \frac{\langle x,y\rangle^2}{\|y\|^2} \geq 0$$

$$\Leftrightarrow \langle x,y\rangle^2 \leq \|x\|^2\|y\|^2,$$

d'on es segueix la desigualtat. Remarca $a^2 \leq b^2 \Leftrightarrow |a| \leq |b| \ \forall a,b \in \mathbb{R}$.

Si $y = 0$, la desigualtat es satisfà de manera trivial. □

4) $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$, d'on es segueix la desigualtat.

Apliquem: $-\|x\|\|y\| \leq \langle x,y\rangle \leq \|x\|\|y\|$

Corol·laris

1') $\forall x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$: $|\langle x,y\rangle| \leq |x_1y_1| + |x_2y_2| + \dots + |x_ny_n| = \sum_{i=1}^n |x_iy_i|$.

2') $\forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \forall i = 1, 2, \dots, n$: $|x_i| \leq \|x\| \leq |x_1| + |x_2| + \dots + |x_n| = \sum_{i=1}^n |x_i|$.

3') $\forall x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$: $\|x-y\| \geq \big| \|x\| - \|y\| \big|$.

Demostració

1') Trivial a partir de la definició i de la desigualtat triangular per $n = 1$:

$|\langle x,y\rangle| = |x_1y_1 + x_2y_2 + \dots + x_ny_n| \leq |x_1y_1| + |x_2y_2| + \dots + |x_ny_n|$.

2') Agafant $e_i = (0, 0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$, aplicant la desigualtat de Schwartz: $|x_i| = |\langle x, e_i\rangle| \leq \|x\| \, \|e_i\| = \|x\|$.

D'altra banda: $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n) = (x_1, 0, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \dots + (0, 0, \dots, 0, x_n)$ i llavors:

$\|x\| = \|(x_1, 0, 0, \dots, 0) + (0, x_2, 0, \dots, 0) + \dots + (0, 0, \dots, 0, x_n)\| \leq \|(x_1, 0, 0, \dots, 0)\| + \|(0, x_2, 0, \dots, 0)\| + \dots + \|(0, 0, \dots, 0, x_n)\|$

$= |x_1| + |x_2| + \dots + |x_n|$,

ja que $\|(0, 0, \dots, 0, \overset{i}{x_i}, 0, \dots, 0)\| = \sqrt{0^2 + 0^2 + \dots + 0^2 + x_i^2 + 0^2 + \dots + 0^2} = \sqrt{x_i^2} = |x_i| \ \forall i = 1, 2, \dots, n$. En resum, la norma de $x \in \mathbb{R}^n$ és $\geq$ que el valor absolut de qualsevol de les seves coordenades i $\leq$ que la suma del valor absolut de les seves coordenades □

3) $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\| \iff \|x\| - \|y\| \leq \|x-y\|$
$\|y\| = \|y-x+x\| \leq \|x-y\| + \|x\| \iff -\|x-y\| \leq \|x\| - \|y\|$ $\Big\} \iff -\|x-y\| \leq \|x\|-\|y\| \leq \|x-y\|$

$\iff \big| \|x\| - \|y\| \big| \leq \|x-y\|$. Remarca: $-a \leq x \leq a \iff |x| \leq a \ \forall x, a \in \mathbb{R},\ a>0$. $\square$

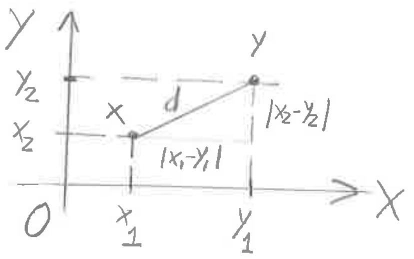
4| Donats $x=(x_1,x_2,\dots,x_n)$, $y=(y_1,y_2,\dots,y_n) \in \mathbb{R}^n$ definim la distància entre x i y com:

$$d(x,y) := \|x-y\| = \sqrt{\langle x-y, x-y\rangle} = \left(\sum_{i=1}^{n} (x_i-y_i)^2\right)^{\frac{1}{2}}$$

c.p.: $n=1$: $x,y \in \mathbb{R}$ $\quad d(x,y) = |x-y|$

$n=2$: $x=(x_1,x_2)$, $y=(y_1,y_2) \in \mathbb{R}^2$: $d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

$n=3$: $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3) \in \mathbb{R}^3$: $d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$



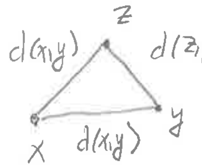
$$d(x,y) = \|x-y\| = \left((x_1-y_1)^2 + (x_2-y_2)^2\right)^{\frac{1}{2}}$$

Propietats.

1) $d(x,y) \geq 0 \ \forall x,y \in \mathbb{R}^n$, $d(x,y) = 0 \iff x=y$.

2) $\forall x,y \in \mathbb{R}^n$: $d(x,y) = d(y,x)$

3) $\forall x,y,z \in \mathbb{R}^n$: $d(x,y) \leq d(x,z) + d(z,x)$ (desigualtat triangular)



Demostració. Són conseqüència de la definició de distància mitjançant $\|\cdot\|$.

1,2) Immediates a partir de la definició: $d(x,y) = \|x-y\| \geq 0$, $d(x,y) = \|x-y\| = 0 \iff x-y=0 \iff x=$

$d(x,y) = \|x-y\| = \|y-x\| = d(y,x) \ \forall x,y \in \mathbb{R}^n$

3) $d(x,y) = \|x-z+z-y\| \leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y) \ \forall x,y,z \in \mathbb{R}^3$

5) Donats $x=(x_1,x_2,\dots,x_n) \in \mathbb{R}^n$, $r \in \mathbb{R}$, $r>0$, definiem bola oberta de radi r i centre x:

$$B_r^n(x) = B^n(x;r) = \{ y=(y_1,y_2,\dots,y_n) \in \mathbb{R}^n : d(x,y) = \|x-y\| < r \} \subseteq \mathbb{R}^n$$

c.p. $n=1$: $x \in \mathbb{R}$, $r \in \mathbb{R}$, $r>0$: $B_r^1(x) = \{ y \in \mathbb{R} : d(x,y) = \|x-y\| = |x-y| < r \} =$

$= \{ y \in \mathbb{R} : x-r < y < x+r \} = (x-r, x+r)$: interval obert amb centre x i radi $r>0$.

$|x-y| < r \iff -r < x-y < r$

$n=2$: $x=(x_1,x_2) \in \mathbb{R}^2$, $r \in \mathbb{R}$, $r>0$: $B_r^2(x_1,x_2) = \{ y=(y_1,y_2) \in \mathbb{R}^2 : \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} < r \}$:

disc de radi $r>0$ i centre $x=(x_1,x_2)$