# TEMA 2

Problemes: 1, 2, 5, 6, 7, 9, 10, 11, 13, 17, 18, 19, 21, 27, 29, 30, 31, 32, 33, 35.

1. Usant el teorema del valor mig per integrals, proveu les següents desigualtats

(a) $4e^5 \le \iint_A e^{x^2+y^2}\,dx\,dy \le 4e^{25}$, on $A=[1,3]\times[2,4]$

S. Recordem el Teorema del Valor Mig (veure apunts de teoria, tema 2):

$\triangleleft$ Sigui $f:\Omega\subset\mathbb{R}^n \longrightarrow \mathbb{R}$, $\Omega$ acotat i arc-connex, i $f\in C^0(\bar\Omega)$, llavors, $\exists\ c\in\bar\Omega$ tal que:

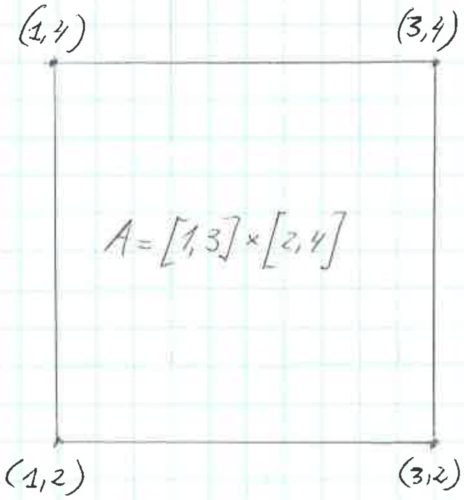
$$\int_\Omega f(x) = f(c)\, m(\Omega),$$

i on $m(\Omega)$ és la mesura del conjunt $\Omega$ (longitud en dimensió 1, àrea en dimensió 2, volumen en dimensió 3, ...). Llavors tenim les acotacions següents:

$$m(\Omega)\inf_{x\in\Omega} f(x) \le \int_\Omega f(x) \le m(\Omega)\sup_{x\in\Omega} f(x).$$

(a) $f(x,y)=e^{x^2+y^2}$, d'on: $f_x(x,y)=2xe^{x^2+y^2}=0 \Leftrightarrow x=0$, $f_y(x,y)=2ye^{x^2+y^2}=0$. Llavors l'únic punt crític de la funció és $(x,y)=(0,0)\notin A=[1,3]\times[2,4]$

- $x=1$: $f(1,y)=e^{1+y^2}$: $f'(1,y)=2y\,e^{1+y^2}=0 \Leftrightarrow y=0$: $(1,0)\notin A$.
- $x=3$: $f(3,y)=e^{9+y^2}$: $f'(3,y)=2y\,e^{9+y^2}=0 \Leftrightarrow y=0$: $(3,0)\notin A$.
- $y=2$: $f(x,2)=e^{x^2+4}$: $f'(x,2)=2x\,e^{x^2+4}=0 \Leftrightarrow x=0$: $(0,2)\notin A$.
- $y=4$: $f(x,4)=e^{x^2+16}$: $f'(x,4)=2x\,e^{x^2+16}=0$ $\Leftrightarrow x=0$, $(0,4)\notin A$.




Veiem que el punt crític de $f$ no pertany a $A$ i que les restriccions de $f$ sobre els costats d'$A$ no tenen punts crítics. Avaluem $f$ als vèrtexs d'$A$:

$f(1,2)=e^5$, $f(3,2)=e^{13}$, $f(1,4)=e^{17}$, $f(3,4)=e^{25}$.

d'on: $\max\limits_{(x,y)\in A} e^{x^2+y^2} = e^{25}$ i $\min\limits_{(x,y)\, A} e^{x^2+y^2} = e^{5}$ ; s'atanyen als punts (3,4) i (1,2) respectivament. D'altra banda: $m(A) = \iint dxdy = (3-1)\times(4-2) = 4$. Així doncs, aplicant el Teorema del Valor Mig (fórmula (1)), A

$$4e^5 \leq \iint\limits_{A=[1,3]\times[2,4]} e^{x^2+y^2}\,dxdy \leq 4e^{25}. \quad \square$$

(b) $\dfrac{1}{e} \leq \dfrac{1}{4\pi^2} \iint\limits_A e^{\sin(x+y)}\,dxdy \leq e$, $A = [-\pi,\pi]\times[-\pi,\pi]$

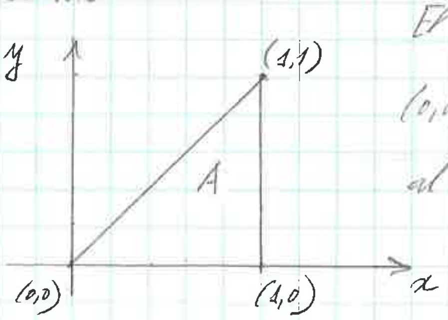
Solució. Clarament: $\max\limits_{(x,y)\in A} e^{\sin(x+y)} = e$, $\min\limits_{(x,y)\in A} e^{\sin(x+y)} = 1/e$, mentre que $m(A) = 4\pi^2$. Aleshores, aplicant (1) resulta:

$$\frac{4\pi^2}{e} \leq \iint\limits_{A=[-\pi,\pi]\times[-\pi,\pi]} e^{\sin(x+y)}\,dxdy \leq 4\pi^2 e,$$

i dividint les dues desigualtats per $4\pi^2$ s'obtenen les acotacions buscades. $\square$

(c) $\dfrac{1}{6} \leq \iint\limits_A \dfrac{dxdy}{y-x+3} \leq \dfrac{1}{4}$, on $A$ és el triangle de vèrtexs $(0,0)$, $(1,1)$ i $(1,0)$

Solució.




El màxim de $y-x+3$ sobre el triangle $A$ s'atany als vèrtexs $(0,0)$ i $(1,1)$ i val 3, mentre que el mínim té lloc al vèrtexs $(1,0)$ i val 2. Per tant:

$$\frac{1}{3} \leq \frac{1}{y-x+3} \leq \frac{1}{2}$$

per tot punt del triangle donat; mentre que l'àrea corresponent és $m(A) = \frac{1}{2}$. Finalment doncs, aplicant la fórmula (1) del Teorema del Valor Mitjà, obtenim:

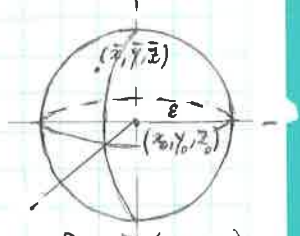
$$\frac{1}{6} \leq \iint\limits_A \frac{dxdy}{y-x+3} \leq \frac{1}{4}$$

, que són les desigualtats buscades. $\square$

$\underline{2})$ Signi $f(x,y,z)$ una funció contínua i $B_\varepsilon$ la bola de centre $(x_0,y_0,z_0)$ i radi $\varepsilon$.
Proveu que se satisfà que $f(x_0,y_0,z_0) = \lim\limits_{\varepsilon\to 0^+} \dfrac{1}{\text{volum}(B_\varepsilon)} \iiint\limits_{B_\varepsilon} f(x,y,z)\,dx\,dy\,dz$

Solució. Pel Teorema del Valor Mig, existeix $(\bar{x},\bar{y},\bar{z}) \in B_\varepsilon = B_\varepsilon(x_0,y_0,z_0)$ t.q.:

$$\iiint\limits_{B_\varepsilon(x_0,y_0,z_0)} f(x,y,z)\,dx\,dy\,dz = f(\bar{x},\bar{y},\bar{z}) \cdot \text{Volum } B_\varepsilon(x_0,y_0,z_0)$$



$B_\varepsilon := B_\varepsilon(x_0,y_0,z_0)$

D'altra banda, com que $f$ és contínua i $(\bar{x},\bar{y},\bar{z}) \longrightarrow (x_0,y_0,z_0)$ quan $\varepsilon\to 0$, prenent límits a l'expressió de dalt hom obté:

$$\lim_{\varepsilon\to 0} \frac{1}{\text{Volum}(B_\varepsilon(x_0,y_0,z_0))} \iiint\limits_{B_\varepsilon(x_0,y_0,z_0)} f(x,y,z)\,dx\,dy\,dz = \lim_{(\bar{x},\bar{y},\bar{z})\to(x_0,y_0,z_0)} f(\bar{x},\bar{y},\bar{z}) = f(x_0,y_0,z_0),$$

que és el resultat que es buscava. □

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$\underline{5})$ Trobeu les següents integrals dobles en els rectangles que s'indiquen

(a) $I = \iint\limits_R x^2y\,dx\,dy$, $R = [0,1]\times[0,1]$.

Solució: $I = \iint\limits_R x^2y\,dx\,dy = \left(\int_0^1 x^2\,dx\right)\left(\int_0^1 y\,dy\right) = \left[\frac{x^3}{3}\right]_0^1 \cdot \left[\frac{y^2}{2}\right]_0^1 = \boxed{\frac{1}{6}}$

(b) $I = \iint\limits_R \dfrac{x^2}{1+y^2}\,dx\,dy$, $R = [0,1]\times[0,1]$

Solució: $I = \iint\limits_{R=[0,1]\times[0,1]} \dfrac{x^2}{1+y^2}\,dx\,dy = \int_0^1 x^2\,dx \int_0^1 \dfrac{dy}{1+y^2} = \left[\dfrac{x^3}{3}\right]_0^1 \cdot \Big[\arctan y\Big]_0^1 = \boxed{\dfrac{\pi}{12}}$

(c) $I = \iint_R y \ln x \, dx\,dy$, $R = [1,e]\times[1,e]$.

Solució. $I = \iint_{R=[1,e]\times[1,e]} y\ln x\,dx\,dy = \left(\int_1^e \ln x\,dx\right)\cdot\left(\int_1^e y\,dy\right) \overset{(*)}{=} \boxed{\frac{e^2-1}{2}}$

(*) $\int_1^e \ln x\,dx = \left[x\ln x - x\right]_1^e = \cancel{e}-\cancel{e}+1$ (primitivització per parts + Barrow).

$\int_1^e y\,dy = \left[\frac{y^2}{2}\right]_1^e = \frac{1}{2}(e^2-1)$

(d) $\iint_R (x^2+y)\,dx\,dy$, $R=[0,1]\times[0,2]$

Solució $I = \iint_{R=[0,1]\times[0,2]} (x^2+y)\,dx\,dy = \int_0^1 dx \int_0^2 (x^2+y)\,dy = \int_0^1 dx\left[x^2y+\frac{y^2}{2}\right]_0^2 =$

$= \int_0^1 (2x^2+2)\,dx = \left[2\frac{x^3}{3}+2x\right]_0^1 = \boxed{8/3}$

(e) $I=\iint_R \frac{1}{(x+2y)^2}\,dx\,dy$, $R=[2,5]\times[1,3]$

Solució. $I = \iint_R \frac{dx\,dy}{(x+2y)^2} = \int_2^5 dx \int_1^3 \frac{dy}{(x+2y)^2} = \int_2^5 \left[\frac{-1/2}{x+2y}\right]_{x=1}^{x=3} dx = \frac{1}{2}\int_2^5\left(\frac{1}{x+2}-\frac{1}{x+6}\right)dx$

$= \frac{1}{2} \cdot \ln\left(\frac{x+2}{x+6}\right)\Big|_2^5 = \frac{1}{2}\left(\ln\frac{7}{11}-\ln\frac{4}{8}\right) = \boxed{\frac{1}{2}\ln\frac{14}{11}}$

(f) $I = \iint_R e^y\sin\left(\frac{x}{y}\right)dx\,dy$, $R=[-\pi/2,\pi/2]\times[1,2]$.

Solució. $I=\iint_{R=[-\frac{\pi}{2},\frac{\pi}{2}]\times[1,2]} e^y\sin\left(\frac{x}{y}\right)dx\,dy = \int_1^2 e^y dy\int_{-\pi/2}^{\pi/2}\sin\left(\frac{x}{y}\right)dx = -\int_1^2 e^y y\left[\cos\left(\frac{x}{y}\right)\right]_{x=-\pi/2}^{x=\pi/2} dx \overset{(*)}{=} \boxed{0}$

(*) $\left[\ldots\right]_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} = \cos\left(\frac{\pi}{2y}\right)-\cos\left(-\frac{\pi}{2y}\right)=0.$

(g) $I = \iint_R (x+y)^{27}\,dx\,dy$, $R=[-1,1]\times[-1,1]$.

Solució. $I = \iint_{R=[-1,1]\times[-1,1]} (x+y)^{27}\,dx\,dy = \int_{-1}^{1} dx \int_{-1}^{1} (x+y)^{27}\,dy = \int_{-1}^{+1}\left[\frac{(x+y)^{28}}{28}\right|_{y=-1}^{y=1} dx$

$$= \int_{-1}^{1} \left[\frac{(x+1)^{28}}{28} - \frac{(x-1)^{28}}{28}\right]dx = \left[\frac{(x+1)^{29}}{29\cdot 28} - \frac{(x-1)^{29}}{29\cdot 28}\right]_{-1}^{1} = \frac{1}{29\cdot 28}\left(2^{29}+(-2)^{29}\right) = \boxed{0}.$$

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6) Calculeu $I=\iint_R x^y\,dx\,dy$ on $R=[0,1]\times[a,b]$, essent $0<a<b$, i deduïu el valor de la integral $\int_0^1 \frac{x^b-x^a}{\ln x}\,dx$.

Solució. $I=\iint_{R=[0,1]\times[a,b]} x^y\,dx\,dy = \int_a^b dy\int_0^1 x^y\,dx = \int_a^b \frac{x^{y+1}}{y+1}\Big|_{x=0}^{x=1} dy = \int_a^b \frac{dy}{y+1} = \ln\frac{b+1}{a+1}.$

D'altra banda, invertint l'ordre d'integració (Fubini):

$$I = \iint_{R=[0,1]\times[a,b]} x^y\,dx\,dy = \int_0^1 dx \int_0^b e^{y\ln x}\,dy = \int_0^1 \frac{1}{\ln x}\left[x^y\right]_a^b dx = \int_0^1 \frac{x^b-x^a}{\ln x},$$

d'on es segueix que: $\boxed{\int_0^1 \frac{x^b-x^a}{\ln x}\,dx = I = \ln\frac{b+1}{a+1}}$

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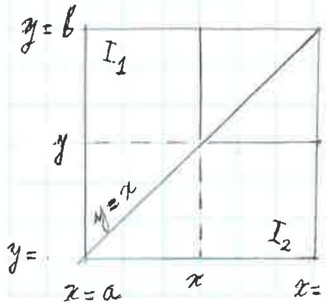
7) Proveu que $2\int_a^b\int_x^b f(x)f(y)\,dy\,dx = \left(\int_a^b f(x)\,dx\right)^2$. (Indicació $\left(\int_a^b f(x)\,dx\right)^2 = \iint_{[a,b]\times[a,b]} f(x)f(y)\,dx\,dy$).

Solució. $2\int_a^b\int_x^b f(x)f(y)\,dx\,dy = \int_a^b f(x)\,dx\int_x^b f(y)\,dy + \overbrace{\int_a^b f(y)\,dy\int_y^b f(x)\,dx}^{\text{canviant les variables } x\leftrightarrow y}$




$\overset{(*)}{=} \iint_{I_1} f(x)f(y)\,dx\,dy + \iint_{I_2} f(x)f(y)\,dx\,dy = \iint_{I_1\cup I_2} f(x)f(y)\,dx\,dy =$ pàgina següent

(*) On $I_1=\{(x,y)\in\mathbb{R}^2: a\le x\le y\le b\}$, $I_2=\{(x,y)\in\mathbb{R}^2 : a\le y\le x\le b\}$, i notem que $m(I_1\cap I_2)=0$, per tant $\iint_{I_1}+\iint_{I_2}=\iint_{I_1\cup I_2}$.

$$= \iint_{I_1 \cup I_2 = [a,b]\times[a,b]} f(x) f(y)\, dx\, dy = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(y)\,dy\right) = \left(\int_a^b f(x)\,dx\right)^2$$

9) Calculeu les següents integrals dobles en els dominis de $\mathbb{R}^2$ que s'indiquen

(a) $I = \iint_A y^3\, dx\, dy$, $A = \left\{ (x,y) \in \mathbb{R}^2 : -\frac{\pi}{2} \le x \le \frac{\pi}{2},\ 0 \le y \le 2\cos x \right\}$

Solució. $I = \int_{-\pi/2}^{\pi/2} dx \int_0^{2\cos x} y^3\, dy = 4 \int_{-\pi/2}^{\pi/2} \cos^4 x\, dx = 8 \int_0^{\pi/2} \cos^4 x\, dx = 8 \int_0^{\pi/2} \cos^2 x\,(1 - \sin^2 x)\, dx$

$$= 8 \int_0^{\pi/2} \left( \cos^2 x - \frac{1}{4} 4 \sin^2 x \cos^2 x \right) dx = 8 \int_0^{\pi/2} \left( \cos^2 x - \frac{1}{4} \sin^2(2x) \right) dx$$

$$= 8 \int_0^{\pi/2} \left( \frac{1 + \cos(2x)}{2} - \frac{1 - \cos(4x)}{8} \right) dx = 8 \left( \frac{\pi}{4} - \frac{\pi}{16} \right) = 8\, \frac{3\pi}{16} = \boxed{\frac{3\pi}{2}}$$

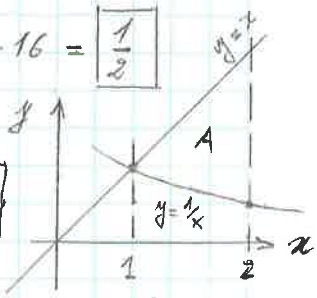
(b) $I = \iint_A x\, dx\, dy$, $A = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le e^x \right\}$

Solució. $I = \iint_A x\, dx\, dy = \int_0^1 x\, dx \int_0^{e^x} dy = \int_0^1 x e^x\, dx \overset{\text{integració per parts}}{=} (x e^x - e^x)\Big|_0^1 = \boxed{1}$.

(c) $I = \iint_A x y\, dx\, dy$, $A = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le \frac{x}{2} \right\}$

Solució. $I = \iint_A x y\, dx\, dy = \int_0^2 x\, dx \int_0^{x/2} y\, dy = \frac{1}{8} \int_0^2 x^3\, dx = \frac{1}{32} \cdot 16 = \boxed{\frac{1}{2}}$

(d) $I = \iint_A \frac{x^2}{y^2}\, dx\, dy$, $A = \left\{ (x,y) \in \mathbb{R}^2 : 1 \le x \le 2,\ \frac{1}{x} \le y \le x \right\}$




Solució. $I = \iint_A \frac{x^2}{y^2}\, dx\, dy = \int_1^2 x^2\, dx \int_{1/x}^{x} \frac{dy}{y^2} = \int_1^2 x^2 \left( x - \frac{1}{x} \right) dx = \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_1^2 = 4 - 2 - \frac{1}{4} + \frac{1}{2}$

$$= \boxed{\frac{9}{4}}$$

e) $I = \iint_A (x+2y)\,dx\,dy, \quad A = \{(x,y) \in \mathbb{R}^2 : -3 \leq y \leq 3,\ y^2-4 \leq x \leq 5\}$

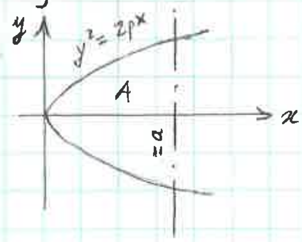
Solució. $I = \int_{-3}^{3} dy \int_{y^2-4}^{5} (x+2y)\,dx = \int_{-3}^{3} dy \left(\frac{x^2}{2} + 2yx\right)\Bigg|_{y^2-4}^{5} = \int_{-3}^{3} \left(\frac{9}{2} + 2y + 4y^2 + 2y^3 - \frac{1}{2}y^4\right) dy$

$= 2 \cdot \int_0^3 \left(\frac{9}{2} + 4y^2 - \frac{1}{2}y^4\right) dy = 2 \cdot \left(\frac{27}{2} + \frac{4}{3}27 - \frac{9}{10}27\right) = \frac{54}{30}(15+40-27)$

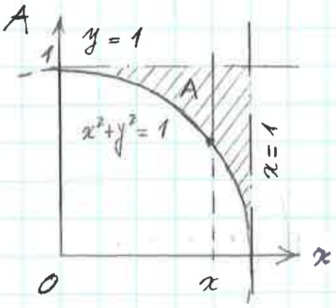
$= \frac{54}{30} \cdot 28 = \boxed{\frac{252}{5}}$.

f) $I = \iint_A y^3\,dx\,dy, \quad A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq a,\ y^2 \leq 2px\}, \ a>0,\ p>0$

Solució. $I = \int_0^a dx \int_{-\sqrt{2px}}^{\sqrt{2px}} y^3\,dy = \int_0^a \underbrace{\left[\frac{y^4}{4}\right]_{y=-\sqrt{2px}}^{y=\sqrt{2px}}}_{0} dy = \boxed{0}$




g) $I = \iint_A \frac{y}{1+x^3}\,dx\,dy, \quad A = \{(x,y) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq\ \leq 1,\ x^2+y^2 \geq 1\}$

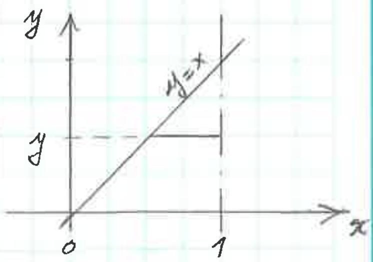



Solució:

$I = \iint_A \frac{y}{1+x^3}\,dx\,dy = \int_0^1 \frac{dx}{1+x^3} \int_{\sqrt{1-x^2}}^{1} y\,dy$

$= \frac{1}{2} \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{6} \ln(1+x^3)\Big|_0^1 = \frac{\ln 2}{6}$

h) $I = \iint_A x^2 \sin(xy)\,dx\,dy, \quad A = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1,\ y \leq x \leq 1\}$.



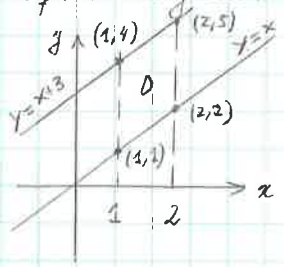

Solució. $I = \iint_A x^2 \sin(xy)\,dx\,dy = \int_0^1 x^2\,dx \int_0^x \sin(xy)\,dy$

$= -\int_0^1 x^2 \cdot \left[\frac{\cos(xy)}{x}\right]_{y=0}^{y=x} dx = -\int_0^1 x\left(\cos(x^2) - 1\right) dx$

$= -\left[-\frac{\sin(x^2)}{2} + \frac{x^2}{2}\right]_0^1 = \boxed{\frac{1-\sin(1)}{2}}$

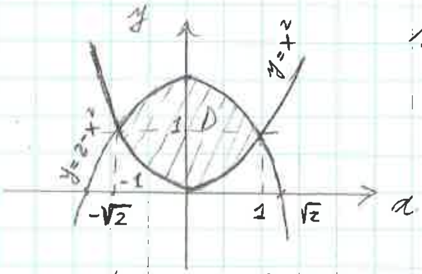
10) Per a les integrals iterades següents escriviu les equacions de les corbes que limiten les regions d'integració i dibuixeu aquestes regions.

(a) $\displaystyle\int_1^2 \left(\int_x^{x+3} f(x,y)\,dy\right)dx$.




Solució: $y=x,\ y=x+3,\ x=1,\ x=2$

(b) $\displaystyle\int_{-1}^{1} \left(\int_{x^2}^{2-x^2} f(x,y)\,dy\right)dx$

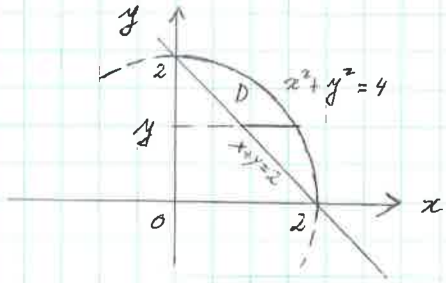



Solució.

$y=x^2,\ y=2-x^2,\ x=1,\ x=-1$ (veure figura).

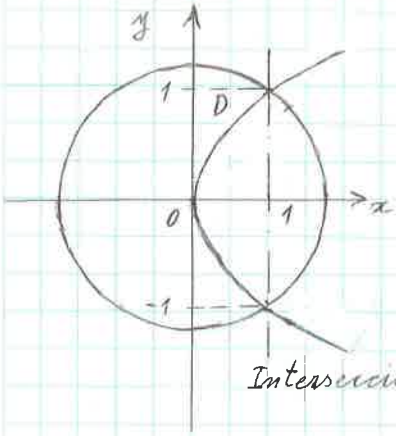
Intersecció: $2-x^2=x^2 \Leftrightarrow x=\pm 1,\ y=1$

(c) $\displaystyle\int_0^2 \left(\int_{2-y}^{\sqrt{4-y^2}} f(x,y)\,dx\right)dy$.




Solució:

$x+y=2,\ x^2+y^2=4,\ y=0,\ y=2$ (veure figura).




(d) $\displaystyle\int_0^1 \left(\int_{\sqrt{x}}^{\sqrt{2-x^2}} f(x,y)\,dy\right)dx$.

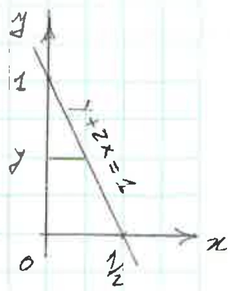
Solució:

$y^2=x,\ x^2+y^2=2,\ x=0,\ x=1$. (veure figura).

Intersecció: $\begin{matrix} x^2+y^2=2 \\ -x+y^2=0 \end{matrix}$ d'on: $x^2+x-2=0$, i llavors $x=\dfrac{-1\pm\sqrt{1+8}}{2}$: $\begin{matrix} -2 \text{ (No)} \\ 1 \text{ d'on } y=\pm 1 \end{matrix}$

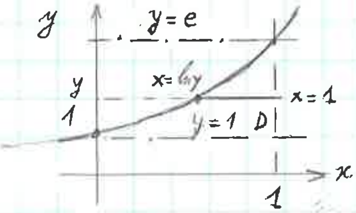
11) Invertiu l'ordre d'integració en les integrals iterades següents

(a) $\displaystyle I=\int_0^{1/2} \left(\int_0^{1-2x} f(x,y)\,dy\right)dx$.



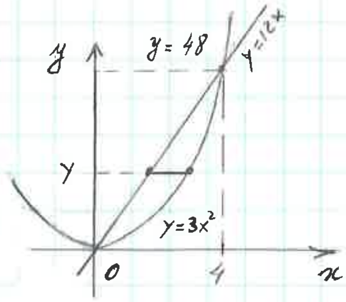

Solució: $\displaystyle I=\int_0^{1/2} \left(\int_0^{1-2x} f(x,y)\,dy\right)dx = \int_0^{1} \left(\int_0^{\frac{1-y}{2}} f(x,y)\,dx\right)dy$. □

(b) $I = \int_0^1 \left( \int_1^{e^x} f(x,y)\, dy \right) dx$




Solució. $I = \int_0^1 \left( \int_1^{e^x} f(x,y)\, dy \right) dx = \int_1^e \left( \int_{\ln y}^1 f(x,y)\, dx \right) dy$. □

c) $I = \int_0^4 \left( \int_{3x^2}^{12x} f(x,y)\, dy \right) dx$.

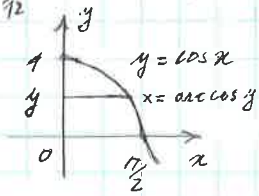



Intersecció: $3x^2 = 12x$
$\Leftrightarrow x=0,\ x=4$
$y=0,\ y=48$

Solució.

$$I = \int_0^4 \left( \int_{3x^2}^{12x} f(x,y)\, dy \right) dx = \int_0^{48} \left( \int_{y/12}^{\sqrt{y/3}} f(x,y)\, dx \right) dy \quad \square$$

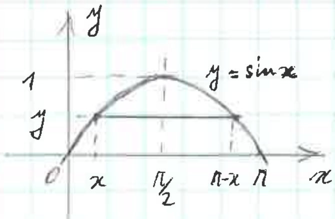
d) $I = \int_0^{\pi/2} \left( \int_0^{\cos x} f(x,y)\, dy \right) dx$




Solució.

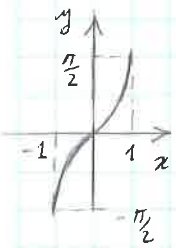
$$I = \int_0^{\pi/2} \left( \int_0^{\cos x} f(x,y)\, dy \right) dx = \int_0^1 \left( \int_0^{\arccos y} f(x,y)\, dx \right) dy. \ \square$$

e) $\int_0^{\pi} \left( \int_0^{\sin x} f(x,y)\, dy \right) dx$



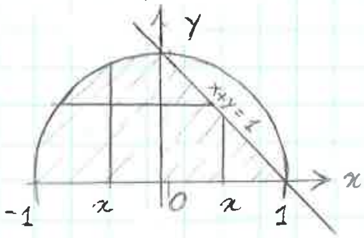

Solució. $I = \int_0^{\pi} \left( \int_0^{\sin x} f(x,y)\, dy \right) dx \overset{(*)}{=} \int_0^1 \left( \int_{\arcsin y}^{\pi - \arcsin y} f(x,y)\, dx \right) dy$ □

(*) Definim: $\arcsin: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ de manera que $v = \arcsin u$, amb $-1 \le u \le 1$, si i només si $u = \sin v$, amb $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$.




funció arcsin

f) $I = \int_0^1 \left( \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\, dx \right) dy$

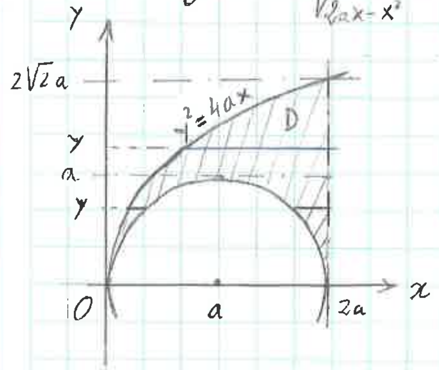



Solució

$$I = \int_0^1 \left( \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\, dx \right) dy = \int_{-1}^0 \left( \int_0^{\sqrt{1-x^2}} f(x,y)\, dy \right) dx + \int_0^1 \left( \int_0^{1-x} f(x,y)\, dy \right) dx$$

□

g) $I=\int_0^{2a}\left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y)\,dy\right)dx\ ,\ a>0.$




Solució.

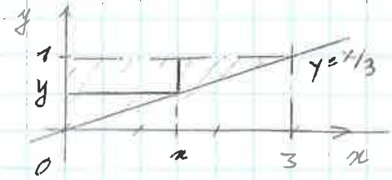
$$I=\int_0^{2a}\left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y)\,dy\right)dx$$

$$=\int_0^{a}\left(\int_{y^2/4a}^{a-\sqrt{a^2-y}} f(x,y)\,dx\right)dy+\int_0^{a}\left(\int_{a+\sqrt{a^2-y}}^{2a} f(x,y)\,dx\right)dy+\int_a^{2\sqrt{2}a}\left(\int_{y^2/4a}^{2a} f(x,y)\,dx\right)dy.\ \square$$

Intersecció:

$\begin{cases}(x-a)^2+y^2=a^2\\ y^2=4ax\end{cases}$ d'on: $x^2-2ax+a^2+4ax=a^2 \Leftrightarrow x(x+2a)=0 \Leftrightarrow \begin{cases}x=0\\ x=-2a<0\ (No)\end{cases}$

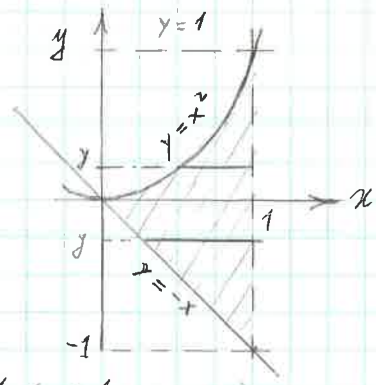
h) $I=\int_0^{3}\left(\int_{x/3}^{1} f(x,y)\,dy\right)dx$




Solució.

$$I=\int_0^{3}\left(\int_{x/3}^{1} f(x,y)\,dy\right)dx=\int_0^{1}\left(\int_0^{3y} f(x,y)\,dx\right)dy\ \square$$

i) $I=\int_0^{1}\left(\int_{-x}^{x^2} f(x,y)\,dy\right)dx$




Solució.

$$I=\int_0^{1}\left(\int_{-x}^{x^2} f(x,y)\,dy\right)dx$$

$$=\int_{-1}^{0}\left(\int_{-y}^{1} f(x,y)\,dx\right)dy+\int_0^{1}\left(\int_{\sqrt{y}}^{1} f(x,y)\,dx\right)dy\ \square$$

13) Calculeu les següents integrals iterades.

(a) $I = \int_1^2 \int_0^1 \int_0^{\pi/2} x^2 y^3 \sin z \, dz\, dy\, dx$

Solució.

$$I = \int_1^2 \int_0^1 \int_0^{\frac{\pi}{2}} x^2 y^3 \sin z \, dz\, dy\, dx = \int_0^1 x^2 dx \int_0^1 y^3 dy \int_0^{\frac{\pi}{2}} \sin z \, dz$$

$$= \left[\frac{x^3}{3}\right]_1^2 \cdot \left[\frac{y^4}{4}\right]_0^1 \cdot \left(-\cos z\right]_0^{\pi/2} = \left(\frac{8}{3} - \frac{1}{3}\right) \cdot \frac{1}{4} = \boxed{\frac{7}{12}}$$

(b) $I = \int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} z\, dz\, dy\, dx$

Solució.

$$I = \int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} z\, dz\, dy\, dx = \frac{1}{2}\int_0^1 dx \int_0^x (x^2+y^2)\, dy = \frac{1}{2}\int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{y=x} dx$$

$$= \frac{1}{2} \cdot \frac{4}{3} \int_0^1 x^3 dx = \boxed{\frac{1}{6}}$$

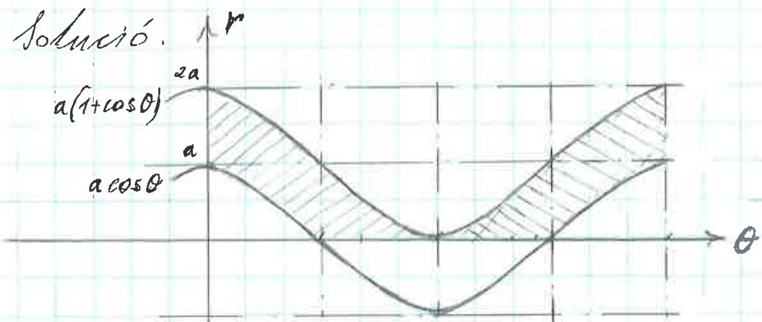
(c) $$I = \int_0^3 \int_0^{2x} \int_0^{\sqrt{xy}} z\, dz\, dy\, dx = \int_0^3 dx \int_0^{2x} dy \int_0^{\sqrt{xy}} z\, dz = \frac{1}{2}\int_0^3 x\, dx \int_0^{2x} y\, dy$$

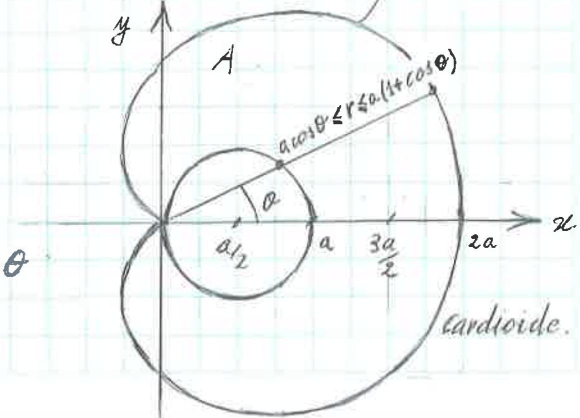
$$= \int_0^3 x^3 dx = \boxed{\frac{81}{4}}$$

17) Calculeu les àrees dels dominis $A \subset \mathbb{R}^2$ definits en coordenades polars, $x = r\cos\theta$, $y = r\sin\theta$, que s'indiquen tot seguit.

(a) A figura definida per $a\cos\theta \leq r \leq a(1+\cos\theta)$ $(a>0)$. (Indicació: Observeu que l'expressió té sentit quan $\cos\theta \geq 0$. Dibuixar les gràfiques de $a\cos\theta$ i $a(1+\cos\theta)$ pot ajudar a veure els valors de $r$ admissibles.)

Solució.



**Dibuix amb Matlab/Octave**

```
>> a=2; % per exemple
>> theta = linspace (0, 2*pi, 201);
>> rho1= a*cos(theta);
>> rho2= a*(1+cos(theta));
>> polar (theta, rho1, 'b-');
>> axis equal
>> hold on
>> polar(thetha, rho2, 'r-')
>> hold off
```

$$I_1 = \int_0^{\pi/2} d\theta \int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr + \int_{3\pi/2}^{\pi/2} d\theta \int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr$$

$$= 2\int_0^{\pi/2} d\theta \int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr = a^2 \int_0^{\pi/2} (1+2\cos\theta)\,d\theta$$

$$= a^2\left(2+\frac{\pi}{2}\right).$$

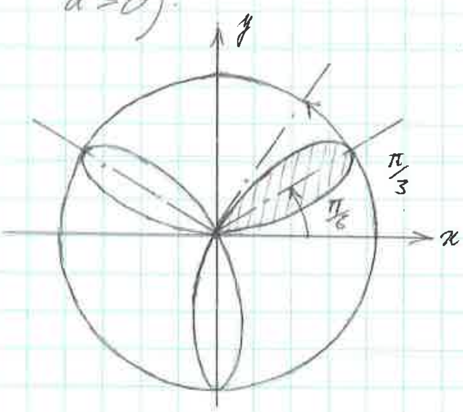
$$I_2 = 2\int_{\pi/2}^{\pi} d\theta \int_0^{a(1+\cos\theta)} r\,dr = a^2 \int_{\pi/2}^{\pi} (1+\cos\theta)^2\,d\theta$$

$$= a^2 \int_{\pi/2}^{\pi} (1+2\cos\theta+\cos^2\theta)\,d\theta = a^2\left(\frac{\pi}{2}-2+\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$= a^2\left(\frac{3\pi}{4}-2\right)$$

$$\therefore\ I = \iint_A dx\,dy = I_1 + I_2 = a^2\left(2+\frac{\pi}{2}+\frac{3\pi}{4}-2\right) = \boxed{a^2\,\frac{5\pi}{4}}.$$

(b) $A$ regió limitada per un pètal de la rosa definit per $r = a\sin 3\theta$ $\left(0 \le \theta \le \frac{\pi}{3},\ a>0\right)$.

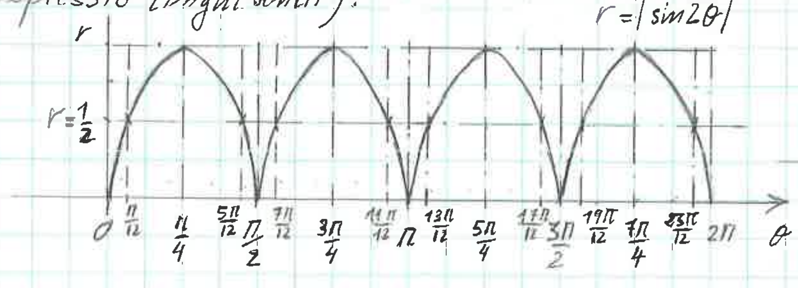


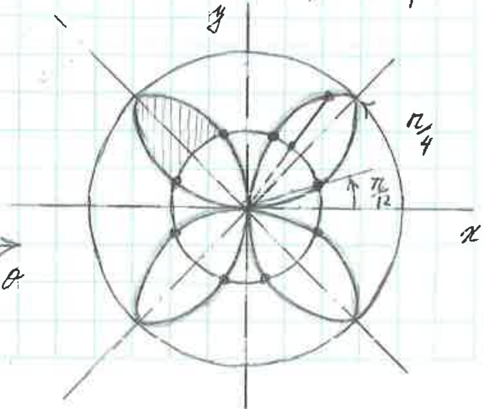

Solució.

$$A = \int_0^{\pi/3} d\theta \int_0^{a\sin(3\theta)} r\,dr = \frac{a^2}{2}\int_0^{\pi/3} \frac{1-\cos(6\theta)}{2}\,d\theta$$

$$= \boxed{\frac{a^2\pi}{12}}.$$

(c) $A$ regió definida per $\frac{1}{2} \le r \le |\sin(2\theta)|$. (Indicació: Cal $|\sin(2\theta)| \ge \frac{1}{2}$ perquè l'expressió tingui sentit).



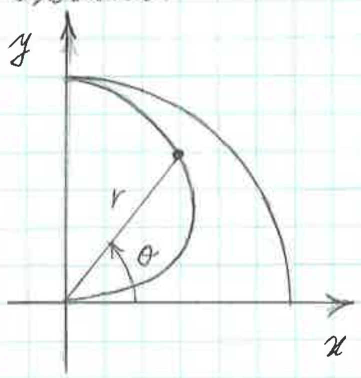
$$A = 4\int_{\pi/12}^{5\pi/12} d\theta \int_{1/2}^{|\sin 2\theta|} r\,dr = 2\int_{\pi/12}^{5\pi/12}\left(\sin^2 2\theta - \frac{1}{4}\right)d\theta = 2\int_{\pi/12}^{5\pi/12}\left(\frac{1-\cos 4\theta}{2} - \frac{1}{4}\right)d\theta$$

$$= 2\int_{\pi/12}^{5\pi/12} \frac{1-2\cos 4\theta}{4}\,d\theta = 2\left[\frac{\theta}{4} - \frac{\sin 4\theta}{8}\right|_{\pi/12}^{5\pi/12} = 2\left(\frac{5\pi}{48} - \frac{1}{8}\sin\left(\frac{5\pi}{3}\right) - \frac{\pi}{48} + \frac{1}{8}\sin\frac{\pi}{3}\right)$$

$$= 2\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right) = \frac{2\pi + 3\sqrt{3}}{12} = \boxed{\frac{\pi}{6} + \frac{\sqrt{3}}{4}}$$

(d) Anàlogament, calculeu la integral doble $\iint_A \arcsin(x^2+y^2)\,dx\,dy$, on $A$ és la regió limitada per la corba $r = \sqrt{\sin\theta}$ $(0 \leq \theta \leq \pi/2)$.

Solució.




$$A = \int_0^{\pi/2} d\theta \int_0^{\sqrt{\sin\theta}} r\arcsin(r^2)\,dr = \left\{\begin{array}{l}\text{Integració per parts:}\\ f = \arcsin r^2 \Rightarrow f' = \frac{2r}{\sqrt{1-r^4}}\\ g' = r \Rightarrow g = \frac{r^2}{2}\end{array}\right.$$

$$= \int_0^{\pi/2} d\theta \left\{\left[\frac{r^2}{2}\arcsin(r^2)\right]_0^{\sqrt{\sin\theta}} - \int_0^{\sqrt{\sin\theta}} \frac{r^3}{\sqrt{1-r^4}}\,dr\right\}$$

$$\overset{(*)}{=} \int_0^{\pi/2} \frac{1}{2}\left(\theta\sin\theta + \cos\theta - 1\right)d\theta = \left\{\begin{array}{l}\text{parts:}\\ f = \theta \Rightarrow f' = 1\\ g' = \sin\theta \Rightarrow g = -\cos\theta\end{array}\right.$$

$$= \frac{1}{2}\left[-\theta\cos\theta\right]_0^{\pi/2} + \int_0^{\pi/2}\left(\cos\theta - \frac{1}{2}\right)d\theta = \left[\sin\theta - \frac{\theta}{2}\right|_0^{\pi/2} = \boxed{1 - \frac{\pi}{4}}.$$

(*) Nota:

$$\int_0^{\sqrt{\sin\theta}} \frac{r^3}{\sqrt{1-r^4}}\,dr = -\frac{1}{2}\sqrt{1-r^4}\Big|_0^{\sqrt{\sin\theta}} = \frac{1}{2} - \frac{1}{2}|\cos\theta| = \frac{1}{2} - \frac{1}{2}\cos\theta$$

18) Calculeu les integrals dobles, fent servir el canvi de variable adequat en cada cas.

(a) $\iint_D xy\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 : 6 \leq 2y - x \leq 12,\ 0 \leq x \leq 4\}$.

Solució. Introduïm el canvi: $\begin{array}{l} x = 4u \\ 6v = 2y - x \end{array} \Leftrightarrow y = 3v + 2u$,

és a dir: $(u,v) \in \mathbb{R}^2 \xrightarrow{T} (x,y) = T(u,v) = (4u, 3v+2u)$, d'on $\det DT(u,v) = \begin{vmatrix} 4 & 0 \\ 2 & 3 \end{vmatrix} = 12$.

D'altra banda: $6 \leq 6v = 2y - x \leq 12$, $0 \leq 4u \leq 4$ $\Leftrightarrow$ $0 \leq u \leq 1$, $1 \leq v \leq 2$, és a dir el domini transformat és, $D' = T^{-1}(D) = [0,1] \times [1,2]$. Aleshores:

$$I = \iint_D xy\,dx\,dy = \iint_{D'=T^{-1}(D)} x(u,v)\cdot y(u,v)\,\left|\det DT(u,v)\right|\,du\,dv = 12\iint_{D'=[0,1]\times[1,2]} 4u(3v+2u)\,du\,dv$$

$$= 48\int_1^2 dv\int_0^1 (3uv+2u^2)\,du = 48\int_1^2 \left[\tfrac{3}{2}u^2v+\tfrac{2}{3}u^3\right]_{u=0}^{u=1} dv = 48\int_1^2\left(\tfrac{3}{2}v+\tfrac{2}{3}\right)dv$$

$$= 48\left[\tfrac{3}{4}v^2+\tfrac{2}{3}v\right]_1^2 = 48\left(3+\tfrac{4}{3}-\tfrac{3}{4}-\tfrac{2}{3}\right) = 48\,\frac{36+8-9}{12} = 4\cdot 35 = \boxed{140}$$

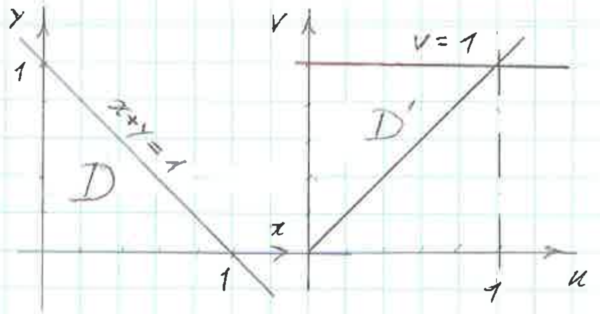
(b) $I=\iint_D \frac{1}{(1+x+y)^5}\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2:\ x\geq 0,\ y\geq 0,\ x+y\leq 1\}$

*Solució.* Fem el canvi: $\begin{matrix} u=x \\ v=x+y\end{matrix} \Leftrightarrow \begin{matrix} x=u \\ y=v-u\end{matrix}$, i.e.: $(u,v)\in\mathbb{R}^2 \overset{T}{\longmapsto} (x,y)=T(u,v)=(u,v-u)$,

$DT(u,v)=\begin{vmatrix} 1 & 0 \\ -1 & 1\end{vmatrix}=1$,

d'altra banda, el domini $D$ es transforma (veure figura) en:

$$D'=T^{-1}(D)=\{(u,v)\in\mathbb{R}^2:\ 0\leq u\leq 1,\ u\leq v\leq 1\}.$$



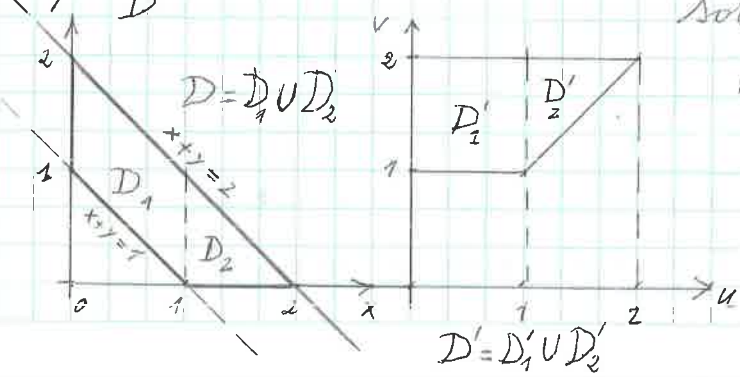

I podem calcular la integral,

$$I=\iint_D\frac{1}{(1+x+y)^5}\,dx\,dy = \iint_{D'=T^{-1}(D)} \frac{1}{(1+x(u,v)+y(u,v))^5}\left|\det DT(u,v)\right|$$

$$= \iint_{D'}\frac{1}{(1+v)^5}\,du\,dv = \int_0^1 du\int_u^1\frac{dv}{(1+v)^5} = \int_0^1\left[\frac{(1+v)^{-4}}{-4}\right]_u^1 du$$

$$= \frac{1}{4}\int_0^1\left[\frac{1}{(1+u)^4}-\frac{1}{16}\right]du = \frac{1}{4}\left[\frac{(1+u)^{-3}}{-3}-\frac{u}{16}\right]_0^1 = \frac{1}{4}\left(\frac{1}{3}-\frac{1}{24}-\frac{1}{16}\right) = \boxed{\frac{11}{192}}$$

(c) $I=\iint_D\frac{dx\,dy}{(x+y)^{n+1}}$, $D=\{(x,y)\in\mathbb{R}^2:\ 1\leq x+y\leq 2,\ x\geq 0,\ y\geq 0\}$.




*Solució.*

Considerem el canvi: $\begin{matrix} u=x \\ v=x+y\end{matrix} \Leftrightarrow \begin{matrix} x=u \\ y=v-u\end{matrix}$,

i.e.:

$(u,v)\in\mathbb{R}^2 \overset{T}{\longmapsto} (x,y)=T(u,v)=(u,v-u)$.

$\det DT(u,v)=\begin{vmatrix} 1 & 0 \\ -1 & 1\end{vmatrix}=1$, mentre que els dominis es transformen com: $D' = T^{-1}(D) = \{(u,v)\in\mathbb{R}^2 :\ u\le v,\ 1\le v\le 2\}$, i podem calcular $I$ fent:

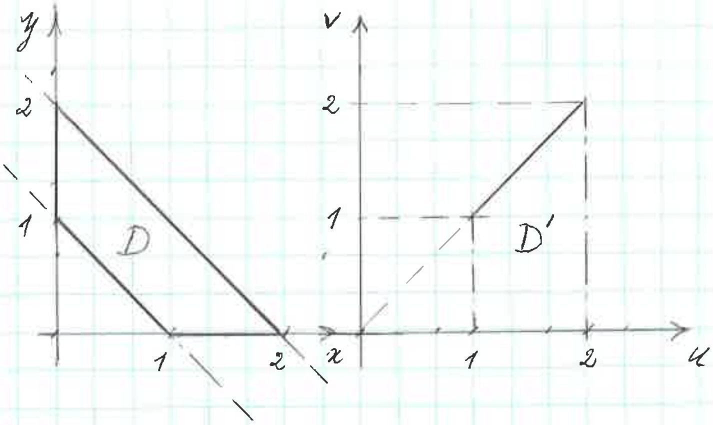
$$I = \iint_D \frac{dx\,dy}{(x+y)^{n+1}} = \iint_{D'=T^{-1}(D)} \frac{1}{\big(x(u,v)+y(u,v)\big)^{n+1}}\,\big|\det DT(u,v)\big| \cdot du\,dv$$

$$= \int_1^2 \frac{dv}{v^{n+1}} \int_0^v du = \int_1^2 \frac{dv}{v^{n}} = \left[\frac{v^{-n+1}}{-n+1}\right]_1^2 = \boxed{\frac{1}{n-1}\left(1-\frac{1}{2^{n-1}}\right)}$$

Alternativament, podem fer el canvi:

$$(u,v)\in\mathbb{R}^2 \xmapsto{T} (x,y) = T(u,v) = (v,\ u-v);$$

el qual transforma el domini $D$ en $D' = T^{-1}(D) = \{(u,v)\in\mathbb{R}^2:\ v\le u,\ 1\le u\le 2\}$, (veure figura al costat).




Tanmateix:

$$\det DT(u,v) = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1.$$

i llavors el càlcul de la integral resulta:

$$I = \iint_D \frac{dx\,dy}{(x+y)^{n+1}} = \int_{D'=T^{-1}(D)} \frac{1}{\big(x(u,v)+y(u,v)\big)^{n+1}}\,\big|\det DT(u,v)\big|\,du\,dv$$

$$= \int_1^2 \frac{du}{u^{n+1}} \int_0^u dv = \int_1^2 \frac{du}{u^n} = \left[\frac{u^{-n+1}}{-n+1}\right]_1^2 = \frac{1}{n-1} - \frac{2^{-n+1}}{n-1} = \boxed{\frac{1}{n-1}\left(1-\frac{1}{2^{n-1}}\right)}$$

(d) $I = \displaystyle\iint_D \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2} dx\,dy,\quad D = \left\{(x,y)\in\mathbb{R}^2:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}.$

Solució. Fem servir coordenades polars adaptades: $\begin{matrix} x = |a|\, r\cos\theta \\ y = |b|\, r\sin\theta \end{matrix}$, que transformen el domini $D$ en $D'$: $0\le r\le 1$, $0\le\theta\le 2\pi$, mentre que $\det\dfrac{\partial(x,y)}{\partial(r,\theta)} = |ab|\,r$. Aleshores:

$$I = \iint_D \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2} dx\,dy = \iint_{D'} \left(1 - x^2(r,\theta)/a^2 - y^2(r,\theta)/b^2\right)^{3/2} \left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= |ab| \int_0^{2\pi} d\theta \int_0^1 r(1-r^2)^{3/2}\,dr = 2\pi|ab|\left[\left(-\tfrac{1}{5}\right)(1-r^2)^{5/2}\right]_0^1 = \boxed{\frac{2\pi}{5}|ab|}$$

(e) $I=\iint_D \arctan\left(x^2+\frac{y^2}{2}\right)dx\,dy$, $D=\left\{(x,y)\in\mathbb{R}^2 : x^2+\frac{y^2}{2}\le 1,\ x\ge 0,\ y\ge 0\right\}$.

Solució. Apliquem, de nou, coordenades polars adaptades, aquest cop de la forma $(r,\theta)\in[0,+\infty)\times\left[0,\frac{\pi}{2}\right] \xrightarrow{T} (x,y)=T(r,\theta)=(r\cos\theta,\ \sqrt{2}\,r\sin\theta)$, amb la qual cosa el domini es transforma en: $D'=T^{-1}(D)=[0,1]\times\left[0,\frac{\pi}{2}\right]$, i podem calcular $I$ tot fent:

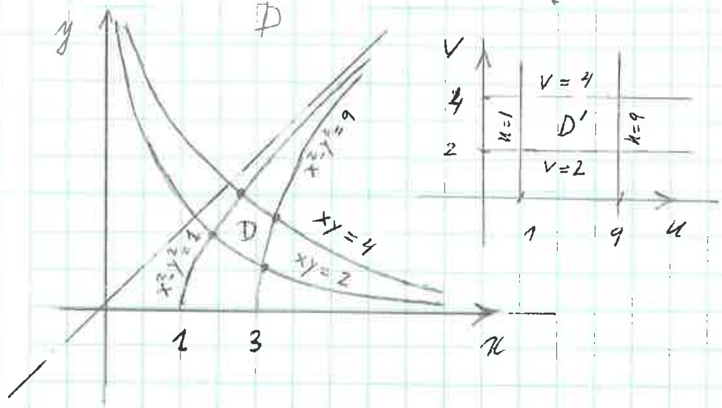
$$I=\iint_D \arctan\left(x^2+\tfrac{y^2}{2}\right)dx\,dy=\iint_{D'=T^{-1}(D)} \arctan\left(x^2(r,\theta)+y^2(r,\theta)/2\right)\underbrace{\left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right|}_{=\sqrt{2}\,r}dr\,d\theta=$$

$$\sqrt{2}\int_0^{\pi/2}d\theta\int_0^1 r\arctan(r^2)\,dr=\frac{\pi\sqrt{2}}{2}\int_0^1 r\arctan(r^2)\,dr=\begin{cases} f=\arctan(r^2)\Rightarrow f'=\frac{2r}{1+r^4}\\ g'=r\Rightarrow g=r^2/2\end{cases}$$

$$=\frac{\sqrt{2}\pi}{4}\Big[r^2\arctan(r^2)\Big|_0^1-\frac{\sqrt{2}\pi}{2}\int_0^1\frac{r^3}{1+r^4}dr=\frac{\sqrt{2}\pi^2}{16}-\frac{\pi\sqrt{2}}{8}\ln(1+r^4)\Big|_0^1=$$

$$=\frac{\sqrt{2}\pi}{2}-\frac{\sqrt{2}\pi}{8}\ln 2=\boxed{\frac{\pi\sqrt{2}}{8}\left(\frac{\pi}{2}-\ln 2\right)}.$$

(f) $I=\iint_D (x^2+y^2)\,dx\,dy$, $D=\left\{(x,y)\in\mathbb{R}^2 : 1\le x^2-y^2\le 9,\ 2\le xy\le 4,\ x\ge 0,\ y\ge 0\right\}$




Solució. Fem el canvi $T$, on $T^{-1}$ ve donat per: $\begin{cases}u=x^2-y^2\\ v=xy\end{cases}$, d'on: $\begin{cases}1\le u=x^2-y^2\le 9\\ 2\le v=xy\le 4\end{cases}$.

Llavors tenim, pel domini transformat:

$$D'=T^{-1}(D)=[1,9]\times[2,4],$$

(veure figura).

D'altra banda, pel determinant del Jacobià, tenim:

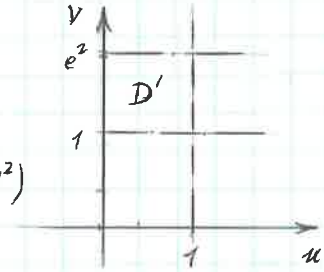
$$\det\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}2x & -2y\\ y & x\end{vmatrix}=2(x^2+y^2),\ \text{d'on: }\ \det\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\det\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1/2}{x^2+y^2},$$

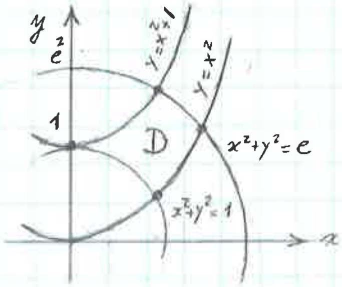
i podem calcular la integral fent,

$$I=\iint_D (x^2+y^2)\,dx\,dy=\iint_{D'=T(D)}\left(x^2(u,v)+y^2(u,v)\right)\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=\frac{1}{2}\iint_{D'=[1,9]\times[2,4]}\frac{x^2(u,v)+y^2(u,v)}{x^2(u,v)+y^2(u,v)}du\,dv$$

$$=\frac{1}{2}\left(\int_1^9 du\right)\cdot\left(\int_2^4 dv\right)=\frac{1}{2}(9-1)\cdot(4-2)=\boxed{8}.$$

$$\iint_D \frac{x+2xy}{x^2+y^2}\,dx\,dy, \quad D=\{(x,y)\in\mathbb{R}^2:\ x^2\le y\le x^2+1,\ 1\le x^2+y^2\le e^2,\ x\ge 0\}.$$




*Introduïm el canvi:*

$$(x,y)\in\mathbb{R}^2 \xrightarrow{T^{-1}} (u,v)=T^{-1}(x,y)=(y-x^2,\ x^2+y^2)$$

*Llavors:*

$0\le u=y-x^2\le 1$

$1\le v=x^2+y^2\le e^2$ d'on, clarament: $D'=T^{-1}(D)=[0,1]\times[1,e^2]$, mentre que el determinant del Jacobià de $T$ es pot calcular a partir del de $T^{-1}$. En efecte:

$$\det DT^{-1}(x,y)=\det\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}-2x & 1\\ 2x & 2y\end{vmatrix}=-4xy-2x=-2(x+2xy),$$

d'on:

$$\det DT(u,v)=\det\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\det\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\det DT^{-1}(x,y)}=\frac{-1/2}{x+2xy},$$

i el càlcul de la integral $I$ es redueix a:

$$I=\iint_D\frac{x+2xy}{x^2+y^2}\,dx\,dy=\iint_{D'=T^{-1}(D)}\frac{x(u,v)+2x(u,v)y(u,v)}{x^2(u,v)+y^2(u,v)}\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

$$=\iint_{D'=T(D)=[0,1]\times[1,e^2]}\frac{x(u,v)+2x(u,v)y(u,v)}{x^2(u,v)+y^2(u,v)}\cdot\frac{1/2}{|x(u,v)+2x(u,v)y(u,v)|}\,du\,dv$$

$$\overset{(*)}{=}\iint_{D'=[0,1]\times[1,e^2]}\frac{\frac12\,du\,dv}{v}=\frac12\left(\int_0^1 du\right)\cdot\left(\int_1^{e^2}\frac{dv}{v}\right)=$$

(*) $x(u,v),\ y(u,v)\ge 0\quad \forall (u,v)\in D'=[0,1]\times[1,e^2]$.

Llavors: $|x(u,v)+2x(u,v)y(u,v)|=x(u,v)+2x(u,v)y(u,v)$.

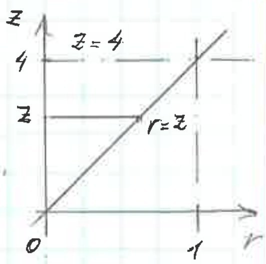
$$=\frac12\left(\ln e^2-\ln 1\right)=\boxed{1}$$

---

19) *Useu coordenades cilíndriques per calcular les següents integrals triples.*

*Nota.* Coordenades cilíndriques: $x=r\cos\theta$, $y=r\sin\theta$, $z=z$, amb $r\in[0,+\infty)$, $\theta\in[0,2\pi]$

$z\in(-\infty,+\infty)$, $\det\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}=r\ge 0$.

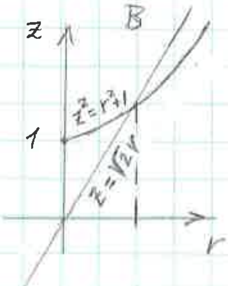



(a) $I=\displaystyle\iiint_B\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3:\ \sqrt{x^2+y^2}\le z\le 4\}$.

*Solució.* $I=\displaystyle\int_0^{2\pi}d\theta\int_0^4 dz\int_0^z r\sqrt{r^2+z^2}\,dr=\frac{2\pi}{3}\int_0^4\left(z^2+r^2\right)^{3/2}\Big|_0^z dz$

$$=\frac{2\pi}{3}(\sqrt{2}-1)\int_0^4 z^3\,dz==\frac{2\pi}{3}(2\sqrt{2}-1)\left[\frac{z^4}{4}\right]_0^4=\boxed{\frac{128\pi}{3}(2\sqrt{2}-1)}.$$

(b) $I = \iiint_B z\, e^{-(x^2+y^2)}\, dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : z^2-1 \le x^2+y^2 \le \frac{z^2}{2},\ z\ge 0\}$




Solució. En coordenades cilíndriques el domini B es transforma en

B': $\sqrt{2}\,r \le z \le \sqrt{r^2+1}$, $0\le r\le 1$.

La integral I en coordenades cilíndriques es calcula com:

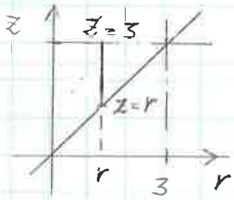
$$I = \int_0^{2\pi} d\theta \int_0^1 r e^{-r^2} dr \int_{\sqrt{2}r}^{\sqrt{r^2+1}} z\,dz = \frac{1}{2}2\pi \int_0^1 r e^{-r^2}(r^2+1-2r^2)\,dr$$

$$= \pi \int_0^1 r(1-r^2)e^{-r^2}dr = \left\{ \begin{array}{l} \text{parts: } f=1-r^2 \Rightarrow f'=-2r \\ g'=re^{-r^2} \Rightarrow g=-e^{-r^2}/2 \end{array} \right\} =$$

$$= \underbrace{\pi\left(\frac{r^2-1}{2}\right)e^{-r^2}\Big|_0^1}_{=\,\pi/2} - \underbrace{\pi\int_0^1 re^{-r^2}dr}_{=\,-\frac{\pi}{2}\left[e^{-r^2}\right|_0^1 \,=\, -\frac{\pi}{2}\left(\frac{1}{e}-1\right)} = \cancel{\frac{\pi}{2}} + \frac{\pi}{2e} - \cancel{\frac{\pi}{2}} = \boxed{\frac{\pi}{2e}}$$

$z^2=r^2+1$ ; intersecció : $r=1$ $(r>0)$
$z^2=2r^2$ $\qquad z=\pm\sqrt{2}$

(c) $I = \iiint_B (x+y-2z)\,dx\,dy\,dz$. $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le z^2,\ 0\le z\le 3\}$.




Solució. En coordenades cilíndriques el domini B es transforma en $B'=\{(\theta,r,z)\in[0,2\pi]\times[0,+\infty)\times(-\infty,+\infty),\ 0\le r\le z,\ 0\le z\le 3\}$, d'on la integral es pot calcular com:

$$I = \int_0^{2\pi} d\theta \int_0^3 r\,dr \int_r^3 (r\cos\theta + r\sin\theta - 2z)\,dz = \int_0^{2\pi} d\theta \int_0^3 r\left(rz\cos\theta + rz\sin\theta - z^2\right)\Big|_{z=r}^{z=3} dr$$

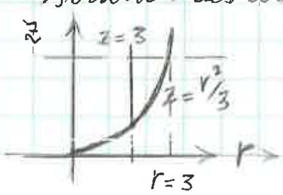
$$= \int_0^{2\pi} d\theta \int_0^3 r\left(3r\cos\theta + 3r\sin\theta - 9 - r^2\cos\theta - r^2\sin\theta + r^2\right) dr$$

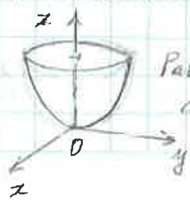
$$= \int_0^{2\pi} d\theta \int_0^3 \left[-9r + 3(\cos\theta+\sin\theta)r^2 - (\cos\theta+\sin\theta-1)r^3\right] dr$$

$$= \int_0^{2\pi} \left[-\frac{81}{2} + (\cos\theta+\sin\theta)\,27 - (\cos\theta+\sin\theta-1)\,\frac{81}{4}\right] d\theta = 2\pi\left(-\frac{81}{2}+\frac{81}{4}\right) = \boxed{-\frac{81\pi}{4}}.$$

(d) $I = \iiint_B (x^2+y^2)\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 3z\le 9\}$

Solució. Les coordenades cilíndriques transformen el domini d'integració B, en B': $\frac{r^2}{3}\le z\le 3$,



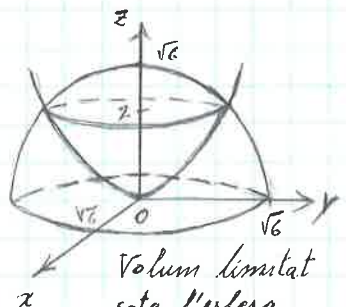




Paraboloide circular.

d'on: $I = \int_0^{2\pi} d\theta \int_0^3 r^3 dr \int_{r^2/3}^3 dz = 2\pi \int_0^3 r^3\left(3-\frac{r^2}{3}\right) dr$

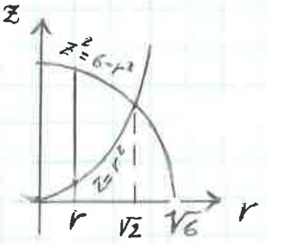
$$= 2\pi\left(\frac{3}{4}r^4 - \frac{r^6}{18}\right)\Big|_0^3 = 2\pi\left(\frac{3\cdot 81}{4} - \frac{3^2\cdot 81}{2\cdot 3^2}\right) = \frac{2\pi}{4}(3\cdot 81 - 2\cdot 81) = \frac{81\pi}{2}$$

(e) $I = \iiint_B z\,dx\,dy\,dz$, $B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 6,\ x^2+y^2 \leq z,\ z \geq 0\}$.




Volum limitat sota l'esfera $x^2+y^2+z^2=6$ i sobre el paraboloide circular $z=x^2+y^2$:

$x^2+y^2 \leq z \leq \sqrt{6-x^2-y^2}$




$z^2=6-r^2$, $z=r^2 \iff r=\sqrt{2}$, $z=2$

Solució. El domini $B$ es transforma, per coordenades cilíndriques en

$B'$ : $r^2 \leq z \leq \sqrt{6-r^2}$, $0 \leq \theta \leq 2\pi$,

i la integral $I$ es pot calcular com

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r\,dr \int_{r^2}^{\sqrt{6-r^2}} z\,dz$$

$$= \frac{2\pi}{2}\int_0^{\sqrt{2}} r(6-r^2-r^4)\,dr = \pi\left(3r^2 - \frac{r^4}{4} - \frac{r^6}{6}\right)\Big|_0^{\sqrt{2}}$$

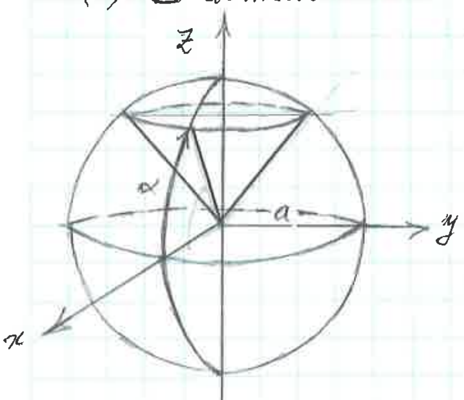
$$= \pi\left(6 - \frac{4}{4} - \frac{4}{3}\right) = \pi\left(5 - \frac{4}{3}\right) = \boxed{\frac{11\pi}{3}}.$$

---

21) Calculeu els volums dels dominis $B \subset \mathbb{R}^3$ definits en coordenades esfèriques, $x = r\cos\varphi\cos\theta$, $y = r\cos\varphi\sin\theta$, $z = r\sin\varphi$ ($0 \leq \theta \leq 2\pi$, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$, $r \geq 0$) que s'indiquen tot seguit.

Remarca. Recordem que: $\det \dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\cos\varphi$.

(a) $B$ domini tallat sobre la bola $r \leq a$ pel con $\alpha \leq \varphi \leq \frac{\pi}{2}$ ($a>0$, $0<\alpha<\frac{\pi}{2}$).




Solució:

$$V = \int_0^{2\pi} d\theta \int_\alpha^{\pi/2} \cos\varphi\,d\varphi \int_0^a r^2\,dr = \boxed{\frac{2\pi a^3}{3}(1-\sin\alpha)}$$

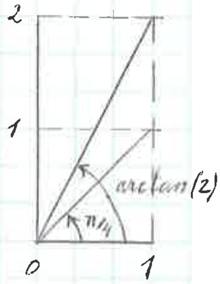
(b) $B$ volum tancat per l'esfera deformada definida per $r = 1+0.2\sin(8\theta)\sin\varphi$. (Sòlids d'aquesta mena s'utilitzen com a models de tumors.)

Solució
$$V = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\,d\varphi \int_0^{1+0.2\sin(8\theta)\sin\varphi} r^2\,dr = \frac{1}{3}\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\,(1+0.2\ (8\theta)\sin\varphi)^3\,d\varphi$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \left(\cos\varphi + 0.2\sin(8\theta)\sin\varphi\cos\varphi + 3\cdot 0'2^2\sin^2(8\theta)\sin^2\varphi\cos\varphi + 0'2^3\sin^3(8\theta)\cdot\sin^3\varphi\cos\varphi\right) d\varphi$$

$$= \frac{1}{3}\int_0^{2\pi}\left(2 + 3\cdot 0'04\cdot\frac{2}{3}\sin^2(8\theta)\right) d\theta = \frac{1}{3}\left(4\pi + 0'08\pi\right) = \boxed{1'36\pi}.$$

(c) Analogament, calculeu la integral triple $I=\iiint_B \frac{1}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$, on B és la regió del primer octant de $\mathbb{R}^3$ acotada pels cons $\varphi=\pi/4$ i $\varphi=\arctan(2)$ i l'esfera $r=\sqrt{6}$ (Recordeu: $\sin(\arctan(a)) = \frac{a}{\sqrt{1+a^2}}$.)




Solució

$$I=\int_0^{\pi/2} d\theta \int_{\pi/4}^{\arctan(2)} \cos\varphi\, d\varphi \int_0^{\sqrt{6}} \frac{1}{r} r^2\, dr = \frac{\pi}{2}\Big[\sin\varphi\Big|_{\pi/4}^{\arctan(2)} \cdot \Big[\frac{r^2}{2}\Big|_0^{\sqrt{6}}$$

$$= \frac{\pi}{2}\cdot\frac{6}{2}\left(\sin(\arctan 2)-\sin\left(\frac{\pi}{4}\right)\right) \overset{(*)}{=} \boxed{\frac{3\pi}{2}\left(\frac{2}{\sqrt{5}}-\frac{\sqrt{2}}{2}\right)}$$

---

(*) $\sin(\arctan a) = \frac{\sin(\arctan a)\cos(\arctan a)}{\cos(\arctan a)} = \tan(\arctan a)\frac{1}{\sqrt{1+\tan^2(\arctan a)}} = \frac{a}{\sqrt{1+a^2}}$,
$(-\frac{\pi}{2}\le a\le \frac{\pi}{2})$

---

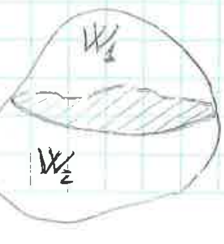
27) Trobeu la massa total del cilindre $V=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 2,\ 0\le z\le 3\}$ si la seva densitat és $\rho(x,y,z)=z e^{-z^2}(x^2+y^2)$.

**Solució.** $m(V)=\iiint_V \rho(x,y,z)\,dx\,dy\,dz \underset{\text{coordenades cilíndriques}}{=} \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r^3\,dr \int_0^3 z e^{-z^2}\,dz =$

$$= 2\pi\Big[\frac{r^4}{4}\Big|_0^{\sqrt{2}} \cdot \Big[-\frac{1}{2}e^{-z^2}\Big|_0^3 = \boxed{\pi(1-e^{-9})}$$

---

29) Sigui $W\subset\mathbb{R}^3$ un cos amb densitat de masses $\rho(x,y,z)$. Si dividim $W$ en dues parts, $W=W_1\cup W_2$, i denotem per $(\bar{x}_1,\bar{y}_1,\bar{z}_1)$ i $(\bar{x}_2,\bar{y}_2,\bar{z}_2)$ els centres de masses de $W_1$ i $W_2$ respectivament, demostreu que el centre de masses $(\bar{x},\bar{y},\bar{z})$ de $W$ és el mateix que si suposem tota la massa de $W_1$ concentrada en $(\bar{x}_1,\bar{y}_1,\bar{z}_1)$ i la de $W_2$ ho està en $(\bar{x}_2,\bar{y}_2,\bar{z}_2)$.




Solució. Introduïm la notació següent $W=W_1\cup W_2$ (amb $\mu(W_1\cap W_2)=\iiint_{W_1\cap W_2} dx\,dy\,dz=0$), $\vec{r}=(x,y,z)$, $\rho(\vec{r})=\rho(x,y,z)$,

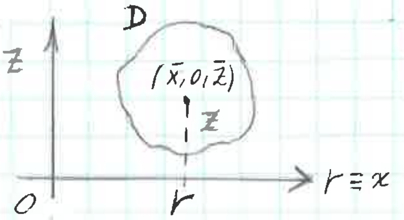
$d\vec{r} = dx\,dy\,dz$. Així, podem escriure, pel CDM del cos $W$:

$$\vec{R}(W) = \big(X(W), Y(W), Z(W)\big) = \frac{1}{m(W)} \iiint_W \vec{r}\,\rho(\vec{r})\,d\vec{r}$$

$$= \frac{1}{m(W)}\left( m(W_1)\cdot \underbrace{\frac{\iiint_{W_1} \vec{r}\rho(\vec{r})\,d\vec{r}}{m(W_1)}}_{\substack{\shortparallel \\ \vec{R}(W_1)\,:\,\text{CDM de } W_1}} + m(W_2)\cdot \underbrace{\frac{\iiint_{W_2} \vec{r}\rho(\vec{r})\,d\vec{r}}{m(W_2)}}_{\substack{\shortparallel \\ \vec{R}(W_2)\,:\,\text{CDM de } W_2}} \right)$$

$$= \boxed{\frac{m(W_1)\cdot\vec{R}(W_1) + m(W_2)\,\vec{R}(W_2)}{m(W)}}$$

30) Sigui $D$ un recinte pla contingut en el semipla $\{y=0, x\geq 0\}$ de $\mathbb{R}^3$. Si denotem per $(\bar{x}, 0, \bar{z})$ el centre geomètric de $D$ (i.e. el seu centre de masses si suposem densitat constant igual a 1), demostreu que el volum del domini de revolució $W$ que obtenim si fem girar $D$ entorn de l'eix $z$ és $\text{Volum}(W) = 2\pi\bar{x}\,\text{Àrea}(D)$, on $2\pi\bar{x}$ és la longitud de la circumferència que obtenim en fer girar $(\bar{x}, 0, \bar{z})$. (Indicació: Useu coordenades cilíndriques i observeu que $(\theta, r, z) \in D^* = [0, 2\pi]\times D$.)




Solució.

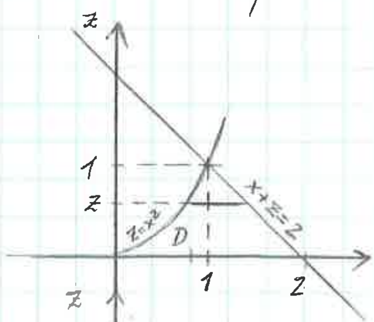
$\{y=0, x>0\}$, $\rho(x,0,z) = \rho \equiv 1$ (densitat ctnt. $= 1$)

$$V = \int_0^{2\pi} d\theta \underbrace{\iint_D r\,dr\,dz}_{\substack{\shortparallel \\ \bar{x}\cdot\text{Àrea}(D)}} \overset{(*)}{=} \int_0^{2\pi} d\theta\, \bar{x}\,\text{Àrea}(D) = \boxed{2\pi\bar{x}\cdot\text{Àrea}(D)}.$$

(*) $$\bar{x} = \frac{\iint_D x\,\rho(x,0,z)\,dx\,dz}{\iint_D \rho(x,0,z)\,dx\,dz} \underset{\rho(x,0,z)\equiv 1}{=} \frac{\iint_D x\,dx\,dz}{\iint_D dx\,dz}$$

d'on: $$\iiint_D x\,dx\,dz = \bar{x}\cdot\text{Àrea}(D).$$

31) Apliquem el resultat del problema anterior al càlcul del volum $W$ si prenem
$D=\{(x,z)\in\mathbb{R}^2 : z\leq x^2;\ x+z\leq 2,\ x\geq 0,\ z\geq 0\}$.

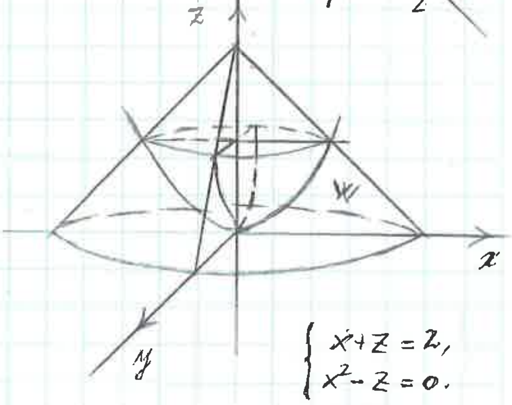



Solució.

$$\text{Àrea}(D)\cdot\bar{x} = \int_0^1 dz\int_{\sqrt{z}}^{2-z} x\,dx = \frac{1}{2}\int_0^1\left[(2-z)^2-z\right]dz$$

$$= \frac{1}{2}\int_0^1 (4-5z+z^2)\,dz = \frac{1}{2}\left(4z-\frac{5}{2}z^2+\frac{z^3}{3}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(4-\frac{5}{2}+\frac{1}{3}\right)\Big|_0^1 = \frac{24-15+2}{12} = \frac{11}{12}$$




D'on, aplicant la fórmula del problema 30:

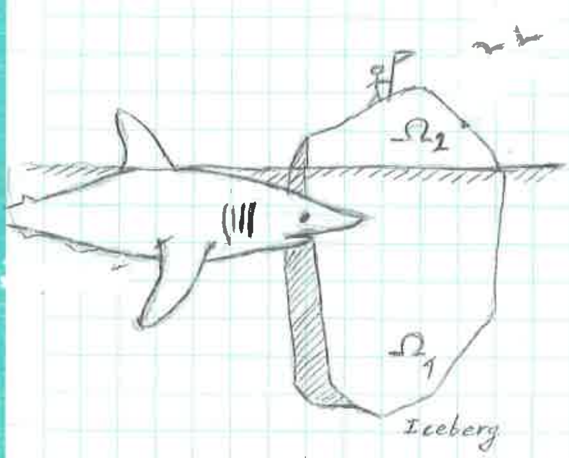
$$\text{Volum}(W) = 2\pi\bar{x}\cdot\text{Àrea}(D) = \boxed{\frac{11\pi}{6}}$$

$$\begin{cases} x+z=2,\\ x^2-z=0.\end{cases}$$

Punt de tall:
$x^2+x-2=0 \Leftrightarrow$ $x=-2$ (No), $x=1$

per tant el punt de tall resulta:
$(x,z)=(1,1)$

32) La densitat de l'aigua és 1 Kg/l i la del gel (aproximadament) 0'9 Kg/l. Pel principi d'Arquimedes, si submergim un bloc de gel en l'aigua el volum d'aigua desplaçat per la part submergida del gel té un pes igual al pes total del gel.




$\Omega = \Omega_1\cup\Omega_2$

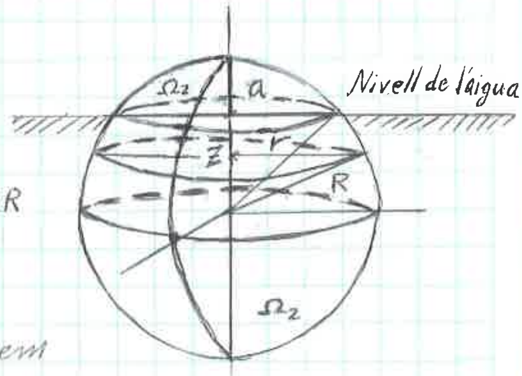
a) Vegeu que la part del gel emergent és el 10% del seu volum.

Solució. Si $\rho = 1$ Kg/l és la densitat de l'aigua i $\sigma = 0.9$ Kg/l és la densitat del gel, d'acord amb el principi d'Arquimedes, si $\Omega_1$ és la part submergida, $\Omega_2$ és la part que sobresurt del bloc de gel $\Omega = \Omega_1\cup\Omega_2$ i $V(\Omega_1), V(\Omega_2), V(\Omega) = V(\Omega_1)+V(\Omega_2)$ $(V(\Omega_1\cap\Omega_2)=0)$ són els respectius volums, hom té:

$$V(\Omega_1)\,\rho = \sigma\,V(\Omega) \Rightarrow \frac{V(\Omega_1)}{V(\Omega)} = \frac{\sigma}{\rho}$$

i per tant: $\dfrac{V(\Omega_2)}{V(\Omega)} = 1-\dfrac{V(\Omega_1)}{V(\Omega)} = 1-\dfrac{\sigma}{\rho} = 1-0'9 = 0'1$ (10%).

(b) Si tenim un bloc de gel esfèric, digueu quina és la part submergida del bloc i quina l'emergent.




Solució. Suposem un bloc esfèric de radi R (veure figura). Volem trobar l'alçada, $0<a<2R$ del casquet esfèric emergent, $\Omega_2$. Per calcular el volum corresponent, $V(\Omega_2)$, apliquem el principi de Cavalieri:

$$r^2 = R^2 - z^2,$$

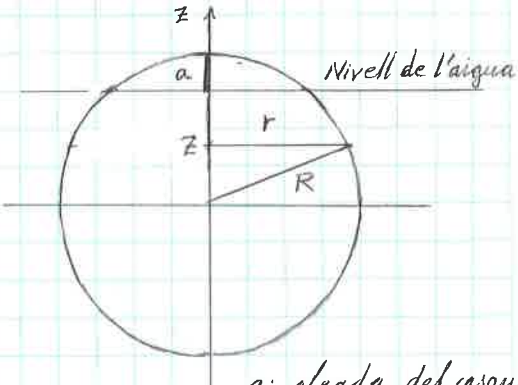
$$S(z) = \pi r^2 = \pi(R^2 - z^2)$$

$$V(\Omega_2) = \int_{R-a}^{R} S(z)\,dz = \int_{R-a}^{R} \pi(R^2 - z^2)\,dz$$

$$= \pi\left(R^2 z - \frac{z^3}{3}\right)\Big|_{R-a}^{R}$$

$$= \pi\left(\cancel{R^3} - \cancel{\frac{R^3}{3}} - \cancel{R^3} + R^2 a + \cancel{\frac{R^3}{3}} - R^2 a + Ra^2 - \frac{a^3}{3}\right)$$

$$= \pi R^3\left(\frac{a^2}{R^2} - \frac{1}{3}\frac{a^3}{R^3}\right).$$



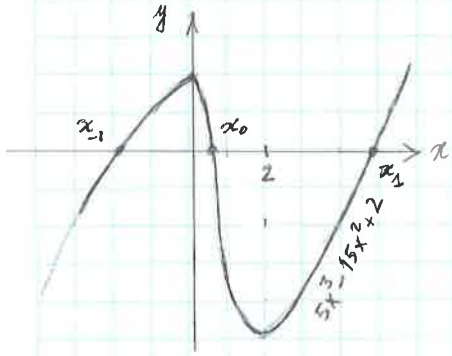

a: alçada del casquet d'esfera emergent.

Llavors, d'acord amb l'apartat anterior:

$$\cancel{\pi R^3}\left(\frac{a^2}{R^2} - \frac{1}{3}\frac{a^3}{R^3}\right) = \frac{1}{10}\,\frac{4}{3}\cancel{\pi R^3},$$

i definint $x \equiv a/R$ s'arriba a l'equació: $5x^3 - 15x^2 + 2 = 0$; la qual té tres solucions reals: $x_{-1} < 0 < x_0 < 2 < x_1$. Com $0 < x = a/R < 2$, la solució que busquem és

$$x_0 = 0.391600211318183^{(*)}.$$




Així doncs, l'alçada del casquet d'esfera emergent, en termes del radi R ve donada per:

$$\boxed{a = 0.391600211318183R}$$

(*) Càlcul amb MATLAB/Octave:

```
>> f = inline('5*x.^3 - 15*x.^2 + 2','x');
>> opt = optimset('TolFun', 1e-16, 'TolX', 0);
>> x0 = 1.5
```

```
>> [x0, f0] = fsolve(f, x0, opt)
x0 = 0.391600211318183
f0 = -4.440892098500063e-16
```

33) La densitat de població d'una certa ciutat pot aproximar-se per la funció $\rho(x,y) = 4000\, e^{-0.01(x^2+y^2)}$ si $x^2+y^2 \leq 49$ i $\rho(x,y) = 0$ altrament, on $x, y$ es mesuren en Km.

(a) Quina és la població de la ciutat.

Solució. Treballant en coordenades polars: $\tilde{\rho}(r) = \rho(r\cos\theta, r\sin\theta) = 4000\, e^{-r^2/100}$;

$$P = \iint_{\mathbb{R}^2} \rho(x,y)\,dx\,dy = \iint_{B_{R=7}(0,0)} \rho(x,y)\,dx\,dy = \int_0^{2\pi} d\theta \int_0^7 r\tilde{\rho}(r)\,dr$$

$$= \int_0^{2\pi} d\theta \int_0^7 4000\, r e^{-r^2/100}\,dr = 2\pi \cdot 4 \cdot 10^3 \left. \left(-50\, e^{-r^2/100}\right)\right|_0^7 = 4\pi \cdot 10^5 \left(1 - e^{-\frac{49}{100}}\right) \simeq \boxed{486.788'0}$$

(b) Quina és la distància $0 < R < 7$ al centre geogràfic de la ciutat de forma que el 50 % de la població viu a una distància més petita o igual que $R$ del centre?

Solució.

$$\int_0^{2\pi} d\theta \int_0^R r\tilde{\rho}(r)\,dr = 8000\pi \int_0^R r e^{-r^2/100}\,dr = 4 \cdot 10^5 \pi \left(1 - e^{-R^2/100}\right)$$

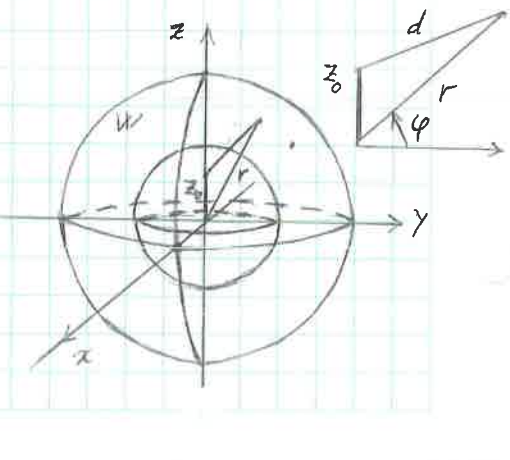
$$= 0'5 \cdot 4 \cdot 10^5 \pi \left(1 - e^{-49/100}\right),$$

d'on, aïllant $R$:

$$R = 10\sqrt{-\ln\left(1 - \frac{1 - e^{-49/100}}{2}\right)} = 10\sqrt{-\ln\left(\frac{1}{2} + \frac{1}{2} e^{-49/100}\right)} \simeq \boxed{4.640 \text{ Km}}$$

---

35) Trobeu el potencial gravitatori $V(0,0,z_0)$ generat pel sòlid $W$ comprès entre dues esferes concèntriques de radis $a < b$, $W = \{(x,y,z) \in \mathbb{R}^3 : a^2 \leq x^2+y^2+z^2 \leq b^2\}$, suposant densitat constant igual a $\rho$. Per simplificar els càlculs considereu només valors $z_0$ amb $0 < z_0 < a$ o $z_0 > b$.




Solució. $\rho(x,y,z) \equiv \rho$ ctnt.

- $0 < z_0 < a$

$$V(0,0,z_0) = -G \iiint_W \frac{\overset{=\rho}{\rho(x,y,z)}\,dx\,dy\,dz}{d\big((0,0,z_0),(x,y,z)\big)} =$$

pàg. següent ...

$$= -G\rho \int_0^{2\pi} d\theta \int_a^b r^2 dr \int_{-\pi/2}^{\pi/2} \frac{\cos\varphi}{\sqrt{z_0^2+r^2-2rz_0\sin\varphi}}\, d\varphi = -G\rho \int_0^{2\pi} d\theta \int_a^b r^2 \left[ - \frac{\sqrt{z_0^2+r^2-2rz_0\sin\varphi}}{rz_0} \right]_{\varphi=-\pi/2}^{\varphi=\pi/2} d\varphi$$

$$= \frac{G\rho}{z_0} 2\pi \int_a^b r\left(|z_0-r|-z_0-r\right) dr = -\frac{G\rho}{\cancel{z_0}} 4\pi \cancel{z_0} \int_a^b r\,dr = \boxed{-2\pi G\left(b^2-a^2\right)}$$

$0 < z_0 < a < r < b$

$\Rightarrow z_0 - r < a - r < 0$

$\forall\, a<r<b.$

(b) $z_0 > b$:

$$V(0,0,z_0) = \dots = \frac{G\rho}{z_0} 2\pi \int_a^b r\left(|z_0-r|-z_0-r\right) dr = -\frac{G\rho}{z_0} 2\pi \int_a^b 2r^2 dr = \boxed{-\frac{4\pi G\rho}{3z_0}\left(b^3-a^3\right)}$$

$0 < a < r < b < z_0$

$\Rightarrow z_0 - r > b - r > 0$

$\forall\, a<r<b.$