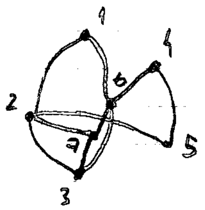
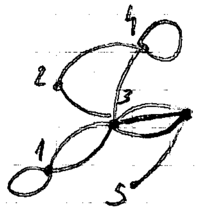
Basic definitions of graphs (Diestel, Chapter 1)

Def / A <u>simple</u> graph is a pair (V,E), where $E \subseteq \binom{V}{2}$. V is the set of vertices of the graph, and E is its set of edges.

Def / A <u>multigraph</u> is a pair (V,E), where E is a multiset of $V \cup \binom{V}{2}$. "..."





- We will consider finite graphs: $|V| < \infty \Rightarrow V = \{1,\dots,n\} = [n]$.
- If we do not say the contrary, we will work with simple graphs.
- Given a graph G, we denote by $V(G)$ and $E(G)$ its sets of vertices and edges.
- Let G be a graph. Assume that $u, v \in V$ define the edge $e \in E$. Then we write $e = uv$ (or $\overline{uv}$), and u and v are <u>adjacent</u>, u and v are <u>incident</u> with e, and u and v are the <u>endvertices</u> of e.

Def / Let $v \in V$. We write $d(v)$ for the number of vertices adjacent with v. (<u>Degree</u> of v)

In particular, we denote by $\delta(G)$ and $\Delta(G)$ the <u>minimum</u> and the <u>maximum</u> degree of vertices of G. A graph where $d(v) = k \ \forall v \in V$ is called <u>k-regular</u>. If $k=3$ it is also called <u>cubic</u>.

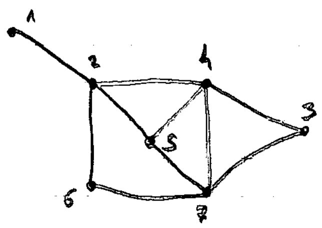
Lemma / $2|E| = \sum_{v \in V} d(v)$ for every graph G (even for multigraphs)
(Handshake lemma)

Subgraphs

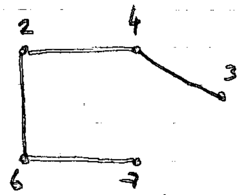
Let $G = (V,E)$ be a graph.

Def / A graph $G' = (V',E')$ is a <u>subgraph</u> of G if $V' \subseteq V$ and $e' \in E'$ if $e' \in E$ and the endvertices of e' belong to V'.

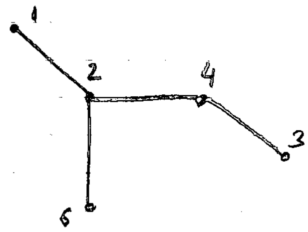
Def / A graph $G' = (V',E')$ is an <u>induced</u> subgraph of G if $V' \subseteq V$ and $e' \in E'$ iff $e' \in E$ and the endvertices of e' belong to V'. We write then $G' = G[V']$.



G



subgraph, but not induced

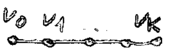


induced subgraph by $V' = \{1,2,3,4,6\}$

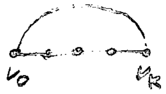
Def / A subgraph $G' = (V',E')$ <u>spans</u> $G = (V,E)$ if $V' = V$. (G' is a spanning subgraph)

Paths and cycles

Def / A <u>path</u> is a non-empty graph of the form $V = \{v_0, v_1, \dots, v_k\}$, $E = \{v_0v_1, \dots, v_{k-1}v_k\}$, where $v_i \neq v_j$ if $i \neq j$. Its <u>length</u> is equal to k. v_0 and v_k are the <u>starting</u> (and <u>final</u>) vertex of the path.



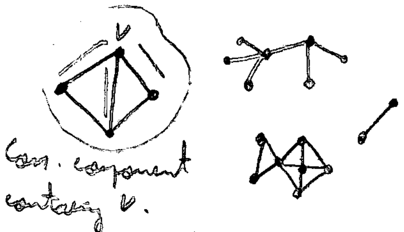
Def/ Let $P_k = (\{v_0,\dots,v_k\},\{v_0v_1,\dots,v_{k-1}v_k\})$ be a path. The graph C_k is $C_k = (\{v_0,\dots,v_k\},\{v_0v_1,\dots,v_{k-1}v_k, v_kv_0\})$ is called the *cycle* of length $k+1$.



Connectivity

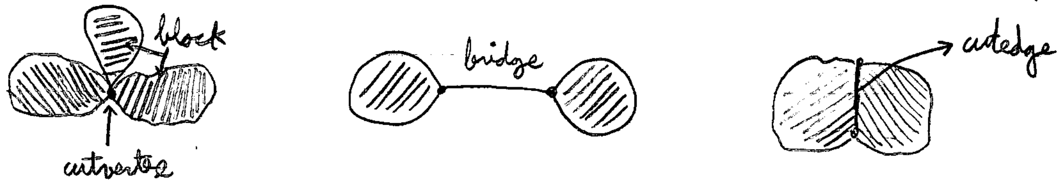
Def/ A graph is *connected* if for all pairs $u,v \in V$, there exists a path starting at u and finishing at v. A graph which is not connected, is called *disconnected*.

Def/ Let $G=(V,E)$ be a disconnected graph, and $v \in V$. The *connected component* containing v is the induced subgraph of G defined by vertices of G which can be linked by a path starting at v.



Def/ let now be $G=(V,E)$ a connected graph. Let X be a set of vertices. We say that X *separates* G if $G[V-X]$ is not connected.

Def/ G is said to be *k-connected* if for all $X \subseteq V$, $|X|<k$, X does not separate G.



Walks on graphs

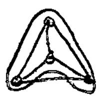
Def/ An *Euler tour* on $G=(V,E)$ is a closed walk on G, such that traverses every edge exactly once.

A graph admits an Euler tour is said *Eulerian*

Theorem/ A connected graph is Eulerian if an only if every vertex has even degree.

D/ Homework!

Def/ A *Hamilton cycle* on G is a cycle that contain every vertex of G exactly once. Such a graph is said *hamiltonian*



Bestiary of graphs.