Ramsey Theory

In the extremal combinatorics part we have studied the following problems:

Turán's Theorem

- Structure: $E(K_n)$
- Substructure: $E(K_k)$
- Statement: $F \subseteq E(K_n)$, $|F| \geq \left(1-\frac{1}{k-1}\right)\binom{n}{2}$

$$\Downarrow$$
$$E(K_k) \subseteq F$$

Roth's Theorem (Change 3 by k in Szemerédi's thm)

- Structure: $[n]$
- Substructure: 3-AP
- Statement: $S \subseteq [n]$, $|S| \geq \varepsilon n$ for $\varepsilon > 0$

$$\Downarrow$$
$$3\text{-AP} \subseteq S$$

So now the question is the following: we have a big structure which is partitioned into a finite number of pieces $\Rightarrow$ Is there one of the pieces containing a substructure?

Proof/ Take $[n]$ and colour the numbers using r colours (namely, you write $[n] = \bigcup_{i=1}^{r} C_i$, where C_i is associated to colour i). Then, it is not possible that $|C_i| = o(n)$ for all i, because if this is the case $n = |[n]| = \sum_{i=1}^{r} |C_i| = o(n)$. This means that there is a set C_{i_0} whose size is $\geq \frac{1}{r} n$, and by Szemerédi's theorem, it must contain a k-AP (if n is large enough, namely when $n >$ than the value given by Szemerédi's theorem).

This is one of the first results in Ramsey Theory.

Def/ Let $r > 0$. An <u>r-colouring</u> of $[n]$ is a function $c: [n] \longrightarrow [r]$. A set $X \subseteq [n]$ is called <u>monochromatic</u> if c is constant in X.

The question is now the following: given r colours, is there a minimum number n_0, such that every r-colouring of $[n]$, $n > n_0$, has a monochromatic k-AP? Denote this n_0 by $W(r,k)$ (called Van der <u>Waerden's number</u>).

Theorem/ (Van der Waerden, 1929) For each $r \geq 1$, $k \geq 1$, $W(r,k) < \infty$.

Proof/ (By Szemerédi's thm)

So, Szemerédi's theorem must be understood as a stronger version of Van der Waerden theorem. We will prove Van der Waerden at the end of this part. We also have the infinite version of this result:

Theorem/ Van der Waerden, 1929) Let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ a partition of $\mathbb{N}$ into r colours. Then, for each fixed k, there exist infinitely many k-AP's in one of the classes C_i (for a certain i)

Proof/ Homework.

We will return to Van der Waerden Theorem later.

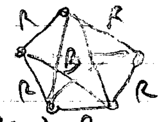
Ramsey Theorem

We consider $E(K_6)$ and we colour them using two colours (RED/BLUE). Then we have the following property

Proposition/ For each 2-colouring of $E(K_6)$ there exist a monochromatic triangle.

Proof/ Take vertex 1. It is incident with 5 edges, hence there is a colour which is more popular than the other (assume it is RED). This colour has at least 3 edges:

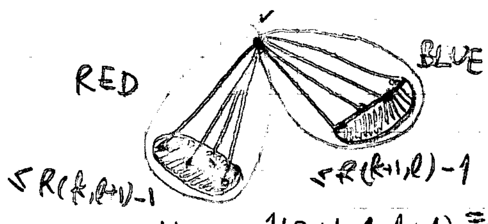
$\Rightarrow$ If a, b or c are RED, we are done. Otherwise $\triangle abc$ is BLUE and we are also done.

Observe that this statement is NOT true for $n=5$:  So, writing $R(k,l)$ the smallest integer R such that any 2-colouring of $E(K_R)$ contain a RED copy of K_k or a blue copy of K_l (Hence, we have proved that $R(3,3)=6$)

Then, we have the following key result:

Theorem / (Ramsey Theorem) For any $k,l\geq 1$, $R(k,l)\leq \binom{k+l-2}{k-1}$ (and hence, $R(k,l)<\infty$)

Proof / We start proving that $R(\overset{RED}{k+1},\overset{BLUE}{l+1})\leq R(k,l+1)+R(k+1,l)$: Assume that n is such that the graph K_n have a 2-colouring (RED/BLUE) without a RED K_k nor a BLUE K_l. Pick a vertex v:



The number of RED edges is at most $R(k,l+1)-1$: if we have more or equal than $R(k,l+1)$ vertices then we have either a BLUE copy of K_{l+1} (and we are done), or a RED copy of K_k. In the second case, then, we create a copy of K_{k+1}.

Hence $-1+R(k+1,l+1)=n\leq 1+(R(k,l+1)-1)+(R(k+1,l)-1)=R(k,l+1)+R(k+1,l)-1$.

Once done this, we apply induction in order to get the bound.

- $R(k+1,2)=k+1=\binom{k+1}{1}$, $R(2,l+1)=\binom{l+1}{1}$
- Assume $R(i+1,j+1)\leq\binom{i+j}{i}$ for all (i,j) with either $i<k$, $j<l$. Then:

$$R(k+1,l+1)\leq R(k,l+1)+R(k+1,l)\leq\binom{k+l-1}{k-1}+\binom{k+l-1}{k}=\binom{k+l}{k}$$

So, in particular, if $k=l$, then:

$$R(k,k)\leq\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\approx\frac{\sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k}}{\left(\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}\right)^2}\approx\frac{1}{\sqrt{\pi k}}4^k\leq 4^k$$

What about lower bounds? Using a counting argument we can obtain the first lower bound for $R(k,k)$:

Theorem / (Erdős, 1949) $2^{k/2}\leq R(k,k)$

Proof / Take K_N (N will be computed later). The total number of 2-colourings of the edges of K_N is $2^{\binom{N}{2}}$.

Observe now that the number of 2-colourings of K_N containing a monochromatic K_k is at most:

$$\underbrace{\binom{N}{k}}_{\text{Choice of a k-set of vertices}}\cdot\underbrace{2^{\binom{N}{2}-\binom{k}{2}}}_{\text{the rest of the edges}}\cdot\underbrace{2}_{\text{2-colours}}$$

Hence, if $2^{\binom{N}{2}}\geq\binom{N}{k}2^{\binom{N}{2}-\binom{k}{2}}\cdot 2$, we have 2-colourings without monochromatic copies of K_k. Observe that $\binom{N}{k}<N^k/2^k$. Then, if $\binom{N}{k}<2^{\binom{k}{2}-1}$ we have colouring without monochromatic copies of K_k. In other words:

$$2^{\binom{k}{2}-1}\leq\binom{R(k,k)}{k}<\frac{R(k,k)^k}{2^k}\Rightarrow R(k,k)^k\geq 2^{\frac{k^2+k}{2}-1}\xRightarrow{k\geq 2}R(k,k)\geq 2^{k/2}$$

In particular, it is not known $\lim_{k\to\infty}\sqrt[k]{R(k,k)}$ (its existence!). This proof is very important because it is the first existence proof in the area of the probabilistic method.

What about the infinite setting? Let X be a set (possibly infinite)

Def / A *c-colouring* of X is a partition of X into c disjoint classes.

Def / Given a c-colouring of $\binom{X}{d} = \{\{x_1, \dots, x_d\} : x_i \in X\}$, we say that $Y \subseteq X$ is *monochromatic* if all elements of $\binom{Y}{d}$ have the same colour.

In this terminology, the result concerning $R(k,k)$ can be stated in the following way: take $X = \{1, \dots, n\}$ and $d = 2$. Then, we have proved that for each choice of k, there is an n such that every 2-colouring of $\binom{X}{2}$ yields a monochromatic set $Y \subseteq X$ of size k.

The point is that this result is also true in the following more general setting:

Theorem / (Ramsey Thm, 1930) Let k, c positive integers and X an infinite set. If $\binom{X}{k}$ is coloured using c colours, then X has an infinite monochromatic subset.

Proof / Induction on k (and c fixed)

- $k=1$: We are colouring X with c colours, so there is a colour which is repeated infinitely many times.

- Assume the result for values $< k$ and let us prove the result for $= k$. For this purpose, we construct an infinite sequence $X_0, X_1, \dots, X_i \subseteq X$, such that, for all i:
  0) $x_i \in X_i$.
  a) $X_{i+1} \subseteq X_i - \{x_i\}$
  b) all sets of size k of the form $\{x_i\} \cup Z$, $Z \subseteq \binom{X_{i+1}}{k-1}$ have the same colour.

In b), we associate this colour to the element x_i. We start taking $X_0 = X$ and x_0 an arbitrary element of $X = X_0$. Now we proceed in the following way: having chosen X_i and $x_i \in X_i$, let us see how to choose X_{i+1}. Each element of the form $\{x_i\} \cup Z$, $Z \subseteq (X_i - \{x_i\}\ k-1)$ is an element of $\binom{X}{k}$, hence it has a certain c-colour. This defines a c-colour in $\binom{X_i - \{x_i\}}{k-1}$. By the induction hypothesis, there is an infinite monochromatic set X_{i+1}, and finally we pick $x_{i+1} \in X_{i+1}$ arbitrarily.

To conclude, as c is finite, one of the c colours is associated with infinitely many x_i's, and this subset of the x_i's defines a monochromatic subset of X.

We are now ready to prove a result we used in the graph minor part, related to well-quasi-ordered sets.

Theorem / Let $(X, \preccurlyeq)$ be an infinite well-quasi-ordered set. Then every infinite sequence in X has an infinite increasing subsequence.

Proof / We prove the following: a quasi-ordering $\preccurlyeq$ on X is well-quasi-ordering iff X contains neither an infinite antichain (every pair of elements are not comparable) nor an infinite strictly decreasing sequence $x_0 \succ x_1 \succ \dots$

$\Rightarrow$) By definition

$\Leftarrow$) Let $x_0, x_1, x_2, \dots$ be an infinite subsequence of X. For each pair $\{x_i, x_j\} \in \binom{X}{2}$, we consider the following 3-colouring:

i) Colour green if $i < j$ and $x_i \leq x_j$
ii) Colour red if $i < j$ and $x_i > x_j$

Then, by the previous theorem, there is an infinite subset of $\{x_0, x_1, \dots\}$ which is monochromatic. Its colour cannot be red or blue, so it must be green $\Rightarrow$ we have constructed an infinite increasing sequence!

Van der Waerden's Theorem revisited.

Recall that Szemerédi's theorem implies that for each choice of r and k there is a value $W(r,k)$ such that if $n \geq W(r,k)$ then any r-colouring of $[n]$ contains a k-AP. We will prove now this result without the use of Sz. theorem.

Our purpose is to show that, for each choice of r and k, $W(r,k) < \infty$. The case $k=1$ and $k=2$ are trivial: for each $r \geq 1$, $W(r,1) = 1$, $W(r,2) = r+1$ (in the second case, by the pigeonhole principle, one colour is repeated, and this gives us the 2-AP).

The first non-obvious case is $k=3$, $r=2$.

Prop/ $W(2,3) \leq 325$.

Proof/ We show that any 2-colouring of $[325]$ has a 3-AP. Observe that $325 = 5 \cdot 65 = 5 \cdot (32 \cdot 2 + 1)$. We have choosen 325 for the following reason: we partition $[325]$ in 65 blocks of length 5: $\{1,\dots,5\}$, $\{6,\dots,10\}$, $\dots$, $\{321,\dots,325\}$. We have 65 such blocks.

$d \in 5 \cdot \{1,\dots,32\}$

Observe now that a block can be coloured in $2^5 = 32$ ways. So, in the first 33 blocks

$a+d \leq 161$

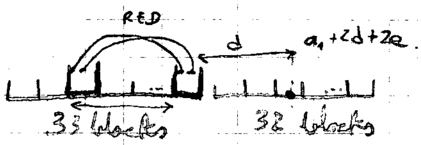
we have two blocks coloured in the same way. Call these blocks $\{a, a+1, a+2, a+3, a+4\}$ and $\{a+d, a+d+1, \dots, a+d+4\}$. The next observation is that in $\{a, a+1, a+2\}$ there are two element of the same colour (say RED). Call them a_1 and a_1+e. Then, some situations may happen: (a_1+d and a_1+d+e are also RED)

a) a_1+2e is RED $\Rightarrow$ We are done! (observe that a_1+2e belongs to the block under study)

b) Hence, if a) is not satisfied, both a_1+2e and a_1+d+2e are BLUE. Let us consider $a_1+2d+2e$. Observe that $a_1+2d+2e \leq 325$ by construction. Then:

i) If $a_1+2d+2e$ is BLUE, we are done.

ii) Otherwise, $a_1+2d+2e$ is RED. Then we have a_1, $a_1+(e+d)$ and $a_1+2e+2d$ in RED.



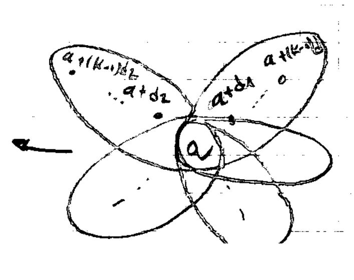
Notice that this proof gives really bad bounds: it can be proved that $W(2,3) = 9$. We will adapt now this proof to the general setting. For this reason, we consider the following definition.

Def/ A <u>sunflower with m petals of size $k-1$</u> is a collection of m $(k-1)$-AP of the form $\{a+d_1, \dots, a+(k-1)d_1\}, \dots, \{a+d_m, \dots, a+(k-1)d_m\}$ such that:

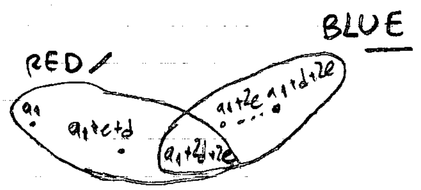
i) Each pair of progressions is disjoint

ii) Each progression is monochromatic and any pair has different colours.

a is called the center of the



$\Rightarrow$ We have used the sunflower with $m=2$ $k=3$ for $W(2,3)$



and $A+d$ is a sunflower of size $k-1$ with m petals.

We exploit the following observation: consider a set A (not a sunflower). Assume also that the translates $A+d, \dots, A+(k-1)d$ are pairwise disjoint and identically coloured) Then the set $A \cup (A+d) \cup \dots \cup (A+(k-1)d)$ contains either a sunflower with $(m+1)$ petals of size $k-1$ or a monochromatic k-AP.

Indeed, if the colour of the center is the same as one of the AP's (say $\{a+d_i, \dots, a+(k-1)d_i\}$, then $\{a, a+d_i, \dots, a+(k-1)d_i\}$ is a k-AP. Otherwise we have the sunflower defined by the petals:

$$\underbrace{\{a+d,\dots,a+(k-1)d\}, \{a+(d+d_1),\dots,a+(k-1)(d+d_1)\},\dots,\{a+(d+d_m),\dots,a+(k-1)(d+d_m)\}}_{\Downarrow\ m+1 \text{ petals of size } k-1}$$

and center a. We also need an additional lemma:

Lemma/ Let $r,k\in\mathbb{N}$ and assume that $W(r,k-1)$ exists. Then, for every $m\in\mathbb{N}$ there is a number $W(r,m,k-1)$ such that if $N\ge W(r,m,k-1)$ and $[N]$ is coloured with r-colours, then there is either a sunflower with m petals of size $k-1$ or a monochromatic k-AP.

Proof/ The case $m=1$ holds from the assumption. Assume the claim true for $m-1$ and let us see for m. Write now $N_1 = W(r,m-1,k-1)$ and $N_2 = 2W(r^{N_1}, k-1)$.

Observe that:

i) Each block of length N_1 contains either a monochromatic k-AP or a sunflower with $m-1$ petals of size $k-1$. If the first situation happens, we are done. Assume that this is not the case.

ii) Take now N_2 blocks of size N_1 (namely, we study $[N_1N_2]$). As there are r^{N_1} different colourings of a block of size N_1, there must be an $(k-1)$-AP of identically coloured blocks: call them $, B+d, \dots, B+(k-1)d$.

As $|B| = N_1 = W(r,m-1,k-1)$, $B+d$ contains a sunflower of $m-1$ petals of size $k-1$; call it $A+d$. Then $A\cup(A+d)\cup\dots\cup(A+(k-1)d) \subseteq B\cup(B+d)\cup\dots\cup(B+(k-1)d)$ contains either a sunflower with m petals of size $k-1$ or a monochromatic k-AP, as we wanted to prove.

Now we are ready to prove VdW theorem: we apply induction in r and k.

$\underline{k=2}$: obvious that $W(r,2)=r+1$.

Assume now that the result is true for each r and $k-1$, and let us prove it for k (and each choice of r). From the previous lemma, $W(r,m,k-1)$ exists for each m. Taking $m=r$, for any colouring of $[W(r,r,k-1)]$ there is a sunflower with r petals of size $k-1$. The colour of the center must agree with the colour of one of the $k-1$ AP as we only have r colours. Hence, we have shown that $W(r,k) \le W(r,r,k-1) < \infty$.

This argument provide really BAD bounds for the numbers $W(r,k)$. For instance, if we try to estimate $W(2,k)$, we get that:

$$W(2,k)\le W(2,2,k-1)\le 2W(2,1,k-1)\cdot W(2^{W(2,1,k-1)},k-1) = 2\cdot W(2,k-1)\cdot W(2^{W(2,k-1)},k-1)\le\dots\le 2^{2^{2^{2^{k+1}}}}$$

The best result known is a 5-tower of exponentials, and it is conjectured that $\le 2^{k^2}$