Planarity

Topological prerequisites

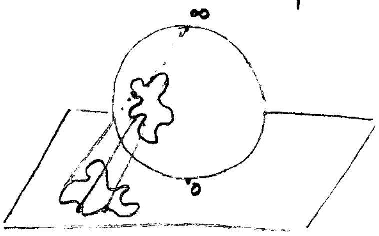
Before starting with graph properties of planar graphs, we need to recall "basic" facts on low-dimensional topology.

Def/ A <u>curve</u> is a subset of $\mathbb{R}^2$ of the form $\{\gamma(x) : x \in [0,1]\}$, such that $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ is a continuous mapping. $\gamma(0)$ and $\gamma(1)$ are the <u>endpoints</u> of the curve.

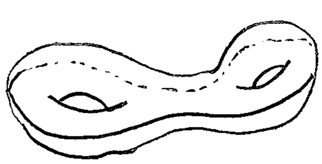
A curve is <u>closed</u> if $\gamma(0) = \gamma(1)$, and <u>simple</u> if it is injective at $(0,1)$.

Theorem/ (<u>Jordan's curve Theorem</u>) Let $\alpha = \{\gamma(x) : x \in [0,1]\}$, $\gamma : [0,1] \to \mathbb{R}^2$, be a closed simple closed curve. Then $\mathbb{R}^2 \setminus \alpha$ has two connected components.

Equivalently, by stereographic projection, a similar result is true in the sphere (so we will work from now ~~on~~ over the sphere).



$\mathbb{S}^2 \simeq \mathbb{R}^2 \cup \{\infty\}$



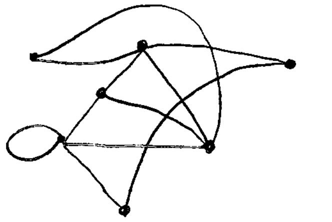
Not true for other surfaces!

Planar graphs and maps (plane graphs)

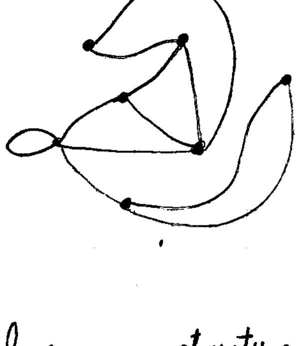
We will assume that all objects we are dealing with are multigraphs.

Def/ Let G be a multigraph. A <u>drawing</u> (or embedding) of G in $\mathbb{R}^2$ (or $\mathbb{S}^2$) is a function which assigns:
 i) a point $f(v) \in \mathbb{R}^2$ (or $\mathbb{S}^2$) for each $v \in V(G)$
 ii) a curve α_e with endpoints $f(u)$ and $f(v)$ if $e \in E(G)$, $e = \overline{uv}$

Def/ A multigraph is <u>planar</u> if it has a drawing in $\mathbb{R}^2$ (or $\mathbb{S}^2$) without edge-crossings. A planar graph joint with a particular embedding is called a <u>plane multigraph</u> (or <u>map</u>)



Drawing of a planar graph



Plane graph $\equiv$ map!

A planar graph + embedding (a map) has more structure than the planar graph by itself: its face structure

Def/ Let G be a planar graph, and let M be an associated map. Then a <u>face</u> of M is a connected component of $\mathbb{S}^2 - M$ (as a topological space). The set of faces of M is $F(M)$.

Whitney's theorem

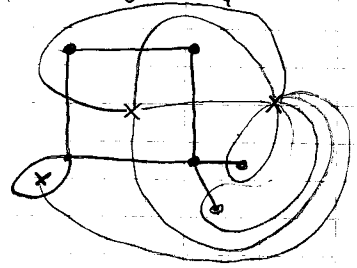
Same planar (multi)-graph

Hence, a map has a triple structure: vertices, edges and faces (and of course, the incidences between them).
$\equiv$ local orientation on vertices.

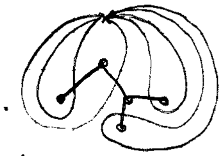
Def/ Let M be a map. The dual map M^* of M is a map defined in the following way:

i) Vertices of M^* are the faces of M.

ii) Two vertices in M^* are connected iff the corresponding faces in M are incident, and the number of multiedges is equal to the number of incidences.



Observations/ 1) We do not have problems to define the dual graph as we are in the sphere.

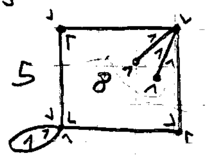


2) The dual of a map defined by a simple graph could be a multigraph (Trees!)

3) Two ~~different~~ maps arising from the same graph could have different associated dual maps.

4) For a map M, M^* is always connected. ($\mathbb{S}$ is connected!)

Def/ Let c be a face of a map M. The length $\ell(c)$ of c is the degree of the associated vertex in the dual map M^*. Equivalently, is the length of the closed walk in M bounding the face.



Some properties

1) Proposition/ For every map M, $2|E(M)| = \sum_{f \in F(M)} \ell(f)$

Proof/ Just apply the handshake lemma for M^*.

2) Proposition/ The following statements are equivalent for a map M:

i) M is bipartite

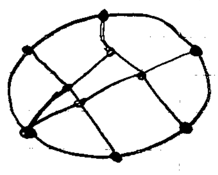
ii) Every face of M has even length

iii) The graph arising from M^* is Eulerian.

Proof/ ii) and iii) are equivalent by Euler's theorem.

i) $\Rightarrow$ ii) A face boundary is based of closed walks. Every odd closed walk contains an odd cycle. Hence, all the contributions in M must be even.

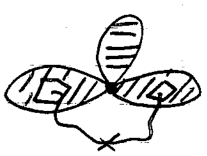
ii) $\Rightarrow$ i) We shall prove that every cycle has even length. Let C be a cycle of M:



Jordan's curve Theorem $\Rightarrow$ C defines 2 connected components.
Take one of them, and sum the face length $\Rightarrow$ is even.
This value is equal to $2\times$ # internal edges + external edges. Hence the number of external edges must be even.

3) Dual of connected (but not 2-connected) maps have the same property.

Proof/ Any path from the face of a block to the face of another block goes through the same vertex.



Euler's relation

Theorem / (Euler 1758) Let M be a connected map with v vertices, e edges and f faces. Then

$$f+v=e+2$$

Proof / Induction on the number of vertices

- $\underline{n=1}$: In this situation the map can just have loops. So, for $e=0$, and $e=1$ Euler's relation is true. Assume true for $e<n$. Let $e=n$. Again, by Jordan's Curve Theorem once a loop is added, we are also adding a new face (a face is cut into two).

- Assume the result true for $v<n$, and let us see for $v=n$. As $n\geq 1$, there is an edge which is not a loop. Contract it: the resulting map has the same number of faces, but the number of edges and vertices has been reduced by $1 \Rightarrow$ The relation is satisfied.

Obs / 1) Duality interchanges f and v in the Euler's relation.
2) Maps arising from the same (multi)graph have the same number of faces.
3) Euler's relation fails when graphs are disconnected. In this case, the formula is $f+v=e+k+1$, where k is the number of connected components (link the components using $k-1$ edges)

Some consequences of Euler's formula:

0) Polyhedra!
1) Let G be a simple planar graph with $|V(G)|\geq 3$. Then $|E(G)|\leq 3|V(G)|-6$.

Proof / We can restrict ourselves to G being connected. Every face in an embedding of G has length ≥ 3. Hence, $2e=\sum_{f\in F(G)} \ell(f) \geq 3|F(G)|$. Consequently, $|F(G)|\leq \frac{2}{3}|E(G)|$.

$$|E(G)| = |F(G)|+|V(G)|-2 \leq \frac{2}{3}|E(G)|+|V(G)|-2 \Rightarrow |E(G)|\leq 3|V(G)|-6.$$

2) If G is a simple triangle-free planar graph, then $|E(G)|\leq 2|V(G)|-4 \Rightarrow$ Bipartite.

Proof / Mimic the last proof.

3) A planar graph has a vertex of degree ≤ 5.

Proof / Assume that all vertices have degree ≥ 6. Then,

- $2|E(G)|\geq 6|V(G)| \Rightarrow |E(G)|\geq 3|V(G)|$ (Degree)
- By 1), the maximum number of faces in a map is $|F(G)|=-|V(G)|+|E(G)|+2\leq 2|V(G)|-4$.

By Euler's relation $\Rightarrow |F(G)|=|E(G)|+2-|V(G)|\geq 2|V(G)|+2$, contradiction!

Characterisation of planarity

The natural question now is: which graphs can be embedded in the plane (or S^2)?

Example / K_5 is not planar: consider an spanning cycle of K_5 (it has 5 edges). It splits the plane in two faces. We need now to draw 5 extra edges, each of them on one of this components. Observe that 2 such edges intersect if their endvertices in C alternate in order. When two edges conflict in such a way, we can only draw one of them.

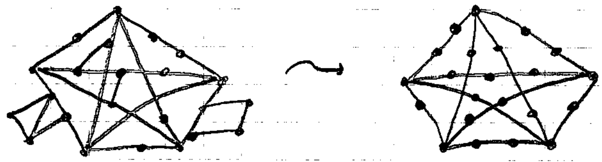
For this reason, at most 2 edges can go inside the spanning cycle (and also out of it)

Example / $K_{3,3}$ is not planar: Euler's relation: as $K_{3,3}$ is bipartite, then $|E(K_{3,3})|\geq 2\cdot 6-4=8$. But

It is also important to notice that any subgraph of either K_5 or $K_{3,3}$ is planar.

The important point in this two graphs is that an "inverse" theorem is also true:

Theorem / (Kuratowski's theorem) A graph is planar if and only if does not contain as a subgraph a subdivision of either K_5 or $K_{3,3}$.

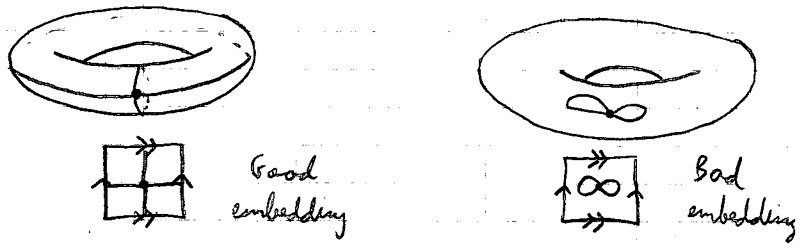


subdivision of K_5

Graphs on surfaces and minors

A natural question is: can we generalize all the previous results to ~~other~~ compact surfaces? The answer, in general, is <u>NO</u>:

i) Of course we have the notion of a graph embeddable on a surface (despite we demand many times [illegible] that faces are contractible, as it happen in the sphere)

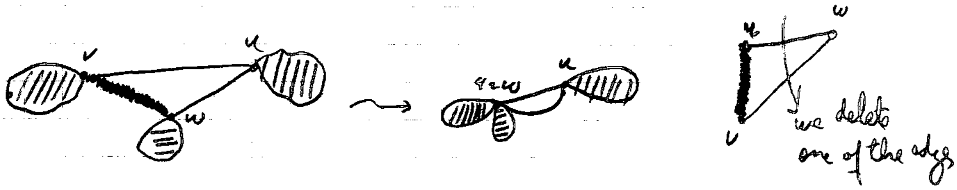


See HW.

ii) We do <u>NOT</u> have a Kuratowski-type result for higher surfaces: graphs on higher surfaces are <u>NOT</u> characterised by a finite set of excluded (subdivisions) of subgraphs.

However, something incredible happen when we relax the subgraph condition:

Def/ Let H and G be graphs. We say that H is a <u>minor</u> of G (and we write $H \preceq G$) if H can be obtained from a subgraph of G by edge contractions (deleting multiple edges, if necessary)



In particular, the notion of minor is more flexible than the one of subgraph (a subgraph is a minor, but a minor could not be a subgraph of the initial graph). In particular we have:

Theorem / (Wagner) A graph is planar if and only if does not contain $K_5, K_{3,3}$ as a minor.

Proof / HW (deduce it from Kuratowski's theorem, which is somehow stronger).

In fact, this is true for a more general reason: let $\mathcal{G}$ be a graph family. We say that $\mathcal{G}$ is <u>closed</u> by the minor relation iff for all $G \in \mathcal{G}$, $H \preceq G \Rightarrow H \in \mathcal{G}$.

So planar graphs are closed by the minor relation, but also graphs that can be embedded in a given surface.

Wagner's conjecture //

Theorem / (Robertson–Seymour) Every graph family closed by the minor relation is characterised by a <u>finite</u> set of excluded minors.

Examples/

1) Forests: those are clearly closed by the minor relation $\Rightarrow$ You just need to exclude $C_3 \Rightarrow Ex(C_3)$
2) Planar graphs: $Ex(K_5, K_{3,3})$.
3) Graphs embeddable on the projective plane $\Rightarrow$ 35 excluded minors (Archdeacon, Huneke '89)
4) Torus: is known more than 800 excluded minors.

Well-quasi order in graphs.

The relation $\preceq$ defined before is reflexive and transitive. Hence, $(\mathbb{G}, \preceq)$ is a quasi-ordered set.

Def/ A quasi-ordered set $(\mathbb{G}, \preceq)$ is well-quasi ordered if for every infinite sequence of elements in $\mathbb{G}$, $(G_1, G_2, G_3, \dots)$ there exist $i<j$ such that $G_i \preceq G_j$. We call such a sequence good.

Exe/ 1) $(\mathbb{N}, \leq)$ is well-quasi ordered.
2) $(\mathbb{Z}, \leq)$ is not well-quasi ordered: $(-1, -2, -3, -4, \dots)$

What about $(\mathbb{G}, \preceq)$? This is precisely the equivalent version (using this language) of Robertson-Seymour theorem:

Thm/ (Robertson-Seymour) $(\mathbb{G}, \preceq)$ is a well-quasi-ordered set.

HW: 1) Prove the equivalences between the results.
2) Prove that this is not true if we change "minor" by "subgraph" or "induced subgraph". (Find a sequence, for example, of trees).

We won't prove R-S theorem, but we shall show that it is true for the class of trees, even in a stronger sense.

We use the following fact for well-quasi orders:

Prop/ If $(X, \leq)$ is well-quasi-ordered, every infinite sequence in X has an infinite increasing subsequence.

Before going to trees, we need a general result. In fact, the proof we will use later is very similar.

Let $(X, \leq)$ be a quasi-ordered set. Let $[X]^{<\omega}$ be the set of subsets of X. $[X]^{<\omega}$ can be equiped with as quasi-order in the following way: $A, B \subseteq X$, we write $A \leq B$ if there exists an injective map $f: A \hookrightarrow B$, $a \mapsto f(a)$ such that $a \leq f(a)$ for all $A \ni a$.

Lemma/ $(X, \leq)$ is well-quasi-ordered $\Rightarrow$ $([X]^{<\omega}, \leq)$ is well-quasi ordered.

Proof/ Assume that $([X]^{<\omega}, \leq)$ is NOT well-quasi-ordered. Hence, there exist bad sequences. We construct a bad sequence as follows: given $n \in \mathbb{N}$, assume inductively that A_i has been defined for $i<n$, and that there exist a bad sequence starting with $A_0, \dots, A_{n-1}$. Choose then A_n such that the subsequence $A_0, \dots, A_n$ belongs to a bad sequence, and $|A_n|$ is as small as possible. Then $(A_n)_{n\in\mathbb{N}}$ is bad and $A_n \neq \emptyset$ for all $n \in \mathbb{N}$.

Take now $a_n \in A_n$, and let $B_n = A_n \setminus \{a_n\}$. Clearly $B_n \leq A_n$ in $[X]^{<\omega}$. Then $(a_n)_{n\in\mathbb{N}}$ is good, so it contains an increasing subsequence $a_{n_0} \leq a_{n_1} \leq a_{n_2} \leq \dots$ Observe now that the sequence

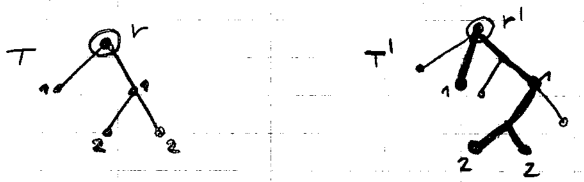
$$A_0, A_1, \dots, A_{n_0-1}, B_{n_0}, B_{n_1}, \dots$$

cannot be bad (by minimality). Hence, necessarily there exist i, j such that $B_{n_i} \leq B_{n_j}$.

Joint with the fact that $a_{n_i} \leq a_{n_j}$, we can finally construct an injective map $f: A_{n_i} \hookrightarrow A_{n_j}$ and we conclude a contradiction.

We apply this argument in the following family:

- $\underline{X}$: set of rooted trees:  $= \mathcal{T}$
- $\underline{\leq}$: let T and T' be rooted trees (roots r and r'). $T \leq T'$ iff T is equal to a subdivision of a subgraph of T', such that the induced order by the root is preserved.



$\Rightarrow$ rooted trees are sets of rooted trees!

Theorem / (Kruskal, 1960) $(\mathcal{T}, \leq)$ is well-quasi-ordered.

Proof / Assume the contrary and construct, as in the previous lemma, a "minimal" bad sequence in $\mathcal{T}$: $(T_n)_{n \in \mathbb{N}}$.

Denote by A_n the subset of rooted trees obtained by $T_n - r_n$. Let $A = \bigcup_{n \in \mathbb{N}} A_n$. We prove now that $(A, \leq)$ is well-quasi-ordered: take $(T^k)_{k \in \mathbb{N}}$ a sequence in A, and let $n(k)$ such that $T^k \in A_{n(k)}$. Take the smallest one $= n(k)$. Then:

$$T_0, T_1, \ldots, T_{n(k)-1}, T^k, T^{k+1}, \ldots$$

is a good sequence by minimality. As in the previous lemma, a good pair must be of the form (T^r, T^s), and hence A is well-quasi-ordered.

Again, by the previous lemma, $[A]^{<\omega}$ is well-quasi-ordered, and $(A_n)_{n \in \mathbb{N}}$ $[A]^{<\omega}$ has a good pair (A_i, A_j). But now we can reconstruct T_i and T_j, and finally (T_i, T_j) is a good pair in $(\mathcal{T}, \leq)$.