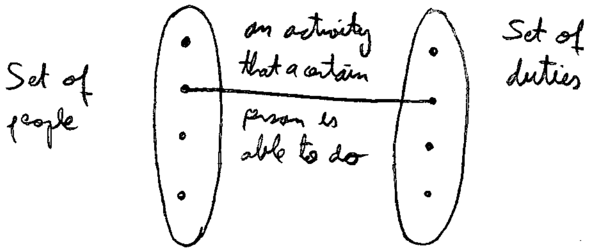
Matching theory (all graphs will be simple)

Motivation:



Q: Are there general results to assure an assignment person-duty such that every task is done by exactly one person?

↳ Choice of a certain "disjoint" set of edges.

Def / Let $G=(V,E)$ be a graph. A <u>matching</u> in G is a set of edges which do not share endpoints. The <u>size</u> of a matching is its number of edges.

Def / Let $G=(V,E)$ be a graph and M a matching in G. M is a <u>perfect</u> matching if every vertex is incident with M.

In a more general context we have the following definition

Def / Let $G=(V,E)$ be a graph. A k-regular spanning subgraph of G is called <u>k-factor</u>.

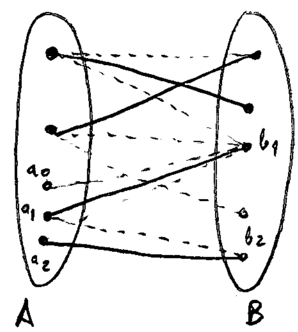
In particular, a perfect matching defines a 1-factor. We will study questions related to matchings, including the characterization of maximum matchings and proving condition for the existence of 1-factors.

Matchings in bipartite graphs.

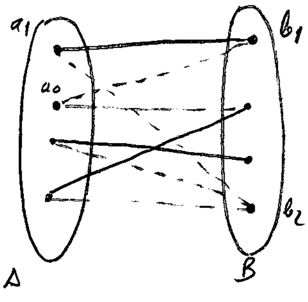
We write in all the section $G=(V,E)$, and $V = A \cup B$ a bipartition of the vertices.

Def / Let $G=(V,E)$ be a bipartite graph, and let M be a matching in G. An <u>alternating path</u> wrt M in G is a path starting at an unmatched vertex in A which contains, alternately, edges from $E(G)-M$ and M.

An alternating path finishing at an unmatched vertex in B is called <u>augmenting path</u>.



$a_0 b_1 a_1 b_2 a_2$ is an alternating path



$a_0 b_1 a_1 b_2$ is an augmenting path

Finding augmenting paths provide a way to increase the size of the initial matching
⇓
flip the edges in order to increase by one the size of the matching.

First <u>question</u>: can we characterize the maximum size of a matching in a bipartite graph? <u>YES</u>!

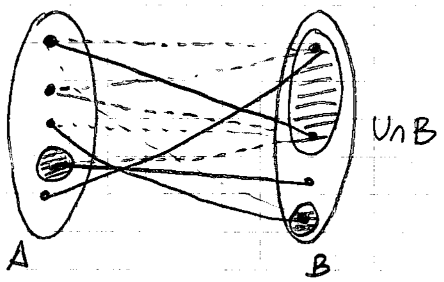
Def / A <u>vertex cover</u> of a graph $G=(V,E)$ is a subset $V' \subseteq V$ such that every edge is incident with V'.

Of course asking for existence of perfect matching is only reasonable when $|A|=|B|$

Theorem / (König, 1931) The size of a maximum matching in a bipartite graph G is equal to the minimum size of a vertex cover.

Proof / Let M be a matching of maximum cardinality. Since every vertex cover in particular covers M, we could try to construct a vertex cover of size $|M|$.

To do so, choose for every edge in M one of its endpoints: in B if some alternating path finishes there, or in A otherwise. Call this set U.

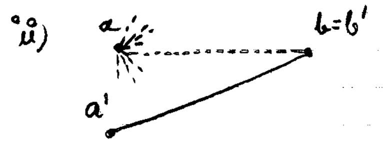


We show that every edge is incident with U. Of course edges from M are all incident with U. So let us assume that $e = ab$ is an edge $\notin M$.

o) a, b are matched $\Rightarrow$ Trivial!

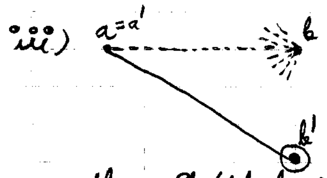
i) a, b are not matched $\Rightarrow$ Not possible, because we could construct $M' = M \cup \{ab\}$ which is a matching with $|M'| > |M|$.

ii)



b is marked, because ab is an alternating path.

iii)



If $a = a' \in U$ we are done. Assume then that $b' \in U$. Hence, there's an alternating path finishing at b' and b belongs to an alternating path: Pb (if $b \in P$) or $Pb'ab$. As M is maximum size, this path cannot be augmenting, and consequently b must be matched!

We have exploited the notion of alternating and augmenting path in this proof.

<u>Second question</u>: When can we assure that there exist a matching such that all vertices in A are matched?

For a subset $S \subseteq A$, denote by $N(S)$ the set of neighbours of S in B. Of course it is necessary that $|N(S)| \leq |S|$ in order to assure such a matching. But in fact, this condition is sufficient:

Theorem / (Hall, 1935) A bipartite graph G has a matching of A iff for all $S \subseteq A$ $|N(S)| \geq |S|$.

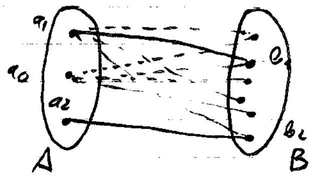
NOT COVERED!

Proof 1 / (Algorithmic) We use augmenting paths. Let M be a matching which does not contain a particular vertex a_0. We construct then an augmenting path in the following form:

i) a_0 is unmatched

ii) b_i is adjacent to some vertex $\{a_0, a_1, \ldots, a_{i-1}\}$

iii) $a_i b_i \in M$.



Observe that step ii) can be always done due to the condition $|N(S)| \geq |S|$, so the sequence will finish in some vertex in B. Let b_k be the last vertex in the sequence, and let

$$P = b_k a_{f(k)} b_{f(k)} a_{f^2(k)} \cdots a_{f^r(k)} \qquad (f^r(k) = 0)$$

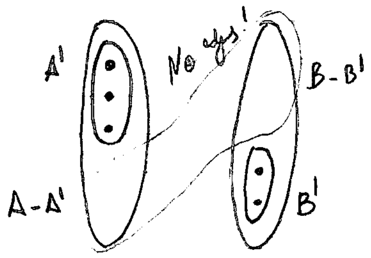
be the corresponding alternating path. Observe that:

a) If b_k is matched, it must be matched to a vertex which has appeared in the sequence (otherwise we could extend the path)

b) Hence, b_k is unmatched and P is an augmenting path!

Another proof using König's theorem:

Proof / Assume that G does not contain a matching of A. Hence, by König's Theorem there exists a cover set of size $< |A|$. Write this cover set $U = A' \cup B'$, $A' \subseteq A$, $B' \subseteq B$:



$$|A'| + |B'| = |U| < |A|$$
$$\Downarrow$$
$$|B'| < |A| - |A'| = |A - A'|$$

And $|N(A-A')| \leq |B'| < |A - A'|$ which is a bad set!!

Consequences: 2 corollaries of Hall's theorem:

Corollary / If G is k-regular with $k \geq 1$, then G has a 1-factor.

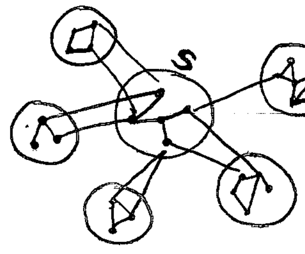
Proof / Just notice that the number of edges incident with a set $S \subseteq A$ is $k|S|$, which is smaller than $k|N(S)|$.

Corollary / (Petersen, 1891) Each $(2k)$-regular graph has a 2-factor.
Proof / Consider an Euler tour and apply the previous corollary when splitting vertices with respect to $\to\circ$ and $\circ\to$.

Matchings in general graphs.

The next question is to study 1-factors in general graphs. For this purpose, let $G = (V, E)$ be a graph and $S \subseteq V$. Denote by $o(G-S)$ the number of connected component of odd size of the graph $G[V-S]$:

$o(G-S) = 3$
$|S| = 5$



It is necessary that $o(G-S) \leq |S|$, because every odd component needs a vertex in S to complete the 1-factor.
$\Downarrow$
In fact the condition is sufficient!

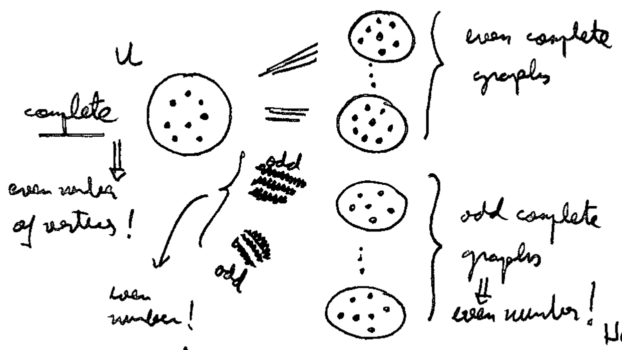
Theorem / (Tutte, 1947) A graph $G = (V, E)$ has a 1-factor iff $o(G-S) \leq |S|$ for all $S \subseteq V$.

Proof / We start with an observation in order to prove the difficult implication. Let G be a graph which satisfies that $o(G-S) \leq |S|$ $\forall S \subseteq V$. Then, if G^* is obtained from G by adding an edge, the number of component of odd order does not increase: once adding a new edge we join component of $G-S$, and two even component generate a unique component; if the component are both odd, once adding an edge the resulting one is even. Something similar can be said when the components have different parity (odd + even $\to$ odd).
Observe also that G has a 1-factor $\Rightarrow$ G^* has a 1-factor.
Hence, we can try to prove the following: it does not exist a graph G such that:

i) Satisfies Tutte's condition
ii) It does not have a 1-factor
iii) Adding any missing edge yields a graph with a 1-factor.

Assume that G exists and let U be the set of vertices in $V(G)$ with degree $|V(G)| - 1$. Two situations might happen:

a) $G[V-U]$ is a disjoint union of complete graphs:

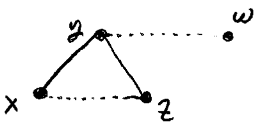


The total number of vertices is even (because adding any edge gives a graph with a 1-factor), hence the number of odd complete graphs must be even.

Each odd complete graph needs to match an odd number of vertices with U (just one, in fact) and match the rest easily. As $o(G-U) \leq |U|$ we can do this.

Hence, U (which defines a complete graph) can be finally matched using the rest of the vertices (not used to match with the odd components).

b) $G[V-U]$ is not a disjoint union of complete graphs: in this case, $G[V-U]$ has two non-adjacent vertices with a common neighbour:

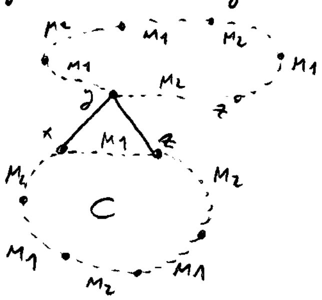


as y does not belong to U, it is not incident with a vertex w.

Observe now that, by assumption, adding either $\overline{xz}$ or $\overline{yw}$, we create graphs with 1-factor. Let M_1 (resp. M_2) the 1-factors obtained when adding $\overline{xz}$ (resp. $\overline{yw}$). We will prove that the set of edges in exactly M_1 or in M_2, joint with $\{\overline{xy}, \overline{yz}\}$ contains a 1-factor avoiding $\overline{yw}$ and $\overline{xz}$. Call this set $\overline{M_1 \Delta M_2} = F$.

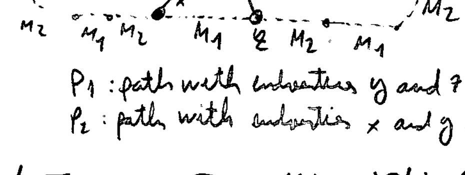
• Observe now that $\overline{yw}$ and $\overline{xz} \in F$. Additionally, each vertex in the matchings M_1 and M_2 have degree 1. So, when joining M_1 and M_2 (killing repeated edges) the resulting graph must be defined in terms of isolated vertices and cycles, which have even length. Distinguish now two cases:

1) The cycle C containing $\overline{xz}$ it does not contain $\overline{yw}$:



just take then the edges in C from M_2 and the rest of the edges of M_1. This new matching covers all vertices, and avoid edges $\overline{xz}$ and $\overline{yw}$.

2) The cycle C containing $\overline{xz}$ it also contains $\overline{yz}$:



P_1 : path with endvertices y and z

P_2 : path with endvertices x and y.

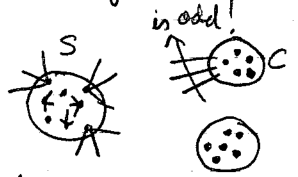
just take then M_1 on the path P_1, M_2 in the path P_2, and edge $\overline{yz}$ to cover y and z.

Obs/ For any $S \subseteq V(G)$, $|S| + o(G-S) \equiv |V(G)| (2)$. Consequently, also the difference $|S| - o(G-S)$. Then, if G do NOT have a 1-factor, and $|V(G)| \equiv 0(2)$ there exist $S \subseteq V(G)$ such that $o(G-S) > |S|$, an in fact $o(G-S) - |S| \geq 2$.

Corollary/ (Petersen' 1891) Every 3-regular graph with no bridges has a 1-factor.

Proof/ Let G be a cubic bridgeless graph. Let us see that Tutte's condition is satisfied

Let $S \subseteq V(G)$ be given, and consider and odd component C of $G-S$:



the sum of the degrees of C is an odd number, but only an even number arises from edges joining vertices in C. So the number of edges joining C and S is odd. Hence it must be greater or equal than 3 (we do not have bridges, hence it cannot be 1). Then the number of such edges is, at least, $3o(G-S)$. But it is also at most $3|S|$, because G is cubic. Hence $o(G-S) \leq |S|$.

In this context one would wonder to know how many 1-factors have a cubic bridgeless graph. This is precisely formulated by the following conjecture:

Conjecture / (Lovász-Plummer, 70's) There exists an universal constant $\varepsilon > 0$ such that, for any cubic bridgeless graph G,

$$2^{\varepsilon |V(G)|} \leq \#\text{ 1-factors} \leq 2^{|V(G)|}$$

After some partial results, now we have an affirmative answer:

Theorem / (Esperet, Kardoš, King, Král, Norine, 2011) One can take $\varepsilon = 1/3656$.