Graph Colouring

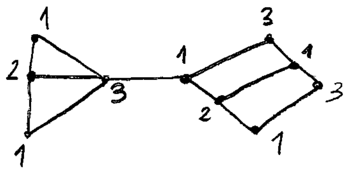
We must start with the basic definition of colouring. We assume all graphs simple.

Def / Let $G=(V,E)$ be a graph. A map $c: V \longrightarrow S$ (where S is a finite set) such that $c(v) \neq c(w)$ when $\overline{vw} \in E$ is called a <u>proper colouring</u>. In this context, element of S are named <u>colours</u>.

Def / The smallest integer k such that there exists a map $c: V \longrightarrow [k]$ is called the <u>chromatic number</u> of G. We write it as $\chi(G)$.

In particular, bipartite graphs are graphs with chromatic number equals to 2.

Example / The chromatic number of K_n is n.



It seems that, in general, given a graph with n vertices, its chromatic number is between 1 (isolated vertices) and n (complete graph, or "almost" complete).

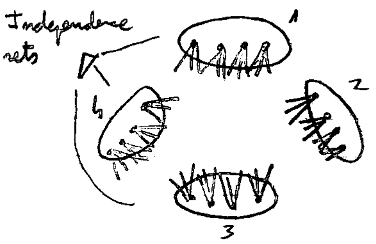
Question:

1) Can we relate this parameter with others in a graph?
2) Can we say that $\chi(G) \leq C$ if we know something additional on G?

We start with the first question. For this purpose, we need some extra definitions:

Def / Let $G=(V,E)$ be a graph. The <u>clique number</u> of G ($\omega(G)$) is the maximum size of a set of pairwise adjacent vertices.

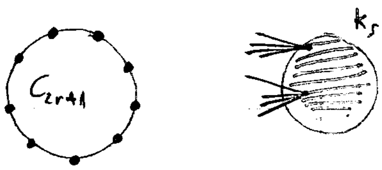
Def / Let $G=(V,E)$ be a graph. The <u>independence number</u> of G ($\alpha(G)$) is the maximum size of a set of independent vertices.



Prop / For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

Proof / The first one is trivial (we need to colour the clique of size $\omega(G)$). For the second, we may prove that $\chi(G)\,\alpha(G) \geq |V(G)|$. Observe that each colour set is an independent set, hence the result follows from this observation.

In particular, we could have graphs with $\chi(G) > \omega(G)$:

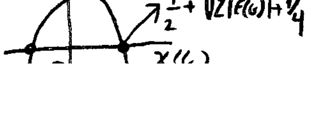


The value $\omega(G)$ is equal to $s+2$ (the cycle does not have triangles), but $\chi(G) \geq s+3$ (we need 3 colours to colour the cycle).

We can also obtain bounds for $\chi(G)$ in terms of $|E(G)|$:

Prop / Let $G=(V,E)$ be a graph. Then $\chi(G) \leq \frac{1}{2} + \sqrt{2|E(G)| + \frac{1}{4}}$.

Proof / Observe that, for a proper colouring of G, the endvertices of a certain edge are coloured with different colours. Observe then that if there is no edge joining vertices of colours i and j, we could then consider a proper colouring of G by identifying colours i and j. Then:



$$|E(G)| \geq \frac{1}{2}\chi(G)\,(\chi(G)-1) \iff \chi(G) \leq \frac{1}{2} + \sqrt{2|E(G)| + \frac{1}{4}}.$$

This is an upper bound in terms of the number of edges. However, we can obtain other upper bounds by algorithmic means:

Prop / (Greedy algorithm) $\chi(G) \leq \Delta(G)+1$ for all graphs G.

Proof / We apply the greedy algorithm: we provide a vertex ordering $v_1, \dots, v_n$ of $V(G)$, and we colour v_i with the smallest indexed colour not already used to colour $v_1, \dots, v_{i-1}$.
As in each vertex ordering, each vertex has at most $\Delta(G)$ earlier neighbours, this algorithm can use less than $\Delta(G)+1$ colours.

This greedy algorithm could be improved if we store the sequence of vertex degrees:

Prop / (Welsh-Powell) If a graph G has degree sequence $d_1 \geq \dots \geq d_n$, then $\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}$.

Proof / Apply again the greedy algorithm.

Theorem / (Brook's Theorem) Let G be a connected graph. If G is neither complete nor and odd cycle, $\chi(G) \leq \Delta(G)$.

Planar graphs and chromatic number

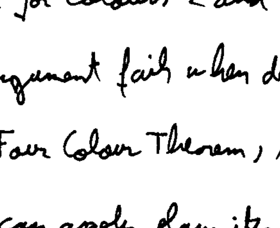
Forcing a certain topology on a graph, would give better improvement for the chromatic number.

Theorem / (Five colour Theorem) Let G be a planar graph. Then $\chi(G) \leq 5$.

Proof / Induction on the number of vertices. The result is true for all graphs with $|V(G)| = 1$. Assume the result true for all graphs which are planar and $|V(G)| < n$, and let us prove for $|V(G)| = n$.
Let G be a planar graph with $|V(G)| = n$, and choose a vertex of G whose degree is ≤ 5. Call it v. Such a vertex exists because G is planar. Distinguish 2 cases:

i) $d(v) < 5$: Then consider the graph $G' = G[V - \{v\}]$, with $|V(G')| = n-1 < n$. By hypothesis it can be coloured with 5 colours. Then, as $d(v) < 5$, we can colour v in G with the colour we haven't used.

ii) $d(v) = 5$. Consider this vertex and its neighbours: we may assume that they are coloured using 5 colours.



- We study the subgraph induced by vertices painted with colours 1 and 3. Observe that if v_1 and v_3 belong to different components of this graph, then we can switch colours on one of the component (For instance, in the one containing v_1), and colour v with colour 1.

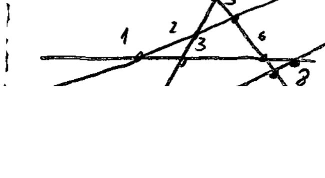
- In the other case, v_1 and v_3 are connected in the subgraph. But then, the corresponding graph for colours 2 and 4 must be disconnected, and we can apply the same argument as before.

Obs / This argument fail when dealing with 4 colours, however, the result is true:

Theorem / (Four Colour Theorem, Appel Haken) Let G be a planar graph. Then $\chi(G) \leq 4$.

We can apply planarity in other contexts.

Example / Consider a set of lines in general position. Prove that the resulting graph has chromatic number ≤ 3.



- Apply an scan argument with a line non parallel to the previous ones: once we find a vertex, it may have at most two predecessors in order $\Rightarrow$ we use then the third colour

Concepts related to vertex colouring

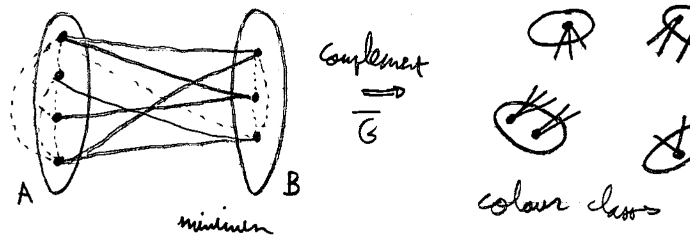
A.- Perfect graphs.

We have seen that the chromatic number can be bounded in terms either of the clique number or the independence number. Additionally, the chromatic number is a local-global condition; hence, one might wonder which graphs have a local behaviour with respect to this parameter.

Def/ A graph G is called <u>perfect</u> if every induced subgraph H of G has chromatic number $\chi(H) = \omega(H)$.

Let us see some examples of perfect graphs:

Example/ 1) Bipartite graphs are clearly perfect.
2) Complete graphs are also perfect, and the complement (isolated vertices) are also perfect.
3) The complement of bipartite graphs are perfect: we just need to prove that if G is a bipartite graph, then $\chi(\overline{G}) = \omega(\overline{G})$ (because every subgraph of the complement of a bipartite graph is the complement of a bipartite graph):



Each colour class have at most 2 vertices (otherwise we will have triangles in G) $\Rightarrow$

$\chi(\overline{G}) = n - \#$ max perfect matching
$= n - \#$ min vertex cover of G.

Now observe that a minimum vertex cover in G defines a clique in $\overline{G}$ with maximum cardinality, hence $\chi(\overline{G}) = \omega(\overline{G})$.

The previous examples relate the perfection of a graph with the perfection of its complementary. And in fact this is the case:

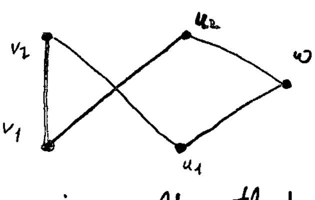
Theorem/ (Berge - Weak Perfect Graph Conjecture - Lovász '72) A graph is perfect iff its complement is perfect.

Something more precise would clarify the structure of such graphs. This is the case in this famous theorem:

Theorem/ (Berge - Strong Perfect Graph Conjecture - Chudnovsky, Robertson, Seymour & Thomas 2002) A graph G is perfect if and only if neither G nor $\overline{G}$ contains an odd cycle of length at least 5 as an induced subgraph.

B.- Graphs with large chromatic number.

In general we have that $\chi(G) \geq \omega(G)$, but one would wonder to know which is the gap between these two values. We show now a construction of graphs without triangles and large chromatic number $\Rightarrow$ Mycielski's construction.



For each vertex v_i in $V(G)$, we make a copy of it (call it u_i), and we join this vertex with all neighbours of v_i. Finally, we add a new vertex w which is adjacent to all vertices $\{u_1, \dots, u_r\} = U$.

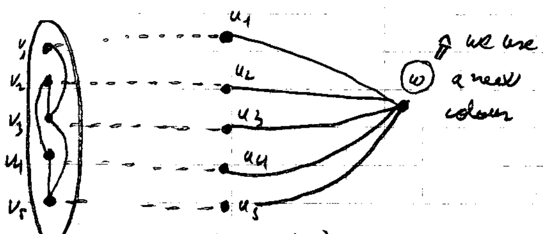
The main result is that applying this operation we would increase the chromatic number of the initial graph.

Theorem / Let G be a graph with $\chi(G)=k$, and triangle-free. Then, Mycielski's construction produces a new graph G', triangle-free, with $\chi(G')=k+1$.

Proof / Write $V(G)=\{v_1,\dots,v_n\}$, $U=\{u_1,\dots,u_n\}$ the copies of vertices in $V(G)$, and w the additional vertex.

i) <u>G' is triangle-free</u>: observe that the set U in G' is independent by construction; so if G' has a triangle, this is because G had one.

ii) <u>$\chi(G')=k+1$</u>: we extend a colouring of G in the following form:



$c(u_i)\equiv c(v_i)$

With this colouring we have that $\chi(G')\le\chi(G)+1$. We show now that $\chi(G)<\chi(G')$. To prove this, we consider a a proper colouring of G' and we get a proper colouring of G with less colours. Assume that the colours are $[s]$, and that $c(w)=s$. Hence, the colours used to paint vertices in U are $[s-1]$.

Let A be the set of vertices in $V(G)$ using colour s. Then, we change its colour by $c(u_i)$ (where $v_i\in A$): this gives a proper colouring of G with $s-1$ colours.

Edge colouring of graphs

One can do similar things when now we colour edges instead of vertices:

Def / Let $G=(V,E)$ be a graph. An <u>edge colouring</u> of G is a map $c: E\longrightarrow[k]$. An edge colouring is <u>proper</u> if $c(e)\neq c(f)$ when e and f share and endpoint. The smallest integer k for which a k-edge colouring exists is the <u>edge-chromatic number</u> of G; we denote it by $\chi'(G)$.

Observations / a) It is clear that $\chi'(G)\ge\Delta(G)$

b) If $L(G)$ is the line graph of G, then $\chi'(G)=\chi(L(G))$

c) $\chi'(G)\le 2\Delta(G)-1$: $\chi'(G)=\chi(L(G))\le\Delta(L(G))+1\le 2(\Delta(G)-1)+1=2\Delta(G)-1$

Theorem / (König, 1916) If G is a bipartite graph, then $\chi'(G)=\Delta(G)$.

Proof / Recall that every k-regular bipartite graph has a 1-factor. Hence, deleting this 1-factor gives a $(k-1)$-regular bipartite graph, and we can repeat the same argument. So we will show that for every bipartite graph G with maximum degree $\Delta(G)=k$, there is a k-regular bipartite graph H containing G.

To start with, add new vertices to the smaller partite set of G (if necessary). If the resulting graph is not regular, this is because there is some vertex whose degree is $<k$ in each partite set. Add an edge with these two vertices as endpoints, and continue until the graph becomes k-regular.

We have shown that $\Delta(G)\le\chi'(G)\le 2\Delta(G)-1$, but in fact we can do much better:

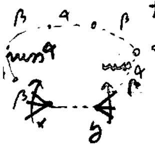
Theorem / (Vizing '64) Let $G=(V,E)$ be a simple graph. Then $\chi'(G)\le\Delta(G)+1$.

Proof / We apply induction on the number of edges of G: for $|E(G)|=0,1$ the result is trivial. Assume that the result is true for graphs with less than k edges and let us prove the result for graphs with k edges.

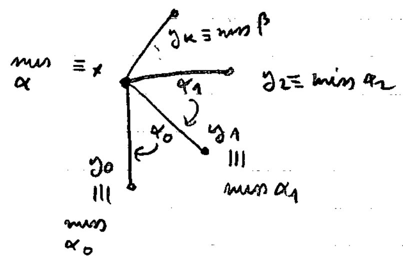
Let G be a graph with $|E(G)|$ edges. Just to simplify notation, if an edge e is coloured with colour α we call e an <u>α-edge</u>. Observe that for every edge $e\in E(G)$, we have an edge colouring of $G'=(V(G), E(G)-e)$ with $\Delta(G')+1$ colours. In such a colouring, the number of colours at a certain vertex is at most $\Delta(G')$, so for each vertex there is a colour $\beta\in\{1,\dots,\Delta(G)+1\}$ which is <u>missing</u> at this vertex.

Finally, for vertices v, w, a colour α and a colour β which is missing at v, w, there exists a unique maximal path (possibly trivial) starting at v and alternating colours α and β and finishing at w. We call this path an <u>α/β-path from v to w</u>.

So assume now that G has no $(\Delta(G)+1)$-edge-colouring. Observe now that given $\overline{xy}=e\in E(G)$ and any colouring of $G'=(V(G), E(G)-\overline{xy})$ in which α is missing at x and β is missing at y, then the α/β-path from y ending at x: otherwise we could interchange colours and painting $\overline{xy}$ with colour α.



Let now be $\overline{xy_0}\in E(G)$, and let $G_0=(V(G), E(G)-\{\overline{xy_0}\})$:

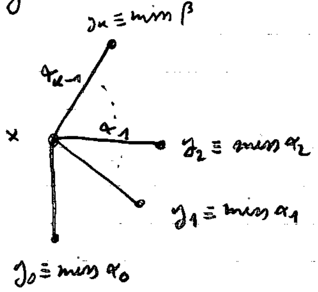


We consider the neighbours of x, and we consider a maximal sequence $y_0, y_1, \dots, y_k$ such that the colour of the edge $\overline{xy_i}$ is the one missed at y_{i-1}.

Then, for each $i=0,\dots,k$, we consider the graph $G_i=(V(G), E(G)-\{\overline{xy_i}\})$, and the colouring c_i:

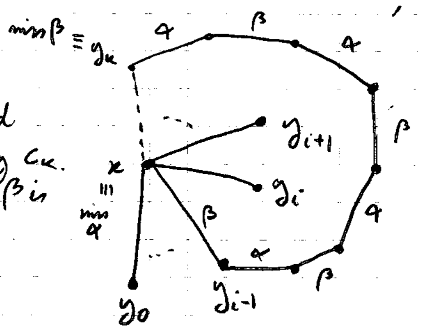
Just move colours. $c_i(e)=\begin{cases} c_0(\overline{xy_{j+1}}) & \text{if } e=\overline{xy_j},\ j\in\{0,\dots,i-1\} \\ c_0(e) & \text{otherwise}\end{cases}$

In particular, the same colours are missed at x. Assume now that β is missing at y_k at the colouring c_0:

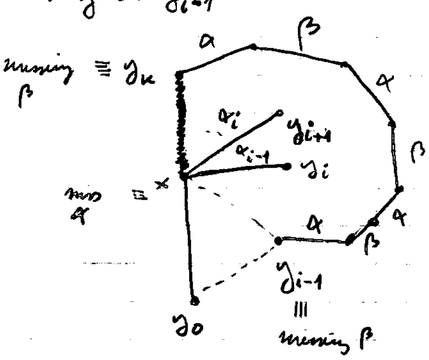


Then, β is still missing at y_k in G_k in c_k. Observe that now x is **NOT** missing β: otherwise we could colour edge $\overline{xy_k}$ with β! Consequently, x is incident with a β-edge, which is xy_r, $r<k$ (the sequence of the y_i is maximal!) Assume that $c_0(xy_i)=\beta$.

↳ And consequently $c_k(xy_{i-1})=\beta$.



We consider the α/β-path starting at x and finishing at y_k, with respect to the colouring c_k. Now observe that in $G_0(c_0)$ and $G_{i-1}(c_{i-1})$, β is missing at y_{i-1}:



Finally, observe that in G_k we cannot have a α/β-path starting at x and finishing at y_{i-1}, because this contradicts the fact that y_k was missing β.

Vizing's Theorem divide graphs into two classes, in terms of $\chi'(G)=\Delta(G)$ or $\chi'(G)=\Delta(G)+1$. Those are called <u>class 1</u> and <u>class 2</u>.