Flows in graphs

To simplify the arguments we will consider simple connected graphs in all this section (However, most of the things we will discuss can be easily generalized to multigraphs).

Def/ A digraf (or oriented graph) (or directed graph) D is a pair $(V, \vec{E})$, where $\vec{E} \subseteq V^2 - \{(v,v) : v \in V\}$.

Roughly speaking, a digraph is obtained from a graph by orienting its edges; we will study a very specific type of digraphs, that we will call network

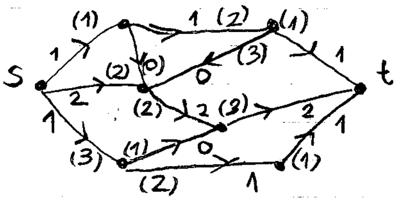
Def/ Let D be an oriented graph, and let $s, t \in V(D)$ (we call s the source and t the sink).
Let $c : \vec{E} \longrightarrow \mathbb{N}$ a map (we call it the capacity function). Then $N := (D, s, t, c)$ is called a network

Observe that c is defined independently of the direction of the edge.

Def/ Let $N = (D, s, t, c)$ be a network. A function $f : \vec{E} \longrightarrow \mathbb{R}^+$ is a flow in N if:

I) For all edge $e \in \vec{E}$, $f(e) \leq c(e)$.
II) For all $v \in V - \{s,t\}$, $f(v,V) := \overbrace{\sum_{\substack{e \text{ edge starting} \\ \text{at } v}} f(e)}^{f^+(v)} - \overbrace{\sum_{\substack{e \text{ edge fin.} \\ \text{at } v}} f(e)}^{f^-(v)} = 0$ (Kirchhoff's law)

Ex/ Here we have a network with source s and sink t, and a certain capacity and an example of flow:



• In all edges, the capacities are bigger than the value of the flow.

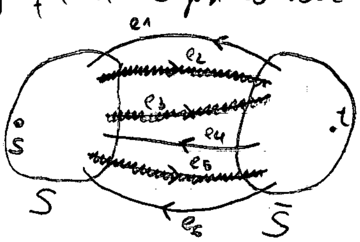
• In every vertex, Kirchhoff's law is satisfied.

So, we will study two questions:
A.- Characterize how is a "maximal" flow, and consequences.
B.- Connection between flows and structural results.

MAX-flow MIN-cut Theorem

Let N be a network and f be a flow. Then

Def/ Let $S \subseteq V$ such that $s \in S$, $t \notin S$. Then the pair $(S, \bar{S})$ is a cut in N, and the sum of the capacities of the edges between S and $\bar{S}$ is the capacity of the cut (we write it $c(S, \bar{S})$). Here, in the capacity of a cut we just consider edges with initial vertex in S. The circulation of the cut is the sum of the flows on each edge (with sign)



$c(S, \bar{S}) = c(e_2) + c(e_3) + c(e_5)$.

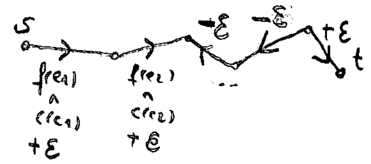
$f(S, \bar{S}) = f(e_2) + f(e_3) + f(e_5) - f(e_1) - f(e_4) - f(e_6)$

The first observation is that, given a cut S on the network N, $f(S,V) = f(S, \bar{S})$ (go from $\{s\}$ to S increasing the size one by one). We call $|f| := f(S, \bar{S})$ the total value of f (it does not depend on S!) Additionally, $|f| = f(S,S) \leq c(S, \bar{S})$ for all choice of S. So a MIN-cut is always an upper bound for a MAX-flow. In fact, we have equality:

Thm / (Ford-Fulkerson, 56) In every network, the maximum total value of a flow equals the minimum capacity of a cut.

Proof/ The proof is algorithmic: we will obtain a sequence of integral flows $f_0, f_1, \dots$, such that $|f_0| < |f_1| < \dots$ (and hence $|f_i| \le |f_{i+1}| + 1$). As all these numbers are bounded by $c(S,\bar{S})$ (for any cut, in particular for the one minimizing $c(S,\bar{S})$) our sequence finishes. Once finished, we would identify the cut.

· <u>Starting point</u>: pick f_0 such that $f_0(e) = 0$ for all $e \in \vec{E}$. Having then defined a flow f_n, we write S_n the set of vertices v of N such that there exist a path starting at s, finishing at v and with the property that $f_n(e_i) < c(e_i)$ for every edge in the path:

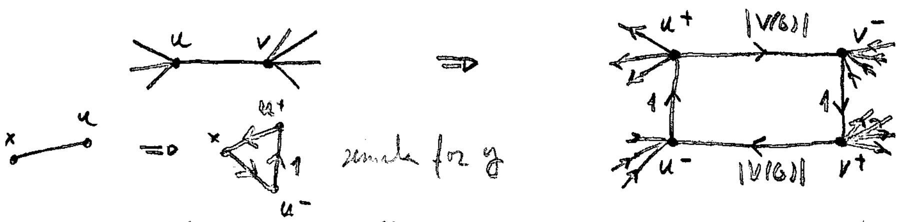


Now, if $t \in S_n$, we can create a new flow in the following way: let $\varepsilon = \min\{c(e_i) - f(e_i),\ e_i \text{ is in a path from } s \text{ to } t\}$, then we sum or substract ε in the path $\Rightarrow$ we are increasing $|f_n|$!

This can be done unless $t \notin S_n$. Then consider the cut $(S_n, \bar{S}_n)$. Then $f_n(e) = c(e)$ for all edge e with starting vertex at S_n and finishing at $\bar{S}_n$. So $|f_n| = c(S_n, \bar{S}_n)$.

Let us see some consequences:

(I) <u>Menger's Theorem</u>: let G be a graph and x, y be vertices in G. We prove the local vertex version of Menger's Thm. We define a transformation of G + an orientation which is suitable:

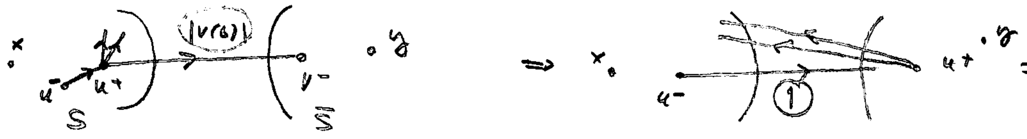


We do this over all vertices which are not x, y.

Let k the maximum flow between x, y (and also the minimum capacity of a cut). We show that this corresponds to $\lambda_G(x,y)$, and $\kappa_G(x,y)$:

a) $\lambda_G(x,y)$: construct this flow using the Ford & Fulkerson algorithm. Then, it always have integral values. As we reach each vertex u^+ using only an edge of capacity 1, it is clear that this k unit of flow are transformed into paths starting at x and finishing at y. Observe that by construction, those paths are vertex disjoint.

b) $\kappa_G(x,y)$: the capacity cut must be defined in terms of edges of the form $u^- \to u^+$:



$\Rightarrow$ we reduce the capacity!!

Then observe finally that every path from x to y in G must contain some vertex associated to the edges of the cut. Hence, the size of this minimal cut is equal to the minimum number of vertices which are necessary to separate x and y: $\kappa_G(x,y)$.

K - Flows on groups.

In this last part we will consider some connections between flows on graphs and structural results. Now we won't consider privileged vertices, in such a way that Kirchhoff law is satisfied for <u>every</u> vertex. Instead of saying that we study flows on networks we say that we study flows on directed graphs (or multigraphs). Those flows are also called circulations.

In general, we can generalize this notion to a general setting:

Def/ Let D be a directed graph and H an abelian group with identity element 0. Then a <u>H-flow</u> is a function $E(D) \to H$, such that $\sum_{\substack{e\ \text{edge} \\ \text{starting } v}} f(e) - \sum_{\substack{e\ \text{edge} \\ \text{finishing at } v}} f(e) = 0$, for all $v \in V$, and f is nowhere zero.

We are specially interested in flows over $\mathbb{Z}$ (or $\mathbb{N}$) and over $\mathbb{Z}/k\mathbb{Z}$. Specially in the first one, we have the following definition:

Def/ A circulation is <u>nowhere-zero</u> if $f(e) \neq 0$ for all $e \in E(D)$.

Def/ A circulation is an <u>k-flow</u> if $\forall e \in E(D)$, $0 < |f(e)| < k$. The <u>flow number</u> of a graph G ($\varphi(G)$) is the minimum of such k (if not, we write $\varphi(G) = \infty$)

Observe in particular that the existence of a k-flow implies the existence of $\mathbb{Z}/k\mathbb{Z}$-flows. (Just take the reduction mod k). Surprisingly, the opposite is also true

Theorem/ (Tutte, 1950) D has a k-flow $\Leftrightarrow$ D has a $\mathbb{Z}/k\mathbb{Z}$-flow.

This theorem provides a way to study the structure of graphs which admit a k-flow with k-small:

Prop/ A graph has a 2-flow iff all its degrees are even.

Proof/ By the previous theorem a graph has a 2-flow iff it has a $\mathbb{Z}/2\mathbb{Z}$-flow.

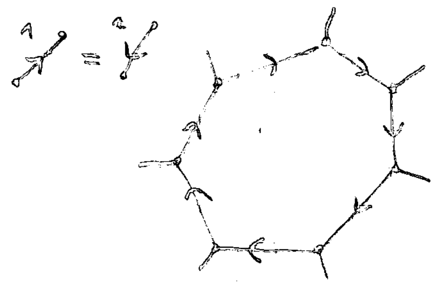
$\Leftarrow$] Consider an Eulerian Tour on G, and write $\bar{1}$ in all edges.
$\Rightarrow$] In every vertex you need an even degree to verify Kirchoff law.

We prove something more involved

Prop/ A cubic graph has a 3-flow if and only if it is bipartite.

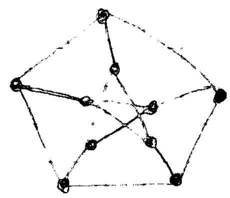
Proof/ We assume that the initial condition is $\mathbb{Z}/3\mathbb{Z}$ instead of 3-flow.

$\Rightarrow$] We show that every cycle has even length:



We consider a cycle and a $\mathbb{Z}/3\mathbb{Z}$-flow on it. Take 2 consecutive edges. They may have different label, otherwise the third incident edge would be label with 0. Hence the length of the cycle is even.

$\Leftarrow$] Easy, use the bipartition!.



In this domain there are still big open conjectures:

Conjecture / (Tutte '54) Every bridgeless (multigraph) has a 5-flow.

Conjecture / (Tutte '66) " " " not containing the Petersen graph as a minor has a 4-flow.

Concerning the 5-flow conjecture, the best result known is due to Seymour:

Theorem / (Seymour, 81) Every bridgeless graph has a 6-flow.