Extremal combinatorics: Introduction & Extremal Set Theory

The main question we will explore in this part is the following: let $\mathcal{F}$ be a combinatorial object satisfying certain restrictions. Then:

1) How large (or how small) can $\mathcal{F}$ be?
2) Which is the "shape" of $\mathcal{F}$ when it reaches its largest (or smallest) size?

We will study problems in extremal combinatorics arising from graph theory and additive combinatorics. But, just to start with, we picture this philosophy with some examples arising from set theory.

Extremal Set Theory

We consider a family $\mathcal{F}$ of subsets of $[n]$ when fixing extra conditions, We will study which is the maximum cardinality of $\mathcal{F}$ (and how is $\mathcal{F}$ when this cardinality is reached).

Intersecting families

We start with an easy definition:

Def/ A family $\mathcal{F}$ of subsets of $[n]$ is intersecting if $\forall A, A' \in \mathcal{F}$, $A \cap A' \neq \emptyset$. An intersecting family $\mathcal{F}$ is k-intersecting if additionally $|A| = k$ for all $A \in \mathcal{F}$.

Let us start with the most basic problem: How large can an intersecting family be? Observe that if $\mathcal{F}$ is intersecting, then $A \in \mathcal{F} \Rightarrow A^c \notin \mathcal{F}$. Consequently $|\mathcal{F}| \leq 2^n/2 = 2^{n-1}$. Observe also that the family $\mathcal{F}_a = \{A \subseteq [n] : a \in A\}$ is intersecting and $|\mathcal{F}_a| = 2^{n-1}$. So this families are extremal ones. However, we could have others:

Ex/ Consider (if n is odd) the set of subsets with size $> \frac{n}{2}$. The cardinal of this family is 2^{n-1} and it is intersecting. Additionally, there is NOT an element in all sets.

Next step is studying k-intersecting families. As before, we could try to fix an element in all elements in the family, giving the value $\binom{n-1}{k-1}$. In fact, this is the best possible when $n \geq 2k$:

Theorem/ (Erdős–Ko–Rado, 38) If $n \geq 2k$ then every k-intersecting family of $[n]$ has at most $\binom{n-1}{k-1}$ elements.

Proof/ We use the Katona's circular Method: the idea is to study the action of cyclic shifts on permutations of $[n]$. For convenience we consider sets over $\{0, 1, \dots, n-1\}$ instead of over $[n]$. Let $\mathcal{F}$ be a k-intersecting family.

Denote by B_s the set $\{s, s+1, \dots, s+k-1\} \subseteq \{0, \dots, n-1\}$, where the indices are taken mod n, and $s = 0, \dots, n-1$.

Claim/ If $\mathcal{F}$ is a k-intersecting family, then at most k sets B_s belong to $\mathcal{F}$.

Let us assume that $B_0 \in \mathcal{F}$. Then, the unique B_s that could belong to $\mathcal{F}$ are the ones with $-(k-1) \leq s \leq k-1$, so $2k-2$ (the indices are taken mod n). But now, the sets B_i and B_{i+k}, with $-(k-1) \leq i \leq 1$ are disjoint. Hence, $\mathcal{F}$ can contain at most 1 set of each pair $\Rightarrow (2k-2)/2 + 1 = k$.

Having observed this, let us count the number of pairs (f, s), where f is a permutation of $\{0,1,\dots,n-1\}$ and $s \in \{0,1,\dots,n-1\}$ satisfies that $f(B_s) := \{f(s), f(s+1), \dots, f(s+k-1)\} \in \mathcal{F}$. By apply the double counting method.

- Fix the permutation: by the claim, for each permutation we have at most k choices for $s \Rightarrow \leq k\,n!$
- Fix the parameters: for each s, and for every element $A \in \mathcal{F}$ we need to compute the number of permutations of the form:

$$A = \underbrace{\begin{matrix} s & s+1 & s+2 & \dots & s+k-1 \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ a_1 & a_2 & a_3 & \dots & a_k \end{matrix}}_{k!} \cup \underbrace{\boxed{\begin{matrix} [n]-\{s,s+1,\dots\} \\ \downarrow \\ [n]\setminus A \end{matrix}}}_{(n-k)!} \Rightarrow \begin{matrix} k!\,(n-k)! \\ \Downarrow \text{Total} \\ |\mathcal{F}| \cdot n \cdot k!\,(n-k)! \end{matrix}$$

Hence, $k \cdot n! \leq |\mathcal{F}| \cdot n \cdot k!\,(n-k)! \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$

Antichains, chains and Sperner's Theorem

We move now to another family of sets.

Def / A set $\mathcal{F}$ of subsets of $[n]$ is an <u>antichain</u> if $\forall A, B \in \mathcal{F}$, $A \neq B \Rightarrow A \not\subseteq B$ and $B \not\subseteq A$.

Example / Take $\mathcal{F}_i$ the set of subsets of size i. This is clearly an antichain of size $\binom{n}{i}$. In particular, the maximum value for this binomial is reached when $i = \lfloor \frac{n}{2} \rfloor$.

In fact, the previous case give the extremal situation:

Theorem / (Sperner '28) F is an antichain iff $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$ (F is defined over $[n]$)

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In order to prove it, we prove the so-called LYM inequality:

Theorem / (LYM inequality) Let F be an antichain over $[n]$. Then $\sum_{A \in F} \binom{n}{|A|}^{-1} \leq 1$

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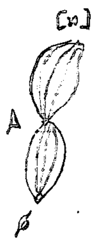
Proof / We will prove that $\sum_{A \in F} |A|!\,(n-|A|)! \leq n!$. In order to do so, we consider the set lattice; a <u>chain</u> on the lattice subset of $[n]$ is a set $\{A_1, A_2, A_3, \dots\}$, $A_i \subseteq [n]$, such that $\forall i, j$, $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Is obvious that extremal chains are of the form $A_1 = \{a_1\}$, $A_2 = \{a_1, a_2\}, \dots, A_n = \{a_1, \dots, a_n\} = [n]$. So the total number of extremal chains is equal to $n!$.



Let us count now the number of pairs $(A, \mathcal{C})$, where A is an element of $\mathcal{F}$, and $\mathcal{C}$ is an extremal chain with $A \in \mathcal{C}$.

- Fix A: A is contained in exactly $|A|!\,(n-|A|)!$ antichains.
- Fix $\mathcal{C}$: every extremal antichain can contain at most 1 element in $\mathcal{F}$, otherwise we would have to elements A, A' such that $A \subseteq A'$.

Hence, $\sum_{A \in F} |A|!\,(n-|A|)! = \#\text{ pairs } (A, \mathcal{C}) \leq 1 \cdot n!$

Now Sperner's Theorem is clear: $|F| \binom{n}{\lfloor n/2 \rfloor}^{-1} \leq \sum_{A \in F} \binom{n}{|A|}^{-1} \leq 1$, from which we conclude the statement. Observe that here equality is reached iff F is the set of subsets of size $\lfloor n/2 \rfloor$.