Enumerative Methods

In this part we will show how to use generating functions in a clear way in order to be able to get automatic results on enumerative results.

Generating Functions and counting

Assume that $(a_n)_{n\in\mathbb{N}}$ is a sequence of integers that count certain combinatorial objects. Then, we can define

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

This is what is called the ordinary generating function associated to the family. The use of generating functions has been proved to be powerful when solving enumerative problems where $(a_n)_{n\in\mathbb{N}}$ satisfies a linear recurrence relation:

Ex/ Fibonacci numbers. Let $a_n = a_{n-1} + a_{n-2}$, where $a_0 = a_1 = 1$. Then the technique of the generating functions reads in the following way:

$$\left.\begin{array}{rcl} a_2 & = & a_1 + a_0 \\ a_3 x & = & a_2 x + a_1 x \\ + \quad \vdots & & \\ a_n x^{n-2} & = & a_{n-1} x^{n-2} + a_{n-2} x^{n-2} \\ \vdots & & \\ \hline \frac{A(x) - a_0 - a_1 x}{x^2} & = & \frac{A(x) - a_0}{x} + A(x) \end{array}\right\} \Rightarrow \begin{array}{l} A(x) - a_0 - a_1 x = x A(x) - x a_0 + x^2 A(x) \Rightarrow \\ \Rightarrow A(x)(1 - x - x^2) = a_0 + (a_1 - a_0) x \Rightarrow \\ \Rightarrow A(x) = \frac{a_0 + (a_1 - a_0)x}{1 - x - x^2} = \frac{1}{1 - x - x^2} = \frac{-1}{x^2 + x - 1} \end{array}$$

Now we write this using simple fractions: $x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$, $\phi = \frac{-1+\sqrt{5}}{2}$, $\bar{\phi} = \frac{-1-\sqrt{5}}{2}$

$$\Rightarrow \left\{\begin{array}{l} \phi \cdot \bar{\phi} = -1 \\ \phi + \bar{\phi} = -1 \\ \phi - \bar{\phi} = \sqrt{5} \end{array}\right. \Rightarrow \frac{-1}{x^2 + x - 1} = \frac{-1}{(x - \phi)(x - \bar{\phi})} = \frac{A}{x - \phi} + \frac{B}{x - \bar{\phi}} = \frac{(A+B)x - (B\phi + A\bar{\phi})}{(x - \phi)(x - \bar{\phi})}$$

Hence $A = -B$, and $-(B\phi + A\bar{\phi}) = -1 \Rightarrow A(\phi - \bar{\phi}) = 1 \Rightarrow A = 1/\sqrt{5}$. So,

$$A(x) = \frac{-1/\sqrt{5}}{x - \phi} + \frac{1/\sqrt{5}}{x - \bar{\phi}} = \frac{1}{\sqrt{5}} \left(\frac{+1}{1 - x/\phi} \frac{1}{\phi} + \frac{-1}{1 - x/\bar{\phi}} \frac{1}{\bar{\phi}} \right) = \left\{ \begin{array}{l} 1/\phi = \frac{1+\sqrt{5}}{2} \\ 1/\bar{\phi} = \frac{1-\sqrt{5}}{2} \end{array}\right.$$

$$= \frac{1}{\sqrt{5}} \left(\sum_{n\geq 0} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} x^n - \sum_{n \geq 0} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} x^n \right) \Rightarrow [x^n] A(x) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right]$$

The previous argument can be extended in the following way:

Theorem/ Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of numbers. Then, the following properties are equivalent:

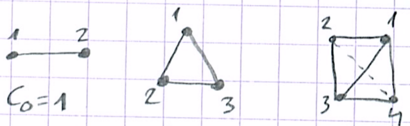
1. The numbers a_n satisfy $a_{n+k} + c_1 a_{n+k-1} + \ldots + c_k a_n = 0$, with $c_1, \ldots, c_k$ fixed values.
2. The generating function $A(x) = \sum_{n\geq 0} a_n x^n$ is of the form:

$$A(x) = \frac{P(x)}{1 - c_1 x - \ldots - c_k x^k}, \text{ where the degree of } P(x) \text{ is} < k$$

3. The numbers a_n satisfy an expression of the form $a_n = P_1(n) \lambda_1^n + \ldots + P_k(n) \lambda_k^n$, where $1 - c_1 x - \ldots - c_k x^k = (1 - \lambda_1 x)^{e_1} \cdots (1 - \lambda_k x)^{e_k}$, and $P_i(n)$ is a polynomial of degree $< e_i$.

This theorem covers essentially all linear recurrences. Let us see a prototype for a non-linear recurrence.

Example / Triangulations of a polygon. Consider the number C_n of triangulations of an $(n+2)$-agon:



$C_0 = 1$ $\qquad$ $C_1 = 1$ $\qquad$ $C_2 = 2$

$$\Rightarrow C_{n+1} = C_0 C_n + C_1 C_{n-1} + \dots + C_n C_0$$

$$\Rightarrow C(x) = \sum_{n\geq 0} C_n x^n.$$

Now the key point is to observe that if $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$, then $A(x)B(x) = \sum_{n\geq 0}\left(\sum_{m=0}^{n} a_m b_{n-m}\right)x^n$. Hence, the previous recurrence gives the following:

$$C(x) = 1 + xC(x)^2 \Rightarrow C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Now we use the following:

$$(1-4x)^{1/2} = \sum_{n\geq 0}\binom{1/2}{n}(-4x)^n, \quad \binom{1/2}{n} = \frac{(\frac{1}{2})(-\frac{1}{2})\cdots(-\frac{2n-3}{2})}{n!} \Rightarrow C_{n-1} = -\frac{1}{2}\binom{1/2}{n}(-4)^n =$$

$$= -\frac{1}{2}\frac{1}{n!}\cdot\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)\cdot(-1)^n 2^{2n} = \frac{1}{2}\frac{1}{n!}\frac{1}{\cancel{2^n}}(1)\cdot(3)(5)\cdots(2n-3)\cdot 2^{2n} =$$

$$= \left[2^{n-1}(n-1)! = 2\cdot 4\cdot\ldots\cdot(2n-2)\right] = \frac{1}{2}\frac{1}{n!}\frac{1}{2}\frac{(n-1)!}{(n-1)!}\frac{1}{\cancel{2^n}}\, 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-3)\cdot \cancel{2^{n}}\cdot 2$$

$$= \frac{1}{2}\frac{1}{2}\frac{(2n-2)!}{n!(n-1)!}2^2 = \frac{1}{n}\binom{2n-2}{n-1} \Rightarrow C_n = \frac{1}{n+1}\binom{2n}{n} \Rightarrow \text{Catalan numbers}$$

In both examples we have seen the following scheme:

$$\text{Combinatorial problem} \longrightarrow \text{Recurrence} \longrightarrow \text{Equation for the GF} \longrightarrow \text{Extraction of coefficients}$$

Symbolic Methods $\qquad$ Lagrange inversion formulas.

Before going to the Symbolic Methods, let us recall some computation tools: let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences with GFs $A(x)$ $B(x)$. Then:

a) $(a_n + b_n) \to A(x) + B(x)$
b) $(\sum_{i=0}^{n} a_i b_{n-i}) \to A(x)B(x)$
c) $(a_{n+k}) \to (A(x) - a_0 - a_1 x - \ldots - a_{k-1}x^{k-1})/x^k$
d) $(n a_n) \to x\frac{d}{dx}A(x)$
e) $(a_0 + \dots + a_n) \to A(x)/1-x$.

The Symbolic Method and OGF

Let A a family of objects (possibly infinite), joint with a function $|\cdot|_A : A \to \mathbb{N}$ (called the size). Assume that $|A_n| = |\{\alpha\in A : |\alpha|_A = n\}| < \infty$ for each n. Under this assumption we can define the GF

$$A(x) = \sum_{n\geq 0}|A_n|x^n = \sum_{\alpha\in A}x^{|\alpha|} = \sum_{n\geq 0}a_n x^n$$

Let us see some basic contructions:

ⓐ <u>Disjoint union</u>: given two classes A and B, the disjoint union (or the sum) is the union of two disjoint copies of A and B. Then:

$$|g|,\ g \in A \dot{\cup} B = \begin{cases} |g|_A \text{ if } g \in A \\ |g|_B \text{ if } g \in B \end{cases}$$

ⓑ <u>Product</u>: given two classes A, B, the product of A and B is the cartesian product $A \times B$. For a pair $(\alpha, \beta) \in A \times B$ its size is

$$|(\alpha,\beta)| = |\alpha|_A + |\beta|_B$$

ⓒ <u>Sequence</u>: for a class A, the sequence of A ($\mathrm{Seq}\{A\}$) is $\mathrm{Seq}\{A\} = \{\varepsilon\} \dot{\cup} A \dot{\cup} A \times A \dot{\cup} A \times A \times A \dot{\cup} \ldots$ where ε is an object of size 0. The size of the element $(\alpha_1, \alpha_2, \ldots, \alpha_t)$ is $|\alpha_1| + \ldots + |\alpha_t|$.

Theorem / The previous contructions can be translated in the GF setting in the following way:

① If $A = B \dot{\cup} C$, then $A(x) = B(x) + C(x)$.
② If $A = B \times C$, then $A(x) = B(x) \cdot C(x)$.
③ If $A = \mathrm{Seq}(B)$, then $A(x) = 1/1 - B(x)$.

Let us see some examples:

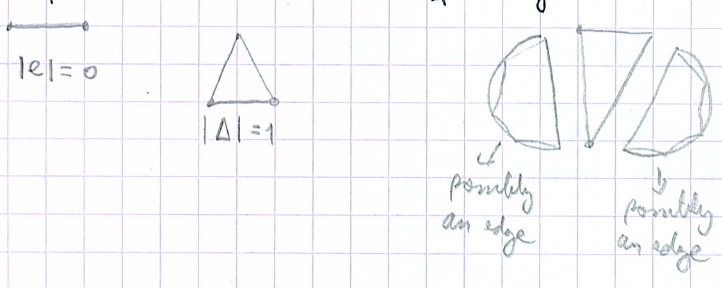
Example / Let P the set of all words with letters in the alphabet $\{0,1\}$. Then $P = \mathrm{Seq}(A)$, where $A = \{0,1\}$. Then:

$$P(x) = \frac{1}{1-A(x)} = \frac{1}{1-2x} \Rightarrow [x^n]P(x) = 2^n.$$

Example / let P be the set of coverings of the intervals $1,\ldots,n$ with pieces of the form $\{\bullet, \bullet\!\!-\!\!\bullet\}$. Then, again, $P = \mathrm{Sq}(A)$, where $A = \{\bullet, \bullet\!\!-\!\!\bullet\}$, and $A(x) = x + x^2$. Hence,

$$P(x) = \frac{1}{1-x-x^2} \Rightarrow \text{Fibonacci numbers.}$$

Example / let C be the class of triangulations of a labelled polygon. Then, in this situation the size of an element is the number of triangles.



$$\Rightarrow C = e \dot{\cup} C \times \Delta \times C$$
$$\Downarrow$$
$$C(x) = 1 + x\,C(x)^2 \Rightarrow \text{Catalan}$$

Example / Compositions and partitions: given an integer n, a <u>composition</u> of n is a solution $\neq 0$ of the equation $x_1 + x_2 + \ldots + x_k = n$, where k goes from 1 up to n (order matters). Then, the class of compositions of integers is $\mathrm{Seq}(\mathbb{N})$, where $\mathbb{N} = \{1,2,3,\ldots\}$. Hence,

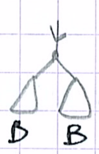
$$C(x) = \frac{1}{1 - I(x)} = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x} \rightsquigarrow \underline{\underline{2^{n-1}}}$$

For the <u>partitions</u>, here the order is <u>NOT</u> important, hence, $P = \mathrm{Seq}\{1\} \times \mathrm{Seq}(2) \times \ldots$ and so, we have that

$$\prod \quad 1$$

Example / Decompositions of a polygon into quadrangles: $Q(x) = 1 + x\,Q(x)^3$.

Example / Let $\mathcal{B}$ be the set of rooted binary trees:

⇒ The size is the number of internal vertices ⇒ $\mathcal{B} = \,$  $\dot\cup$ 

external v. / internal vertex

⇒ $\mathcal{B} = \mathcal{E} \,\dot\cup\, \mathcal{B} \times \mathcal{N} \times \mathcal{B}$

⇒ $B(x) = 1 + x\,B(x)^2$.

Labelled structures and EGF

Now consider a combinatorial class A where each element is formed by atoms, and each atom has a different label. The size now is the number of atoms: if the size of α is n, then the labels used are the ones in $[n]$. We write ε for the element with size equal to 0, without a label. In this situation we will consider $A_n = \{\alpha \in A : |\alpha| = n\}$, and again, we assume that $|A_n| < \infty$ for all n. Then, we consider the exponential generating function

$$A(x) = \sum_{\alpha \in A} \frac{1}{|\alpha|!} x^{|\alpha|} = \sum_{n \geq 0} a_n \frac{x^n}{n!} = \sum_{n \geq 0} \frac{|A_n|}{n!} x^n$$

We show now that this term $n!$ is convenient in order to deal with the labels.

ⓐ Disjoint union: we need to take care of the labels: let A and B be combinatorial classes, then we have the following:

$$\gamma \in A \,\dot\cup\, B \Rightarrow |\gamma| = \begin{cases} |\gamma|_A & \text{if } \gamma \in A \\ |\gamma|_B & \text{if } \gamma \in B \end{cases}$$

⇒ the same as in unlabelled families ⇒ $A(x) + B(x)$

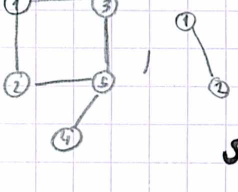
ⓑ Labelled product: this is the key part here. We need a definition. Let α be an element with size n which is NOT labelled with elements in $\{1,2,\ldots,n\}$. Then $\rho(\alpha)$ is an element (identic to α) where the labels are now in $\{1,\ldots,n\}$, and in increasing order. For instance, $\rho(4,8,3,6,2) = (3,5,2,4,1)$. We say then that $\rho(\alpha)$ is well labelled.

Given now two structures A, B (labelled), and two elements $\alpha \in A$, $\beta \in B$, the labelled product is:

$$\alpha * \beta = \{(\alpha', \beta') : \rho(\alpha') = \alpha,\ \rho(\beta') = \beta,\ (\alpha', \beta') \text{ is well-labelled}\}$$

Let us see how is translated into GFs. Assume that α has size n and β has size m. then:

$$\frac{x^n}{n!} \cdot \frac{x^m}{m!} = \frac{(n+m)!}{n!\,m!} \frac{x^{n+m}}{(n+m)!} = \underbrace{\binom{n+m}{n}}_{\text{Number of ways to choose the labels for } \alpha'} \frac{x^{n+m}}{(n+m)!}$$



So, $A * B = \bigcup_{\substack{\alpha \in A \\ \beta \in B}} (\alpha * \beta)$, and $A(x)\,B(x)$ is the corresponding GF.

ⓒ Sequence: Let A be a labelled class. Then the $\mathrm{Seq}(A)$ (sequence of A) is $\varepsilon \,\dot\cup\, A \,\dot\cup\, A * A \,\dot\cup \cdots$ Again, the GF is $1/(1 - A(x))$

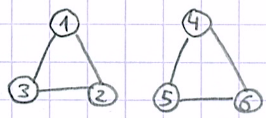
ⓓ Set: given a labelled class A, the class of sets of A is $\{a_1, \ldots, a_k\} \subseteq A$ (formed by el. of the form). Then, this is equal to $A * \overset{k}{\cdots} * A /\sim$, where two elements are the same by $\sim$ if they have the same element (in different order). This gives $\frac{1}{k!} A(x)^k$.

If we do not restrict now to the size k, we have the Set construction: $\mathrm{Set}(A) = \varepsilon \,\dot\cup\, A \,\dot\cup\, A * A/\sim \,\dot\cup\, A * A * A/\sim \,\dot\cup \cdots \Rightarrow 1 + A(x) + \frac{1}{2!} A(x)^2 + \frac{1}{3!} A(x)^3 + \cdots = \exp(A(x))$

Let us see some examples:

Ex / Permutations and cyclic permutations: let $\mathcal{S}$ and $\mathcal{C}$ denote the class of permutations and cyclic permutations, and write $\mathcal{S}_n$ and $\mathcal{C}_n$ for the ones of size n:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 5 & 6 & 4 \end{matrix} \qquad \underbrace{(1,3,2)}_{\text{cyclic per}} \underbrace{(4,5,6)}_{\text{cyclic per.}} \Rightarrow$$



Then we know that $|\mathcal{S}_n| = n!$, $|\mathcal{C}_n| = (n-1)!$. Then, the corresponding generating functions are:

$$S(x) = \sum_{n\geq 0}^{\infty} \frac{1}{n!} \cdot n! \, x^n = \frac{1}{1-x}, \quad C(x) = \sum_{n=0}^{\infty} (n-1)! \frac{1}{n!} x^n = -\log(1-x).$$

In particular, a permutation is a set of cycle permutations. Hence, $\mathcal{S} = \mathrm{Set}(\mathcal{C})$, and so:

$$S(x) = \frac{1}{1-x} = \exp(C(x)) = \exp(-\log(1-x)) = \frac{1}{1-x} \quad //$$

Ex / Derangements and involutions: a derangement is a permutation without fixed points, or equivalently a set of cycle permutations avoiding the one of size 1.

So, write $\mathcal{C}_{\geq 1}$ for cycle permutations of size ≥ 1. Then,

$$D(x) = \exp(C(x) - x) = \frac{\exp(-x)}{1-x}$$

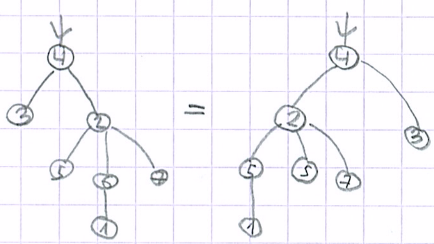
An involution is a permutation π such that $\pi^2 = Id$. This is equivalent to the fact of having just cycles of lenght 1 or 2.

$$I(x) = \mathrm{Set}(\{\circ, \circ\!-\!\circ\}) = \exp\left(x + \frac{x^2}{2!}\right)$$

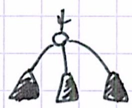
Ex / Set partitions: given the set $[n]$, a set partition of it is a partition into disjoint subsets. each part is also called a block. Then, $\mathcal{P}$ denots the class of partitions.

So $\mathcal{P} = \mathrm{Set}(\mathrm{Set}(\bullet) - \{\emptyset\}) \Rightarrow P(x) = \exp(\exp(x) - 1)$. In particular, if we focus ourselves to partitions with exactly k blocks we have $\frac{1}{k!}(\exp(z) - 1)^k$.

Ex / Labelled trees: let $\mathcal{T}$ be the class of rooted labelled trees, without an embedding o the plane.



$$\Rightarrow \mathcal{T} = \bullet \times \mathrm{Set}(\mathcal{T}):$$



$$\Rightarrow T(x) = x\, e^{T(x)}$$

Now, observe that if $\mathcal{U}$ is the same family without a root, then $xU'(x) = T(x)$. We can solve this equation:

$$U(x) = \int_0^x \frac{T(s)}{s}\, ds = \left\{ \begin{array}{l} T(s) = u \\ T'(s)\,ds = du \end{array} \right\} \Rightarrow \int_0^x e^{T(s)}\, ds = \int_{T(0)}^{T(x)} 1 - u \, du = T(x) - \frac{1}{2} T(x)^2 \quad //$$

$$\Downarrow$$

$$T'(s) = \frac{e^u}{1-u}$$

Lagrange inversion formula

We have found some situations in which we have counting formulas of the form $A(x) = \Phi(x, A(x))$. Can we say something about the coefficients of $A(x)$?

Theorem / (Lagrange inversion Formula) Assume that $A(x)$ satisfies an equation of the form $A(x) = x\,\Phi(A(x))$, where the function $\Phi(t)$ satisfies that $\Phi(0) \neq 0$. Then,

$$[x^n]\,A(x) = \frac{1}{n}[t^{n-1}]\,\Phi(t)^n \,;\quad [x^n]\,A(x)^k = \frac{k}{n}[t^{n-k}]\,\Phi(t)^n$$

We will apply this result in some of the previous combinatorial families.

Ex/ Rooted and unrooted trees. Recall that we obtained that $T(x) = x e^{T(x)}$, $U(x) = T(x) - \frac{1}{2}T(x)^2$. Hence, starting from $T(x)$, we have that $\Phi(t) = \exp(t)$. Hence:

$$[x^n]\,T(x) = \frac{1}{n}[t^{n-1}]\,e^{tn} = \frac{1}{n}[t^{n-1}]\sum_{r=0}^{\infty}\frac{(tn)^r}{r!} = \frac{1}{n}\cdot\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}.$$

Hence, the number of rooted labelled trees with n vertices is n^{n-1}, and the number of unrooted ones is n^{n-2}. Let us obtain this last value by getting $[x^n]\,T(x)^2$:

$$[x^n]\,T(x)^2 = \frac{2}{n}[t^{n-2}]\,e^{tn} = \frac{2}{n}[t^{n-2}]\sum_{r=0}^{\infty}\frac{(tn)^r}{r!} = \frac{2}{n}\,\frac{n^{n-2}}{(n-2)!} =$$

$$= \frac{2}{n!}\,n^{n-2}(n-1) = \frac{1}{n!}\left(n^{n-1} - n^{n-2}\right) \Rightarrow [x^n]\,T(x) - \frac{1}{2}T(x)^2 = \frac{1}{n!}\left(n^{n-1} - \frac{1}{2}\left(2n^{n-1} - 2n^{n-2}\right)\right)$$

Ex/ Catalan numbers and generalizations. Recall that the Catalan function satisfies $C(x) = 1 + xC(x)^2$. Now, we need to modify a little the shape of this equation in order to apply the Lagrangian formalism: write $D(x) = C(x) - 1$. Then,

$$D(x) = x\left(D(x)+1\right)^2 \Rightarrow \Phi(t) = (1+t)^2 \Rightarrow [x^n]\,D(x) = \frac{1}{n}[t^{n-1}]\,(1+t)^{2n} =$$

$$= \frac{1}{n}[t^{n-1}]\sum_{r=0}^{2n}\binom{2n}{r}t^r = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n}\,\frac{(2n)!}{(n-1)!\,(n+1)!} = \frac{1}{n+1}\binom{2n}{n}$$

We can use the same methodology when we have the generalization of the Catalan function $C_k(x) = 1 + xC_k(x)^k$. Then, write $D_k(x) = -1 + C_k(x)$, then,

$$D_k(x) = x\left(D_k(x)+1\right)^k \Rightarrow [x^n]\,D_k(x) = \frac{1}{n}[t^{n-1}]\,(1+t)^{kn} = \frac{1}{n}\binom{kn}{n-1} =$$

$$= \frac{1}{n}\,\frac{(kn)!}{(n-1)!\,(kn-n+1)!} = \frac{1}{(k-1)n+1}\binom{kn}{n}.$$