Graph connectivity

Our objective in this part is double (we will assume simple graphs in all this section):

1) Study the structure of connected, 2-connected and 3-connected graphs.
2) Justify the name "connectivity" ⇝ Menger's Theorem

But before that, we should start recalling some definitions:

Def/ Let $G=(V,E)$ be a graph. We say that $X \subseteq V$ is a <u>separating set</u> or <u>vertex cut</u> if $G[V-X]$ has more than one component. The <u>connectivity of G ($K(G)$) is the minimum size</u> of a <u>separating set</u>, and the graph G is then $K(G)$-connected. $\boxed{\text{We may assume that } |V(G)| > K(G)}$

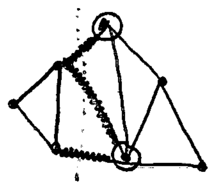
or has only one vertex!: K_1

edge!!

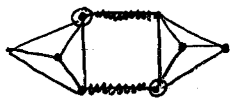
Of course this definition can be also adapted in the context of edges:

Def/ Let $G=(V,E)$ be a graph. We say that $Y \subseteq E$ is a <u>disconnecting set of edges</u> if the graph $G'=(V, E-Y)$ has more than one component (also called <u>edge cut</u>). The <u>edge connectivity</u> of G, ($K'(G)$) is the minimum size of an edge cut. (As before, $|E(G)| > K'(G)$).

Ex/



G
$K(G)=2$
$K'(G)=3$



G'
$K(G)=2$
$K(G')=2$

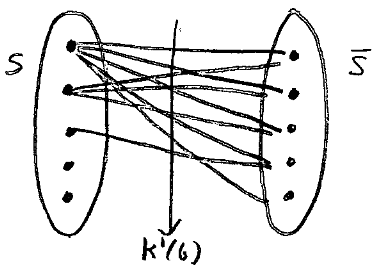
$\begin{cases} K(K_n) = n-1 \\ K'(K_n) = n-1 \end{cases}$

So now the questions are the following:
a) Which is the relation between $K(G)$ and $K'(G)$?
b) When $K'(G)=K(G)$?

Theorem/ (Whitney'32) If G is a simple graph, $K(G) \leq K'(G) \leq \delta(G)$.

Proof/ • $K'(G) \leq \delta(G)$: the edges incident to a vertex v of minimum degree define an edge cut (possibly not of smallest size)

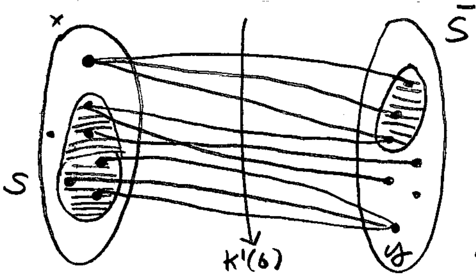
• $K(G) \leq K'(G)$: Of course $K(G) \leq |V(G)|-1$. Let Y be an edge cut of smallest size, and write $S, \bar{S}$ for the two defined components (which are non-empty):



a) All vertices in S are connected with all vertices in $\bar{S}$. Then

$$K'(G) = |S||\bar{S}| \underset{\underset{|S|+|\bar{S}|=|V(G)|}{\uparrow}}{\geq} |V(G)|-1 \geq K(G)$$

b) Not all vertices of S are linked to all vertices of $\bar{S}$: choose $x \in S$ and $y \in \bar{S}$ not incident with an edge. Let T be the set of vertices: neighbours of x in $\bar{S}$, and vertices of $S-\{x\}$ with neighbours in $\bar{S}$.



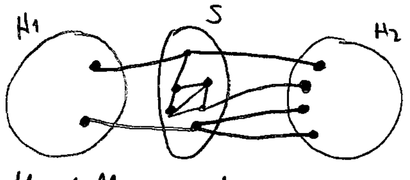
Then every path from x to y pass through T, hence T is a separating set. Then:

$K(G) \leq |T| \leq K'(G)$

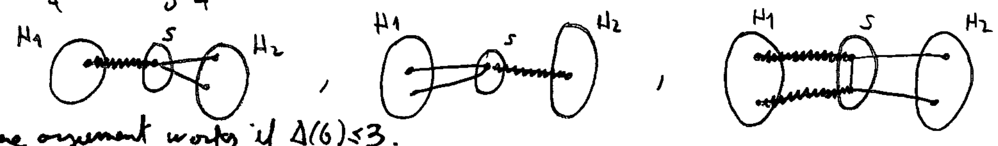
Concerning the second question, there exist several ways to characterize this fact. We just show one example.

Theorem/ If G is a 3-regular graph, then $K(G) = K'(G)$.

Proof/ Let S be a min. vertex cut. We will construct an edge cut of the same size. As $K(G) \leq K(G')$, we will conclude that $K(G) = K'(G)$



Let H_1 and H_2 two connected component of $G-S$. Observe that every vertex in S is adjacent once with some vertex at H_1 and in H_2. But a vertex in S cannot be adjacent with H_1 and H_2 twice. We define an edge set S' in the following form:



Then S' is an edge cut with the desired size.

The same argument works if $\Delta(G) \leq 3$.

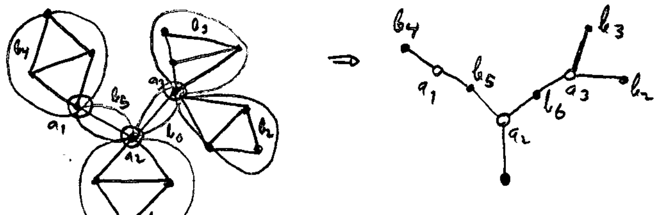
Structural results.

Let us move to describe how a (1-2-3)-connected graph is. We will consider vertex-con.

a) <u>Connected graphs</u>: connected graphs can be described in terms of its blocks (Recall, maximal 2-connected subgraphs)

Let A denote the set of cutvertices of a graph G, and B the set of its blocks:

NOT POSSIBLE!



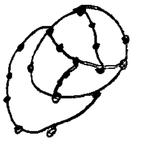
Two blocks can intersect in at most 1 cut vertex

$\Downarrow$

The graph of incidences between blocks and cutvertices is a <u>tree</u>

b) <u>2-connected graphs</u>: We can express how to construct all 2-connected graphs by means of easy operations

Proposition/ A graph is 2-connected if and only if can be constructed from a cycle by successively adding paths with endpoints in the one already constructed.

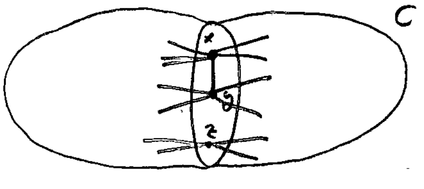


Proof/ $\Leftarrow$] Easy. For $\Rightarrow$], take a cycle in your 2-connected graph, and consider the maximal subgraph which is constructed as described by the proposition. If different from the initial graph, we easily conclude by contradiction.

c) <u>3-connected graphs</u>: in this situation we also have a precise way to construct such graphs. However, we first need a lemma.

Lemma/ If G is a 3-connected graph, $|V(G)| \geq 4$, then G has an edge e such that the graph obtained by contracting e from G is again 3-connected.

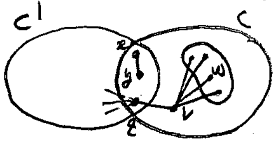
Proof/ Assume that such an edge does not exist. Then, for every edge $\bar{xy}$, the graph obtained by contracting $\bar{xy}$ has a cut set of size 2; call the resulting vertex v_{xy}:



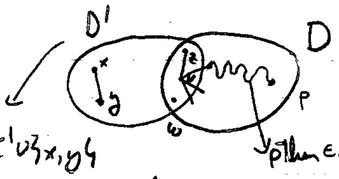
Observe that x, y and z have neighbours in both component (v_{xy} and z must have neighbours in both components, and reversing the operation we conclude the statement).

We choose now $\bar{xy}$, the corresponding vertex z, such that the size of the component C is as small as possible $\Rightarrow$ We will find a contradiction.

We do the following; pick a neighbour of z in C (call it v):



By assumption, contracting $\overline{zv}$ gives a graph which is NOT three connected. Additionally, all neighbours of v belong to C. Hence there is a third vertex w, such that $\{v_{zv}, w\}$ separates G.



$D' \supseteq C' \cup \{x,y\}$

path $\in C$!!

Finally, observe that the subgraph induced by $V(C') \cup \{x,y\}$ is connected. So, deleting w from this subgraph (if it belongs there) we cannot disconnect it, since $\{z,w\}$ would separate G. Finally $C' \subseteq D'$, and D is smaller than C!

This lemma can be used in order to prove the following structural result:

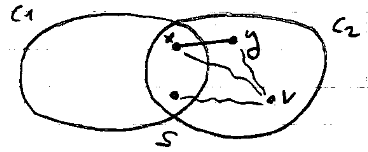
Theorem / (Tutte '61) A graph G is 3-connected iff there exists a sequence $G_0, \dots, G_n$ of graphs with the following properties:

i) $G_0 = K_4$, $G_n = G$

ii) G_{i+1} has an edge $\overline{xy}$, with $d(x), d(y) \geq 3$, and G_i is obtained from G_{i+1} by contracting $\overline{xy}$, for $i < n$.

Proof / $\Rightarrow$] The previous lemma. $\Leftarrow$] We must show that if G_i is 3-connected, then so is G_{i+1}. Assume the contrary (G_{i+1} has a vertex cut of size 2):

obtained by contracting $\overline{xy}$.



Observe that x is part of the cut (otherwise G_i is 2-connected). Then the component containing y has size 1, otherwise contracting $\overline{xy}$ we would obtain a 2-connected graph.

Menger's Theorem

In this part we would justify the name "connectivity". For this reason, we introduce a cornerstone theorem in graph theory: Menger's Theorem. To state it, we need some definitions:

Def / Let x, y be vertices of a graph G. We say that $S \subseteq V(G)$ is an <u>x,y-separator</u> (or <u>x,y-cut</u>) if $G[V-S]$ has no path starting at x and finishing at y. We denote by <u>$K(x,y)$</u> the minimum size of an x,y-cut, and by <u>$\lambda(x,y)$</u> the maximum number of pairwise internally disjoint paths starting at x and finishing at y.

One may wonder to get which is the relation between $K(x,y)$ and $\lambda(x,y)$:

Theorem / (Menger's Theorem) Let $G=(V,E)$ be a graph, and $x,y \in V$ such that $\overline{xy} \notin E$. Then $K(x,y) = \lambda(x,y)$

local version

We won't prove this now, because this will be a consequence of the MAX flow MIN cut theorem that we will prove later. However, we will see some consequences of this result.

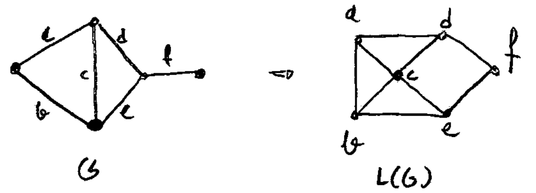
Def / Let x, y be vertices of a graph G. We say that $U \subseteq E(G)$ is an <u>x,y-edge separator</u> (or <u>x,y-edge cut</u>) if $G' = (V(G), E(G) \setminus U)$ has no path starting at x and finishing at y. We denote by <u>$K'(x,y)$</u> and <u>$\lambda'(x,y)$</u> the minimum size of an x,y-edge cut, and the maximum number of pairwise edge internally disjoint paths starting at x and finishing at y.

We would like to have a "Menger's Theorem" in the context of edge connectivity, and this is the case:

Theorem / (Menger's Theorem, edge version) Let $G=(V,E)$ be a graph, and $x,y \in V$. Then $K'(x,y) = \lambda'(x,y)$

Local

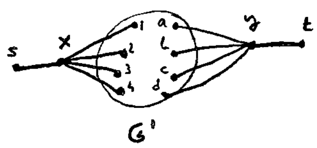
Proof/ In order to prove the theorem, we use the so-called <u>line graph</u> $L(G)$ associated to G:



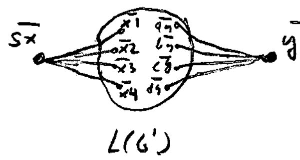
For a graph G, the line graph $L(G)$ is defined as:

- $V(L(G)) \equiv$ edges of G.
- $E(L(G)) \equiv$ two vertices in $L(G)$ are linked by an edge if the associated edges in G share a vertex.

So, let us prove the theorem: we start modifying graph G adding two new vertices s,t and two new edges $\overline{sx}$ and $\overline{ty}$. This does not change the value of $\kappa'(x,y)$ and $\lambda'(x,y)$



$\xrightarrow[\text{line graph } L(G)]{\text{Consider the}}$



Then, $\kappa'_{G}(x,y)=\kappa_{L(G')}(x,y)$ and $\lambda'_{G}(x,y)=\lambda_{L(G')}(x,y)$.

So, we easily conclude the result (because the result is true for vertex-connectivity in the line graph).

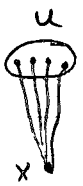
$\boxed{\text{* global version}}$

We have some consequences of this result, but first we need the following lemma:

Lemma/ (Expansion lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbours in G, then G' is k-connected.

Proof/ We show that a separating set $S \subseteq V(G')$ must have size at least k. If $y \in S$, then $S-\{y\}$ separates G, and consequently $|S| \geq k+1$. If $y \notin S$ and $N(y) \subseteq S$, then again $|S| \geq k$. Finally, if the previous situations do not hold, then y and $\{v \in V(G) : v \notin S, v \text{ is a neighbour of } y\}$ belong to the same connected component of $G'[V(G')-S]$, and hence S must also separate $G \Rightarrow |S| \geq k$.

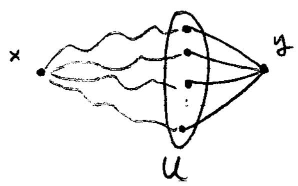
We use this lemma in order to prove the so-called Fan lemma:



Def/ Let $G=(V,E)$ be a graph. A vertex $x \in V$ and a set $U \subseteq V$ define a x,U-fan if there is a set of paths from x to U such that any two of them share only the vertex x.

Theorem/ (Fan lemma, Dirac '1960) A graph G k-connected then G has at least $k+1$ vertices and for every choice of $x \in V(G)$, $U \subseteq V(G)$, $x \notin U$, and $|U| \geq k$ then there exists a x,U-fan of size k.

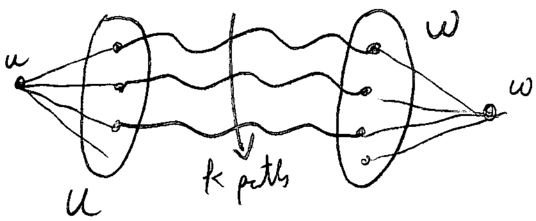
Proof/ $\Rightarrow$] Assume that G is k-connected.



We create a new vertex which is incident with all vertices in U. As $|U| \geq k$, the resulting graph is k-connected. Hence, by Menger's Theorem, $\kappa(x,y)=\lambda(x,y) \geq k$. Then, we just need to consider the paths finishing at U. $\rightarrow \overline{xy} \notin E(G)$!

Of course the same idea can be exploited in order to get the following improvement:

Theorem/ Let G be a k-connected graph, and U, W be subsets of $V(G)$, $U \cap W = \emptyset$, $|U|,|W| > k$. Then there exists k-disjoint paths starting at U and finishing at W.



* global version

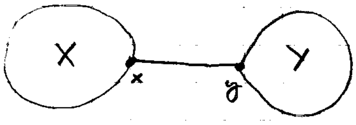
All these results are "local", meaning that they provide information only respect to pairs of vertices. Observe that in the case of edge-connectivity we have that if for each pair $x, y \in V(G)$, $\kappa'(x,y) = \lambda'(x,y) \geq k$, then necessarily $\kappa'(G) \geq k$.

Theorem / (Menger's Theorem for edge-connectivity, Global version) A graph G is k-edge-connected iff for each pair of vertices x, y, $\lambda'(x,y) = \kappa'(x,y) \geq k$.

However, we need to refine our argument when dealing with vertex-connectivity: the local version just deals with $\lambda(x,y)$ and $\kappa(x,y)$ when $\overline{xy} \notin E(G)$.

Lemma / Deletion of an edge reduces connectivity by at most 1.

Proof / Of course, if $G=(V,E)$ is a graph, and $G'=(V, E-\{\overline{xy}\})$ $(x,y \in V)$, then $\kappa(G') \leq \kappa(G)$. The relation does <u>NOT</u> hold if G' has a separating set S with size $< \kappa(G)$ (and hence not being separating set for G). Then $G[V-S]$ is connected, and $\overline{xy}$ is a bridge:



$G[V-S]$

- If $|X| \geq 2 \Rightarrow S \cup \{x\}$ is a separating set.
- If $|Y| \geq 2 \Rightarrow S \cup \{y\}$ is a separating set.
- Otherwise, $|S| = |V(G)|-2$. Then $\kappa(G) \geq |V(G)|-1$, and consequently $G = K_n$, which also satisfies what we wanted.

Finally, the global version of Menger's Theorem for vertices is also true: if now we consider a pair of vertices $x, y \in V$, $\overline{xy} \in E(G)$, then:

$$\lambda_G(x,y) = 1 + \lambda_{G-\overline{xy}}(x,y) = 1 + \kappa_{G-\overline{xy}}(x,y) \geq 1 + \kappa(G-\overline{xy}) \geq \kappa(G)$$