Material covered in Discrete Mathematics II (2013/2014)

Additionally to the material covered on the lectures, you should know how to solve all homework exercises.

Part I: Structural Graph Theory

Basic definitions of graphs

Essentially basic notions and definitions on graph theory, including:

- Definition of graph, multigraph, simple graph, adjacency of vertices, incidence of edges.
- Degree of a vertex $d(v)$, maximum vertex degree and minimum vertex degree $(\delta(G), \Delta(G))$, k-regular graphs ($k = 3$ is the cubic case).
- Handshaking Lemma for graphs.
- Subgraph, induced subgraph and subgraph spanning a graph.
- Path, length of a path.
- Connected graph, connected components, separating set, k-connectivity.
- Euler tour and eulerian graph, Hamilton cycle and Hamiltonian graph.
- Census of graphs: loop and multiedge, star, cycle C_k, path P_k, complete graph K_n, bipartite graph $K_{n,m}$, wheel W_n, Wagner's graph V_8, Petersen graph.

Planarity and graph minors

- Low dimensional topology: curve on the plane (or sphere), simple closed curve, Jordan's Curve Theorem (NO proof).
- Planar graph, drawing of a planar graph and plane graph (maps). Faces of a map. Dual map: handshaking lemma for dual maps, properties relating a map and its dual.
- Euler's relation. Consequences: maximum number of edges in a planar graph, number of edges in a triangle-free planar graph, existence of a vertex of degree ≤ 5 in a planar graph.
- Characterization of planarity: neither K_5 nor $K_{3,3}$ are NOT planar. Subdivision of a graph, Kuratowski's Theorem (NO proof).
- Graphs on surfaces: some little examples.
- Graph minors: definition, Wagner's Theorem (proof using Kuratowski's Thm), minor-closed families of graphs, Robertson-Seymour Theorem (NO proof), examples of minor-closed families of graphs.
- Well-quasi orders in graphs: definition, examples, version of Roberston-Seymour Theorem on graphs and equivalence with the previous statement of the theorem. Properties of well-quasi-ordered sets (Proposition+Lemma; the proof of the Proposition uses Ramsey Theory, see later), Kruskal's Theorem for trees (proof using the previous lemmas).

Matching theory

- Basic definitions: matching, perfect matching, k-factor.
- Matchings in bipartite graphs: alternating path and augmenting path. Vertex cover. Knig's Theorem. Hall's Theorem. Consequences of Hall's Theorem: bipartite k-regular graphs have 1-factors, $(2k)-$regular graphs have 2-factors.
- Matchings in general graphs: Tutte's condition $(o(G - S) \leq |S|)$ and Tutte's Theorem. Petersen's Theorem for cubic graphs without bridges.

Graph connectivity

- Basic definitions: separating set of vertices (vertex cut) and separating set of edges (edge cut, disconnecting set of edges). Connectivity $\kappa(G)$ and edge-connectivity $\kappa'(G)$ of a graph G.
- Relation between $\kappa(G)$ and $\kappa'(G)$ (Whitney's Theorem): $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. Consequences when the graph is cubic.
- Structure of connected graphs with respect to the blocks: tree-like structure.
- 2-connected graphs: ring construction.
- 3-connected graphs: Lemma proving that we can always contract an edge and make that the resulting graph is 3-con, Tutte's Theorem characterizing how to construct a 3-connected grarph starting from K_4.
- Menger's Theorem: definitions (x, y- separator, parameters $\kappa(x, y)$, $\lambda(x, y)$), statement of Menger's Theorem local-vertex version (proof using flows). Menger's Theorem local-edge version (proof from local-vertex version). Global versions of the theorem. Expansion Lemma and consequences: Dirac's Fan Lemma.

Graph coloring

- Basic definitions: coloring and proper coloring of the vertices of a graph. Chromatic number, clique number and independence number.
- First bounds between the chromatic number and the clique number and the independence number. Bound of the chromatic number in terms of the number of edges. Greedy algorithm (bound of the chromatic number in terms of the maximum degree of the graph). Welsh-Powell theorem which generalizes the greedy algorithm. Brook's Theorem (NO proof).
- The Five Color Theorem. Coloring of an arrangement of lines in the plane.
- Graphs with large chromatic number: Mycielski's construction.
- Perfect graphs: definition, examples (Bipartite graphs, complement of bipartite graphs), Berge Weak and Strong Perfect graph conjectures (Now theorems), NO proof.
- Edge coloring of graphs: edge coloring and edge chromatic number.Trivial bounds for the edge chromatic number. Vizing's Theorem.

Flows in graphs

- Basic definitions: digraph, orientation of an edge, source and sink of a digraph, capacity function, network, flow, Kirchhoff's Law.
- MAX flow- MIN Cut: definition of cut, capacity and circulation. Ford-Fulkerson's Theorem. Application: proof of Menger's local-vertex version.
- k-flows on groups: nowhere-zero flows and k-flows. Flow number of a graph. Tutte's theorem for flows (we can restrict to cyclic groups). Some properties: a graph has a 2-flow iff all its degrees are even. A cubic graph has a 3-flow iff it is bipartite.
- Some conjectures on flows on graphs.

Part II: Extremal Combinatorics

In this part you need to recall all the basics related to big O-little o notatation.

Classical extremal combinatorics

- Classical extremal combinatorics on sets: Erdős-Ko-Rado (intersecting and k-intersecting families), Sperner's Theorem (chains and antichains).
- Some motivational examples in the graph setting.
- Dirac's Theorem stating the existence of Hamiltonian paths in terms of the minimum degree of a graph.
- Mantel's Theorem and Turán's Theorem (and of course, all the topics related with Turán graphs).
- Erdős-Stone theorem (sketch of the proof later), and Erdős-Simonovits corollary (with proof).
- Erdős Theorem for $\text{ex}(n, C_4)$, and Klein's Theorem (HW).
- Kővari-Ss-Tur'an theorem (HW).

Szemerdi's regularity lemma and applications

- Definitions: ε-pair and ε-partition of a graph. The statement of the Szemerédi's Regularity Lemma (NO proof).
- First application of the SZR: counting lemma for triangles, using the key idea on vertices with big neighbourhoods.
- Triangle Removal Lemma, using $\varepsilon - \delta$ notation and using $o - O$ notation.
- Szemerédi's Theorem (NO proof) and Roth's Theorem (proof using the Triangle Removal Lemma).
- Behrend's construction for a dense set without $3-$AP.
- Sketch of the proof of Erdős- Stone theorem (a complete proof won't be asked, possibly some of the ideas behind it).

Ramsey Theory

- Philosophy behind Ramsey Theory, and connection with extremal combinatorics.
- Van der Waerden's Theorem. Proof using Szemerédi's Theorem. Van der Waerden's numbers.
- Ramsey Theory on graphs: bounds for Ramsey numbers, bounds for diagonal Ramsey numbers. Erdős bound for $R(k, k)$ by counting.
- Ramsey Theory on infinite sets: c-colorings of sets and mono-chromatic sets. Ramsey Theorem for colorings of infinite sets. Application on well-quasi-ordered sets.
- Proof of Van der Waerden's Theorem: sunflowers, preliminary lemmas and final proof using Generalized Van der Waerden's numbers.

Part III: Methods in Discrete Mathematics

Probabilistic methods

- The basics on probability theory over finite sets: definition of probability space (in a simplified way), random variables, indicator random variables, independence of events.
- The use of the union bound and first applications: Erdős Theorem for the lower bound of $R(k,k)$, Theorem for the lower bound of $W(2,k)$, tournaments, property S_k and Erdős Theorem for the existence of tournaments with property S_k.
- The expectation and the first moment method: definition of the expectation and properties. Philosophy behind the first moment method. Applications: Existence of a big bipartite subgraph in each graph, Erdős theorem for sum-free subsets, balanced vectors and crossing number of a graph.
- The alteration method: philosophy behind the method. Applications: lower bound for the independence number of a graph. Markov's inequality and the model $\mathcal{G}(n,p)$. Erdős probabilistic construction of a graph with large girth and large chromatic number.
- The variance and the second moment method. Applications

Enumerative methods (Introduction)

- Generating functions and counting.
- The Symbolic Method and examples: labelled trees (rooted, unrooted), Catalan numbers, dissections of polygons.
- Laplace inversion formula.