

Proof of Van der Waerden's Theorem

In this note we cover the complete proof of Van der Waerden's Theorem, with all the details which have been missed at the lecture. Additionally, some slight corrections are included.

The statement of the result we want to prove is the following:

Theorem 1 (Van der Waerden's Theorem, 1927) *Let r, k be positive integers. Then there exists a number $W(r, k)$ such that if $N \geq W(r, k)$ then any r -coloring of $[N]$ contains a monochromatic k -AP.*

The numbers $W(r, k)$ are called *Van der Waerden's numbers*. So proving the theorem consists of showing that for every choice of r, k , the value $W(r, k) < \infty$. For instance, it is obvious that $W(r, 1) = 1$ (1-AP are trivial), and $W(r, 2) = r + 1$ (by the Pigeonhole Principle, once having $r + 1$ elements there are two of them which have the same color. This two elements define a 2-AP).

We need an extra definition which generalizes the notion of monochromatic AP. From now on, we assume a certain coloring on $[N]$ (for N large enough). Namely, the following definition must be understood taking the object defined inside a bigger interval which is colored:

Definition 2 *A sunflower with m petals of size $k - 1$ is a set of integers of the form $\{a\} \cup A_1 \cup \dots \cup A_m$, where $A_i = \{a + d_i, a + 2d_i, \dots, a + (k - 1)d_i\}$ ($i = 1, \dots, m$), with the following properties:*

- $A_i \cap A_j = \emptyset$ for $i \neq j$ (*Disjointness of the petals*).
- All elements in A_i are colored with the same color (*Monochromaticity of the petals*).
- If $i \neq j$, the color used to color elements of A_i and the color used to color elements of A_j are different (*Different colors for the petals*).

In such a situation, the sets A_1, \dots, A_m are called the petals of the sunflower and a is the center of the sunflower.

Note that in a sunflower we have lot of control on the color (monochromatic) and the structure ($(k - 1)$ -AP) of each petal, but we do NOT control which is the color of the center. Observe that in particular, if the color of the center is equal to the color of one of the petals, then this sunflower defines a monochromatic k -AP.

We start proving a proposition that will be useful in the complete proof of VdW theorem.

Proposition 3 *Consider a r -coloring of $[N]$, and let $A, A + d, \dots, A + (k - 1)d \subseteq [N]$, with the induced coloring (recall that a set $A + k$ is equal to $\{a + k : a \in A\}$). Assume also that*

- $A + d, \dots, A + (k - 1)d$ are colored in exactly the same way.
- $A + d$ is a sunflower with m petals of size $k - 1$ (and in particular all its dilates $A + 2d, \dots, A + (k - 1)d$).

Then $A \cup (A + d) \cup \dots \cup (A + (k - 1)d)$ contains either a k -AP or a sunflower with $m + 1$ petals of size $k - 1$.

Observe that the statement on the lecture was slightly wrong (we need to control the structure of the dilates, and not the structure of A !).

Proof: Let us pick $A + d$, which we know that it is a sunflower with m petals of size $k - 1$. If the center of this sunflower has the same color as one of the petals, we are done, because we have a k -AP.

Assume now that the color of the center of $A + d$ is different to the color of each one of the petals in $A + d$. Let us now construct a sunflower with $m + 1$ petals of size $k - 1$ in $A \cup (A + d) \cup \dots \cup (A + (k - 1)d)$. Just to fix some notation, write $A + d = \{a + d\} \cup A_1 \cup \dots \cup A_m$, where $A_i = \{a + d + d_i, a + d + 2d_i, \dots, a + d + (k - 1)d_i\}$ ($i = 1, \dots, m$), where each A_i is a petal of $A + d$.

Consider now the following set: $B = \{a\} \cup B_1 \cup B_2 \cup \dots \cup B_m \cup B_{m+1}$, where $B_i = \{a + (d + d_i), a + 2(d + d_i), \dots, a + (k - 1)(d + d_i)\}$, for $i = 1, \dots, m$, and $B_{m+1} = \{a + d, \dots, a + (k - 1)d\}$ (namely, B_{m+1} is created by taking the centers of the sunflowers $A + d, \dots, A + (k - 1)d$).

By construction it is obvious that $B \subseteq A \cup (A + d) \cup \dots \cup (A + (k - 1)d)$. Let us prove that it is in fact a sunflower with $m + 1$ petals of size $k - 1$:

- The color of $a + r(d + d_i)$ ($r = 1, \dots, k - 1, i = 1, \dots, m$) is the same as the color of $a + d + rd_i$ (dilates by d do not change de color), and the color of $a + d + rd_i$ is the same as the color of $a + d + d_i$ (the elements of the same petal in $A + d$ have the same color). This shows that all elements in B_i have the same color ($i = 1, \dots, m$).
- The color used in the set B_i is different to the color used in the set B_j if $i \neq j$ ($i, j = 1, \dots, m$): the color used in B_i is the color used to color the element $a + d_i$, which is different to the color used to color the element $a + d_j$.

The new petal B_{m+1} is also monochromatic because the colorings of $A + d, \dots, A + (k - 1)d$ is the same, and its color is different to the colors used to paint B_1, \dots, B_m because by hypothesis the center of $A + d$ (recall, the element $a + d$) had a color different to the color of $a + d + d_i$, $i = 1, \dots, m$.

So, we have proved that $B = \{a\} \cup B_1 \cup B_2 \cup \dots \cup B_m \cup B_{m+1}$ is a sunflower with $m + 1$ petals of $k - 1$, as we wanted to prove. \square

The key strategy in order to prove VdW is to apply and induction argument on the length of the AP, and combine it with the existence of certain sunflowers. More precisely, we have the following key lemma:

Lemma 4 *Let k be a positive integer. Assume that $W(r, k - 1)$ exists for every choice of r . Then, for every choice of r, m there exists a positive integer $W(r, m, k - 1)$ such that if $N \geq W(r, m, k - 1)$ then any r -coloring of $[N]$ contains either a monochromatic k -AP or a sunflower with m petals of size $k - 1$.*

Proof: We apply induction on m . For $m = 1$ the statement holds trivially: a sunflower with 1 petal of size $k - 1$ defines a monochromatic $(k - 1)$ -AP (or even more, a k -AP if the center has the same color as the elements of this petal). As, by hypothesis, $W(r, k - 1)$ exists, in particular $W(r, 1, k - 1) \leq W(r, k - 1) < \infty$.

So now let us assume that $W(r, m - 1, k - 1)$ exists, and let us prove that $W(r, m, k - 1)$ also exists. For this purpose, write $N_1 = W(r, m - 1, k - 1)$ and $N_2 = 2W(r^{N_1}, k - 1)$. By induction hypothesis, N_1 exists, and hence N_2 it also exists (because we are assuming the existence of $W(r, k - 1)$ for each choice of r). It is enough then to prove that $W(r, m, k - 1) \leq N_1 N_2$.

So take the interval $[N_1 N_2]$ and color it using r colors. Consider it as the concatenation of $W(r^{N_1}, k - 1)$ blocks of size N_1 each (these blocks define the *first part* of the partition), followed by extra $W(r^{N_1}, k - 1)$ blocks of size N_1 each (these last blocks define the *second part* of the partition). In other words, both parts (first and second) of the partition as $N_2/2$ blocks of size N_1 (this is the reason of the factor 2 in the definition of N_2).

Let us focus our attention to the second part of the partition. If we have a block containing a monochromatic k -AP, we are done.

Assume the contrary. Recall that in the second part of the partition we have $N_2/2 = W(r^{N_1}, k - 1)$ blocks. As the size of a block is N_1 , a block can be colored in r^{N_1} different ways. This means that in the second part of the partition we have $(k - 1)$ blocks identically colored forming a $(k - 1)$ -AP, say $B + d, \dots, B + (k - 1)d$ (so here we are applying the hypothesis that $W(r, k - 1)$ (for each r) exists not on the elements of $[N_1 N_2]$, but on its blocks). Now, observe that $|B + d| = N_1 = W(n, m - 1, k - 1)$, so $B + d$ contains a sunflower $A + d$ of with $m - 1$ petals and size $k - 1$ (because by hypothesis it does not contain a monochromatic k -AP).

We just need a final step: consider now $B = (B + d) - d$ and $A = (A + d) - d$ which is contained in B . Observe that B can possibly belong to the first part of the partition (again, this is the reason why we take 2 in the definition of N_2). We do not control how is the coloring of B , but by Proposition 3 we know that $A \cup (A + d) \cup \dots \cup (A + (k - 1)d)$ must contain either a monochromatic k -AP or a sunflower with m petals of size $k - 1$, which is exactly what we wanted to prove. \square

So now we have all the ingredients to prove VdW: we will apply induction in the following way: we want to prove that $W(r, k)$ exists, assuming that $W(r, k - 1)$ exists. Then using Lemma 4 we prove that $W(r, m, k - 1)$ exists (for every choice of m), and then we use this value to bound $W(r, k)$. Let us see the details:

Proof: (Proof of VdW Theorem) We apply induction on k . We have shown that $W(r, 1)$ and $W(r, 2)$ exists for every choice of r . So let us assume that $W(r, k - 1)$ exists for every choice of r and let us see that $W(r, k)$ exists for every choice of r .

In order to do so, by Lemma 4 the value $W(r, m, k - 1)$ exists for every choice of r and m . In particular, $W(r, r, k - 1)$ exists for every choice of r . Let us show that $W(r, k) = W(r, r, k - 1)$. This last number tells us that for every $N \geq W(r, r, k - 1)$, any r -coloring of $[N]$ contains a monochromatic k -AP or a sunflower with r petals of size $k - 1$. But then, if we are in the second situation, the color of the center must be equal to the color of the center (we have r petals and a total of r colors). So, in any of the cases, we have a monochromatic k -AP, as we wanted to prove. \square