Problem Sheet 9

Ramsey Theory and Van der Waerden's Theorem

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Discrete Mathematics II, Winter 2013-2014
Deadline: 14th January 2014 (Tuesday) by 10:00, at the end of the lecture.

Problem 1: Prove the infinite version of Van der Waerden's Theorem: let $c : \mathbb{N} \to [r]$ be an r-colouring of $\mathbb{N}$. Show that, for each k, there exists one colour i such that $c^{-1}(i)$ containing infinitely many k - AP's (*Hint:* use the finite version of the theorem).

Problem 2: Show that infinite Van der Waerden's Theorem is *not* true when we require the existence of a infinite monochromatic AP. To do so, construct a counterexample of a 2-colouring without an infinite monochromatic AP.

Problem 3: *Generalized Ramsey numbers.* The Ramsey number $R(s_1, \dots, s_r)$ is the minimum number N such that any colouring of K_N with r colours has, for at least one colour i, $i = 1, \dots, r$, a monochromatic copy of K_{s_i} .

- Prove that $R(s_1+1, s_2+1, \dots, s_{r-1}+1, s_r+1) \leq R(s_1, s_2+1, \dots, s_{r-1}+1, s_r+1) + R(s_1+1, s_2, \dots, s_{r-1}+1, s_r+1) + \dots + R(s_1+1, s_2+1, \dots, s_{r-1}+1, s_r) - (r-2)$.
- Prove that $R(s_1+1, s_2+1, \dots, s_{r-1}+1, s_r+1) < \infty$ for each choice of $s_1, \dots, s_r$.
- Apply the previous arguments to get an upper bound for $R(3, \overset{r}{\dots}, 3) := R_r(3)$.

Problem 4: Prove that $R(3,4) = 9$.

Problem 5: *Fermat's equation in modular groups.* We want to study Fermat's equation in $\mathbb{Z}/p\mathbb{Z}$: $x^n + y^n \equiv z^n(p)$.

- Show that any r-colouring of $[m]$, with $m \geq R_r(3)$ has a monochromatic solution of the equation $x+y=z$ (This is what is called a *Schur triple*). See Problem 3 for the definition of $R_r(3)$ (*Hint:* construct a convenient colouring of K_m).
- Let $H_n = \{x^n : x \in (\mathbb{Z}/p\mathbb{Z})^*\}$. Show that H_n is a subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$. Finally show that $(\mathbb{Z}/p\mathbb{Z})^* = a_1 H_n \cup \dots \cup a_r H_n$, where the union is disjoint (namely, once we have a subgroup, we can decompose the group into disjoint union of *cosets* of the subgroup).
- Show that if $p-1 \geq R_n(3)$, then we have a solution (x,y,z) of Fermat equation $x^n + y^n \equiv z^n(p)$, where $xyz \not\equiv 0(p)$ (*Hint:* find a Schur triple).