Problem Sheet 8

Szemerédi's Regularity Lemma. Roth's Theorem

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Discrete Mathematics II, Winter 2013-2014
Deadline: 7th January 2014 (Tuesday) by 10:00, at the end of the lecture.

Problem 1: Show that an ε-regular partition of a graph is also an ε-regular partition of its complement.

Problem 2: If A and B define an ε-regular pair with $d(A,B)=\delta$, and A', B' satisfy that $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$ for some $\gamma \geq \varepsilon$, then the pair (A',B') is $\varepsilon \cdot \max\left\{\frac{1}{\gamma}, 2\right\}-$ regular with density in $[\delta-\varepsilon, \delta+\varepsilon]$.

Problem 3: *A construction of Erdős and Turán.* The best upper bound for $r_3(n)$ before Behrend's construction was given by Erdős and Turán. For this purpose, consider the infinite set A of integers which do not have the digit "2" when writing them in basis 3.

1. Prove that A does not have $3-$AP. Show that $|A \cap [n]| = \theta(n^{\log(2)/\log(3)})$ (*Hint:* Take $n = 3^l$).
2. Prove that this set is *maximal*: there is not a set A' such that $A \subseteq A'$, $A \neq A'$ such that A' is $3-$ AP free.
3. Show that the order of magnitude of the smallest size of a set which is maximal with respect to the property of not having 3-AP lies between $n^{1/2}$ and $n^{\log(2)/\log(3)}$ (*Hint:* Apply a greedy algorithm for the lower bound).

In the last point, any improvement between $n^{1/2}$ and $n^{\log(2)/\log(3)}$ would be a new result.

Problem 4: (Ajtai-Szemerédi) A *right isosceles triangle* in $[2n]^2$ is a set of points of the form $\{(x,y),(x+h,y),(x,y+h) : x,y,h \in [n]\}$.

1. Show that for every $\varepsilon > 0$, there exists $n > 0$ such that any subset of $[n]^2$ with at least εn^2 points has a right isosceles triangle (*Hint:* use Solymosi's strategy: construct a convenient graph and apply the Triangle Removal Lemma as we did in Roth's Theorem).
2. Prove Roth's Theorem from this result (*Hint*: look at the corner of the triangle).

Problem 5: Prove the following stronger version of Roth's Theorem (due to Frankl, Graham and Rödl): for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ with the following property. Suppose that A is an abelian group of odd order, $|A| > n_0$. Then every subset $B \subset A$ with $|B| > \varepsilon|A|$ contains three distinct elements x, y, z with $x+y=2z$ (*Hint:* again, try to obtain a convenient tripartite graph and apply the Triangle removal Lemma).