Problem Sheet 7

Extremal Graph Theory

Juanjo Rué
Discrete Mathematics II, Winter 2013-2014
Deadline: 17th December 2013 (Tuesday) by 10:00, at the end of the lecture.

Problem 1: Let G be a graph with n vertices.

- If $|E(G)| > \binom{n-1}{2}$, then G is connected. Construct a disconnected graph G with $\binom{n-1}{2}$ edges.
- If $|E(G)| > n-1$, then G contains a cycle. Show that this bound is the best possible.

Problem 2:

- Let $T_{r,n}$ the Turán graph in n vertices and r partite sets. Check that $|E(T_{r,n})| \leq \left(1-\frac{1}{r}\right)\frac{n^2}{2}$ when r does not divide n. Check also that, in fact, $|E(T_{r,n})| = \left(1-\frac{1}{r}\right)\binom{n}{2} + O(n)$.
- Show that Turán graphs $T_{r,n}$ are the unique graphs that reaches the value $ex(n, K_{r+1})$ (*Hint:* see in which steps of the proof of Turan's Theorem we have equality instead of inequality).

Problem 3: *Sidon sets and* $\mathrm{ex}(n, C_4)$. Let $(G, +)$ be an abelian group. We say that a subset S of G is a *Sidon set* if the equation $x+y = z+t$ is satisfied with $x, y, z, t \in S$ then $\{x, y\} = \{z, t\}$ (namely, sums are *not* repeated). Equivalently, the previous condition is equivalent to the fact that all differences $x + (-x')$, with x, $x' \in S$ and $x \neq x'$ are different.

1. Prove that if G is finite, then $|S| = O(|G|^{1/2})$.
2. Let g be a multiplicative generator of the cyclic group $(\mathbb{Z}/p\mathbb{Z})^* \simeq \mathbb{Z}/(p-1)\mathbb{Z}$. Prove that the set $\{(x, g^x) \in \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\}$ is a Sidon set in $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Prove that this set translates to a Sidon set in $\mathbb{Z}/(p^2-p)\mathbb{Z}$. Which is its cardinality?
3. Using the previous Sidon set, construct a graph without cycles of length four. Which is its number of edges?
4. Deduce that $\mathrm{ex}(p^2-p, C_4) = \Theta(p^3)$ for each prime p large enough.
5. Finally prove that $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$ (*Hint:* you could use the following strong result from analytic number theory: for each m large enough, there is a prime in the interval $(m, m+m^\theta)$, with $\theta = 0{,}525$. In fact, it is thought that one can take $\theta > 0$, but this is an open question).

Problem 4: Prove Kővari - Sós - Turán Theorem (*Hint:* Adapt Erdős proof for $\mathrm{ex}(n, C_4)$).

Problem 5: Let P be a set of n points in $\mathbb{R}^2$. Show that there are at most $O(n^{3/2})$ pairs of points that have Euclidean distance exactly equals to one (*Hint:* Construct an appropiate unit distance graph which does not contain $K_{2,3}$ as a subgraph, and then apply Kővari - Sós - Turán Theorem).