Problem Sheet 6

Flows on graphs

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Discrete Mathematics II, Winter 2013-2014

Deadline: 3rd December 2013 (Tuesday) by 10:00, at the end of the lecture.

Problem 1: Prove, using the MAX flow - MIN cut Theorem, König's Theorem for the maximum size of a matching on a bipartite graph.

Problem 2: Let G be a connected graph.

- Prove that G has a k-flow if and only if the blocks of G have a k-flow.
- Prove that G has a $\mathbb{Z}$-flow if and only if G is a bridgeless graph (*Hint:* for the existence of a certain k-flow on G, restrict yourself to the blocks of the graph. Later, explote the structural result for 2-connected graphs that we have seen in the connectivity part of the lectures).

Problem 3: *The Flow Polynomial.* Let G be a multigraph, and H be a finite abelian group. We denote by $P(G, H)$ the number of H-flows of G (Observe that when applying Kirchhoff's Law over a vertex, loops do not contribute. So we can put whatever we want over a loop).

1. If $|V(G)| = 1$ and all its edges are loops, prove that $P(G, H) = (|H| - 1)^{|E(G)|}$.
2. For an edge $e \in E(G)$ which is not a loop, denote by G_1 and G_2 the graphs obtained by deleting e and contracting e, respectively (In this case, when contracting e we do NOT erase the possible multiple edges). Prove that $P(G, H) = P(G_2, H) - P(G_1, H)$ (*Hint*: it is better to prove that $P(G_2, H) = P(G, H) + P(G_1, H)$ by interpreting combinatorially which are the things counted in both sides of the equality).
3. Conclude that $P(G, H)$ is a polynomial on $|H| - 1$.

This is the so-called *Flow polynomial of G* (which just depends on the cardinal of H). From this we deduce directly the stronger version of Tutte's Theorem: G has a k-flow if and only if G has a H-flow with $|H| = k$.

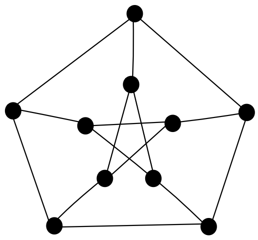
Problem 4: Let G be a bridgeless connected planar graph. Prove that G has a 4-flow (*Hint*: Use the 4-colour Theorem + duality of planar graphs in order to build a flow on the graph).

Problem 5: Prove that if G has a Hamiltonian cycle (namely, a connected 2-factor), then G has a 4-flow (*Hint*: Work using $\mathbb{Z}/4\mathbb{Z}$).

Problem 6: Let G be a bridgeless cubic graph. Show that G has a 4-flow if and only if G is 3-edge colorable (*Hint*: you should consider to work with the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$).

In particular, this result shows that every cubic 3-edge-colourable graph is bridgeless.

Problem 7: *The Petersen Graph and the 4-flow Conjecture.* Recall that the Petersen graph is the cubic graph shown in the following picture:



Prove that the Petersen Graph has a 5-flow, and prove that it is *not* 3-edge colorable.

Thus, by Problem 6, the Petersen graph cannot have a 4-flow. It is conjectured that the Petersen Graph is the unique obstruction for not having a 4-flow. More precisely, the *4-flow Conjecture* (Tutte'66) states that a bridgeless graph G has a 4-flow if and only if it does not contain the Petersen graph as a minor.