Problem Sheet 5

Graph colouring. Chromatic number and edge chromatic number.

Juanjo Rué
Discrete Mathematics II, Winter 2013-2014

Deadline: 26th November 2013 (Tuesday) by 10:00, at the end of the lecture.

Problem 1: Let G be a connected graph.

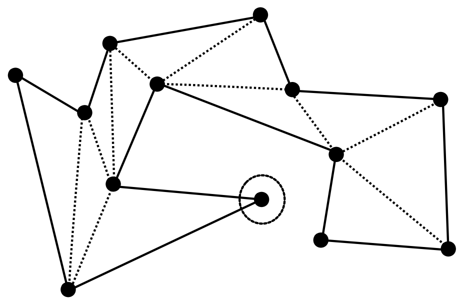
- Obtain the chromatic number of G in terms of the chromatic number of its blocks.
- Obtain (and prove) a formula for the number of spanning trees of G in terms of the number of spanning trees of each block of G.

Problem 2: Given finite sets $S_1, \dots, S_m$, write $U = S_1 \times \cdots \times S_m$. Let G be the graph with vertex set U, and two vertices are adjacent if and only if they differ in every coordinate. Determine $\chi(G)$.

Problem 3: *The Art Gallery Problem*: let P be an n-vertex polygon drawn in the plane (we call it the Art Gallery). The polygon is *not* necessarily convex. We will show that for every P, there exists a choice of $\lfloor \frac{n}{3} \rfloor$ vertices of the polygon (the position of the guards) such that for all interior point p of P, there exist a straight line joining p with one of the guards without crossing any edge of the polygon. We will follow Chvátal's proof.

1.- Prove that every Art Gallery can be triangulated (i.e., decompose the interior in triangles, using straight lines). (*Hint*: prove that you can always find an internal chord linking to vertices on the Art Gallery).

2.- Prove that every triangulation of an Art Gallery has a vertex of degree 2. See the figure for a certain Art Gallery, a triangulation of it and one of the vertices with degree 2:



3.- Apply induction and conclude the statement.

4.- Obtain an art gallery where $\lfloor \frac{n}{3} \rfloor$ guards are needed.

Problem 4: Let G be a graph. Prove that $\chi(G)\chi(\overline{G}) \geq |V(G)|$, and use it to prove that $\chi(G) + \chi(\overline{G}) \geq 2|V(G)|^{1/2}$. Provide a construction achieving these bounds whenever $|V(G)|^{1/2}$ is an integer.

Problem 5: Let G be a regular graph with a cut-vertex. Prove that $\chi'(G) > \Delta(G)$.

Problem 6: Prove that $\chi'(K_{r,s}) = \Delta(K_{r,s})$ by giving an explicit edge-colouring.

Problem 7: Let G be a graph with $\Delta(G) = 3$. Use Brook's Theorem over $L(G)$ to prove that $\chi'(G) \leq 4$. Do *not* use Vizing's Theorem to prove this result.