

Problem Sheet 12

More on the first moment method and the alteration method

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Discrete Mathematics II, Winter 2013-2014

Deadline: 4th February 2014 (Tuesday) by 10:00, at the end of the lecture.

Problem 1: Prove that if the graph G satisfies that $|E(G)| \geq 4|V(G)|$, then $\text{cross}(G) = \Theta\left(\frac{|E(G)|^3}{|V(G)|^2}\right)$. To do so, consider the graph arising when connecting adjacent and nearly-adjacent vertices over a cycle.

Problem 2: Show that for any graph G , we have that $|E(G)| = O(|V(G)| + |V(G)|^{2/3} \text{cross}(G)^{1/3})$. (*Hint:* distinguish two cases: $|E(G)| \geq 4|V(G)|$ and $|E(G)| < 4|V(G)|$).

Problem 3: Compute the expected value of the number of isolated vertices (i.e. incidents to no edge) in the random graph $\mathcal{G}(n, p)$. Show that if $p = c \frac{\log n}{n}$ for $c > 1$, then the probability that the random graph $\mathcal{G}(n, p)$ has isolated vertices tends to 0 when n tends to infinity.

Problem 4: consider the probability space where Ω is the set of subsets of $[n]$, and each element in $[n]$ is in the random set with probability $p = n^{-\alpha}$, for a certain positive value of α .

1. Let X_n be the random variable which counts the number of elements in a random set A of $[n]$. Show that $\mathbb{E}[X_n] = n^{1-\alpha}$.
2. Let $Y_{n,k}$ be the random variable which counts the number of k -AP in a random set A of $[n]$. Show that $\mathbb{E}[Y_{n,k}] = O(n^{2-k\alpha})$.
3. Apply the alteration method with a suitable choice of α in order to obtain a subset of $[n]$ without k -APs.