Problem Sheet 11

The probabilistic method: the union bound and the first moment method

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Discrete Mathematics II, Winter 2013-2014
Deadline: 28th January 2014 (Tuesday) by 10:00, at the end of the lecture.

Problem 1: Adapt the probabilistic proof in the lecture in order to get a lower bound for $W(r,k)$.

Problem 2: Let $\Omega = \mathcal{S}_n$ be the set of permutations of n letters, and consider the uniform probability model on it (namely, for all $\sigma \in \Omega$, $\mathbb{P}(\{\sigma\}) = \frac{1}{n!}$). Let X_n be the random variable which counts the number of fixed points on a permutation. Find $\mathbb{E}[X_n]$.

Problem 3: Show that there is a 2-coloring of $K_{m,n}$ with at most $\binom{m}{a}\binom{n}{b}2^{1-ab}$ monochromatic copies of $K_{a,b}$.

Problem 4: Let $k \geq 1$ and let $\mathcal{A}$ be a subset of $\binom{[n]}{k}$, such that $|\mathcal{A}| < s^{k-1}$. Prove that there is a coloring of $[n]$ using s colors so that no element in $\mathcal{A}$ is monochromatic.

Problem 5: *Turan's theorem for the independence number.* Let $G = (V,E)$ be a fixed graph. For $v \in V$. denote by d_v the degree of v. We will find a lower bound for $\alpha(G)$ in terms of the degree sequence d_v.

With this purpose, consider a bijection $\sigma : V \to |V|$ chosen uniformly at random (namely, we are ordering the vertices of G randomly).

- For a given permutation, show that the set of vertices $v \in V$ such that $\sigma(v) > \sigma(w)$ if w is a neighbor of v is an independent set.
- Let X be the random variable which counts the number of elements in this random independent set. Show that $\mathbb{E}[X] = \sum_{v\in V} \frac{1}{d_v+1}$ (*Hint:* use the linearity of the expectation, in order to write X as a convenient sum of indicator functions).
- Conclude that there is a subset of V which is independent and whose size is greater or equal than $\sum_{v\in V} \frac{1}{d_v+1}$.