Problem Sheet 10

More on Ramsey Theory

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Discrete Mathematics II, Winter 2013-2014
Deadline: 21th January 2014 (Tuesday) by 10:00, at the end of the lecture.

Problem 1: *Explicit constructions (Paley graph).* Let p be a prime of the form $1+4k$ (recall that in this situation the equation $x^2 \equiv -1\,(p)$ has solution). The Paley graph P_p is defined in the following way:

- $V(P_p)$: is the set $\mathbb{Z}/p\mathbb{Z}$.
- $E(P_P)$: two vertices $x, y \in P_p$ are linked if and only if $x-y$ is a quadratic residue modulo p (namely, $x-y$ can be written as z^2, for a certain z).

Prove the following:

1. P_{17} does not have K_4 as a subgraph.
2. Conclude that $R(4,4) > 17$.

In fact, it can be proved that $R(4,4) = 18$.

Problem 2: *Explicit constructions (Turán's graph).* Exhibit a convenient coloring of the Turán graph $T_{(k-1),(k-1)^2}$ in order to show that $R(k,k) > (k-1)^2$ (This construction is far away from the lower bound $2^{k/2}$ we proved in the lectures).

Problem 3: Construct a colouring of [8] without 3− AP's, and prove that $W(2,3) = 9$ (*Hint*: you can of course check all the 2^9 possible 2-colorings, but it is better that you pay attention at the possible colorings of 3 and 5 and consider then some few cases...).

Problem 4: Show that for every positive integer k there exists a number $M(k)$ such that if the set $\{1, 2, \ldots, M(k)\}$ is partitioned into two subsets, at least one of them contains a set of the form $\{x_1, \ldots, x_k, x_1 + \cdots + x_k\}$ (*Hint*: Consider a complete graph on vertices $\{0, \ldots, M(k)\}$, and take $M(k) = R(k+1, k+1)$. Using the initial partition of $\{1, 2, \ldots, M(k)\}$, devise a 2-coloring of the graph so that a complete monochromatic subgraph on $k+1$ vertices yields the desired set).

Problem 5: *An Anti-Ramsey result.* Let $k \geq 1$, and $f : \binom{\mathbb{N}}{k} \to \mathbb{N}$. Assume that for each choice of $i \in \mathbb{N}$ there is a $M \in \mathbb{N}$ such that $|\{\mathbf{x} \in \binom{\mathbb{N}}{k} : f(\mathbf{x}) = i\}| \leq M$. Show that there exists a subset $H \subseteq \mathbb{N}$ such that f is one-to-one on $\binom{H}{k}$ (*Hint:* $\binom{\mathbb{N}}{k}$ is a countable set, hence can be enumerated. Color $\binom{\mathbb{N}}{k}$ using M colors in a convenient way, and finally apply the infinite version of Ramsey Theorem).