Mock Exam

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Discrete Mathematics II, Winter 2013-2014
Deadline: 14th January 2014 (Tuesday) by 10:00, at the end of the lecture.

Problem 1 (2 points):

1. State the definition of perfect graph (0.5 points).

A graph G is said to be perfect if every induced subgraph H of G satisfies that $\chi(H) = \omega(H)$.

Grading: 0.5 points for the correct definition. 0.1 point will be substracted if "subgraph" is stated instead of "induced subgraph". In any case, it is not necessary to state what χ and ω are.

2. Prove that the complement of a bipartite graph is perfect (1.5 points).

Let G be a bipartite graph, and we denote by $\overline{G}$ its complement. Observe that every induced subgraph of $\overline{G}$ is also the complement of a bipartite graph. So, proving perfectness of the complement of a bipartite graph G is equivalent to proving that for every bipartite graph G, $\chi(\overline{G}) = \omega(\overline{G})$.

So, fix a bipartite graph G and consider its complement $\overline{G}$. Color it using $\chi(\overline{G})$ colors, and consider each color class. We claim that each color class has at most two vertices: recall that each color class is an independent set (namely, there cannot be edges between vertices painted with the same color). If a color class have more than 3 vertices, then G will have a triangle. And this is not possible because G is a bipartite graph.

In other words, we have shown that $\chi(\overline{G})$ is equal to $|V(G)|$ minus the maximum size of a matching in G: the number of color classes for a coloring of $\overline{G}$ is minimized when the number of color classes with two elements is maximal, and each pair of vertices in each color class define then a matching in G.

Now, applying König's Theorem for the maximum size of a perfect matching in bipartite graphs, we have that:

$$\chi(\overline{G}) = |V(G)| - \text{maximum size of a matching in } G = |V(G)| - \text{minimum size of a vertex cover in } G$$

Finally, observe that a minimum vertex cover in G defines a complete subgraph in $\overline{G}$ with maximum cardinality, hence the last value is exactly equal to $\omega(\overline{G})$, as we wanted to prove.

Grading: Additive with respect to all the ingredients of the proof:

- 0.4 points are given if it is stated (0.3) and explained (0.1) that we can restrict to show that $\chi(\overline{G}) = \omega(\overline{G})$ for every bipartite graph G.
- 0.3 points are given if it is stated (0.1) and explained (0.2) that color classes in a coloring of $\overline{G}$ must have at most 2 vertices.
- 0.5 points are given if it is stated the relation with matchings in graphs (0.3), and if König's Theorem is applied (0.2).
- 0.3 points are given if it is stated the relation between minimal vertex covers in G and cliques in $\overline{G}$ (0.2), and of course if the proof is complete (0.1), by concluding what we wanted to prove.

It is not necessary to state König's Theorem, what a vertex cover is, what a clique is, ... Additionally, the grade will be positive if Berge's Weak Perfect Graph conjecture (now theorem) is used to reduce the problem to bipartite graphs.

Problem 2 (2 points):

1. State Turán's Theorem (0.25 points) and Erdős-Stone Theorem (0.25 points).

   Turán Theorem: $\text{ex}(n, K_{r+1}) = |E(T_{r,n})| = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2) = \left(1 - \frac{1}{r}\right)\binom{n}{2} + o(n^2)$, where $T_{r,n}$ is the Turan graph on n vertices and r parts.

   Erdös-Stone Theorem: $\text{ex}(K_{s,\overset{r+1}{\dots},s}) = \text{ex}(n, K_{r+1}) + o(n^2) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2) = \left(1 - \frac{1}{r}\right)\binom{n}{2} + o(n^2)$.

   Grading: In each case, several versions are valid ($n^2/2$, $\binom{n^2}{2}$, ...). 0.1 points are subtracted in each case for minor errors.

2. State Erdős-Simonovits Corollary (0.5 points) and prove it by using Turán's Theorem and Erdős-Stone Theorem (1 point).

(Note: You do NOT have to prove neither Turán's Theorem nor Erdős-Stone Theorem).

Erdős-Simonovits Corollary: For any graph H, $\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right)\binom{n}{2} + o(n^2) = \left(1 - \frac{1}{\chi(H)-1}\right)\frac{n^2}{2} + o(n^2)$.

Proof: we observe the following:

- If G and G' are graphs such that G is a subgraph of G', then $\text{ex}(n, G) \le \text{ex}(n, G')$: when excluding G' we are also excluding G.
- If G and G' are graphs such that $\chi(G) < \chi(G')$, and G has n vertices, then $|E(G)| \le \text{ex}(n, G')$: G' cannot be a subgraph of G, so the number of edges needed in order to have a copy of G' (when having n vertices) must be greater than $|E(G)|$.

We apply these observations in our case. For the graph H, write $r = \chi(H)$. Firstly, H is a subgraph of $K_{s,\overset{r}{\dots},s}$ for s large enough (for instance, one can take $s = |V(H)|$), hence $\text{ex}(n, H) \le \text{ex}(n, K_{s,\overset{r}{\dots},s})$. Secondly, H is not a subgraph of the Turan graph $T_{r-1,m}$ for all choices of m, hence $|E(T_{r-1,n})| \le \text{ex}(n, H)$.

Finally, taking these two inequalities and applying Turan Theorem + Erdős-Stone Theorem we get the result as claimed.

Grading: 0.5 points for the correct statement. Minor errors will substract 0.2 points. It is not necessary to say explicitly that $\chi(H)$ is the chromatic number of H.

For the proof: additive with respect to the steps needed:

- Upper bound: 0.2 points for the observation (0.1 if stated and 0.1 if justified), 0.2 points for the application in our context.
- Lower bound: 0.3 points for the observation (0.1 if stated and 0.2 if justified), 0.1 points for the application in our context.
- 0.2 point for the sandwidth argument.

Problem 3 (2 points): Prove that if the graph G has a Hamiltonian cycle (namely, a connected 2-factor), then G has a 4-flow.

We shall prove that if G has a Hamiltonian cycle, then G has a $\mathbb{Z}/4\mathbb{Z}$-flow: by Tutte's Theorem, the existence of $\mathbb{Z}/4\mathbb{Z}$-flow is equivalent to the existence of 4-flow in a graph.

Hence, let G be a graph and let C be a Hamiltonian cycle on it. Then, G can be constructed from C by adding additional edges linking vertices on C (C is a cycle, and the edges of G which do not belong to C are chords of this cycle). We call these additional edges $e_1, e_2, \ldots, e_r$.

We construct a $\mathbb{Z}/4\mathbb{Z}$ on G in the following way:

- We start orienting C in one of the two possible directions, and we put a value of 1 over each edge. This trivially defines a 4-flow on C.

Once done this, we observe that for every edge which is not in C there are two cycles containing this edge and the rest of the edges are just edges of C. We recursively construct now a flow on the graph on the following way: we take e_1 and we choose one of this two cycles we have mentioned before. Orient it and put a value of 2 over each edge. In particular, e_1 that does not lay on C will have be carrying a flow of 2 units, and the flow over the edges of C is either 1 or 3. Observe that Kirchhoff's law is satisfied over each vertex (specially note that this is true in the endvertices of e_1), hence we are defining a new flow on this subgraph of G.

Assume that we have applied this procedure using edges $e_1, \ldots, e_{s-1}$, and we include now the edge e_s. Then it is obvious that the flow over each edge on C is either 1 or 3, because we are working with values in $\mathbb{Z}/4\mathbb{Z}$.

This final argument shows that at the end we have constructed a flow (it satisfies Kirchhoff's law in each vertex) with all values belonging to $\{1, 2, 3\}$ in $\mathbb{Z}/4\mathbb{Z}$. Finally, but Tutte's Theorem for flows the same result is true changing $\mathbb{Z}/4\mathbb{Z}$ by $\mathbb{Z}$.

Grading: Additive with respect to all the ingredients of the proof:

- 0.6 points are given if it is stated that we will show the result for $\mathbb{Z}/4\mathbb{Z}$ instead of 4 (0.4) by means of Tutte's Theorem for flows (0.2).
- 0.2 points are given if it is stated that a graph if hamiltonian if it can be defined as a cycle containing all the vertices, and a set of chords linking pairs of vertices.
- 0.2 points are given if it is stated that we put a unit of flow on the hamilton cycle (conveniently oriented).
- 0.8 points are given if it is observed that each chord belongs to two cycles (0.2), and it is exploited this idea to construct a new flow (sending 2 units of flow) on the subgraph (0.3). 0.3 are finally given if it is justified that doing this we get a flow (Kirchhoff's law) (0.1), which is non-zero (0.1) and with values in $\{1, 2, 3\}$ (0.2 or 0.1 if it is stated the non-zero).
- 0.2 points are given if it is said that this last procedure can be done for all chords.

Pictures explaining how the cycle is, how the chords are, how is the orientation, ... will substitute words and will be positively graded using the previous criteria.

Problem 4 (2 points): Prove that the Petersen graph is not planar.

It is sufficient to draw the Petersen graph on show that has either $K_{3,3}$ or K_5 as a minor. Then, applying Wagner's Theorem (namely, if a graph has either $K_{3,3}$ or K_5 as a minor then it is not planar), we get the result as claimed.

Grading: additive with respect to the steps on the proof:

- 0.5 points are given for a correct picture (or definition) of the Petersen graph.
- 0.5 point are given if it is stated that either $K_3, 3$ or K_5 is a minor.
- 0.5 point are given if it is shown (by pictures is valid) that either $K_{3,3}$ or K_5 is a minor.
- 0.5 points are given if Wagner's Theorem is cited in order to complete the proof.

It is not necessary to prove Wagner's Theorem at this point. Solutions using Euler's formula are also possible.

Problem 5 (2 points): Let M be a planar map defined from a simple graph G, where all faces have length 3.

- Prove that the corresponding dual map M^* has not bridges (0.25 points).

  Having bridges in the dual map is equivalent to having loops in the initial map (we saw this in the lectures). As we are starting from a map whose skeleton is a simple graph, we conclude the claim.

  Grading: show the relation with loops (0.15) and remark that we are starting from something that is simple (0.1).

- Prove that the corresponding dual map M^* has a 2-factor (0.5 points).

  Using the previous point we have obtained that the graph related to the map M^* is cubic and bridgeless. Hence, Petersen's theorem for cubic bridgeless graphs states that such a graph have a perfect matching (namely, a 1-factor). Consider then the edges that are not in the 1-factor. We claim that these edges define a 2-factor: that is true because each vertex in the defined subgraph have degree 2 (3-1, because we are not considering one of the edges incident with each vertex).

  Grading: state that the corresponding graph is cubic and bridgeless (0.1). State that by Petersen's Theorem (0.1) we have a 1-factor. State that the complementary defines a 2-factor (0.2), and justify that all vertices have degree 2 (0.1).

  It is not necessary to prove Petersen's theorem.

- Show that the vertices of the initial map M can be coloured using 2 colours (NOT a proper colouring) in such a way that every face is incident with vertices of both colours (1.25 points).

  We exhibit a coloring of the vertices which satisfies this property. In order to do so, we use the 2-factor decomposition of the graph, and its embedding. So, we consider M^* and this 2-factor.

  We observe the following:

  1. the cycles split the plane (or the sphere) in different 2-dimensional regions, and each cycle split the plane into two connected components (Jordan's curve theorem).
  2. We can color the regions in which the plane is splitted using two colors, in such a way that two regions that are incident they do not have the same color. To do this, just apply induction on the number of cycles: for one cycle it is obvious that we can do this (again, Jordan's curve Theorem states that we have two connected components). So, assuming that the result is true when having $s-1$ cycles, let us see what happen when having s cycles.

     Observe that at least one of these cycles has the following property: one of the two connected regions it defines does not contain any cycle (otherwise we won't be able to have a finite number of cycles). Delete this cycle and get a configuration with $s-1$ cycles. Color the regions by induction, and finally add again the cycle we have deleted. But now (again, by the Jordan's curve theorem) we are creating a new region (recall that this special cycle has a component without cycles), and we can simply color it in the color not used in the other region incident with this cycle we have been considering.
  3. Each 2-dimensional region corresponds to a *set* of vertices in M: each region defined by the cycles contains some faces of M^*, and by duality we have the statement.

  Hence, using these three points, we have defined a coloring of the vertices (associating the color of the region to each vertex). Finally, this coloring satisfies the property we wanted to show because the faces of M are the vertices of M^*. All these vertices belong to some cycle in the 2-factor, and hence each vertex in M^* is incident with two colors (again, by Jordan's curve Theorem).

Grading: 1 point is given for saying a coloring satisfying these properties, and 0.25 are given if it is justified correctly. To get 1 point, 0.25 points are given if it is stated that we have a "cycle structure" splitting the plane. 0.25 points are given if it is stated that the regions can be properly colored using 2 colors, and 0.25 are given if this claim is proven (by induction or other means). 0.25 points are given if finally it is said that this coloring induces a coloring on the set of vertices of the map M^*.

(Note: you can solve each part independently).